Linear Goal Programming and Experience Rating.

Antonio Heras[1], José L. Vilar, José A. Gil, Pilar García
Departamento de Economía Financiera y Actuarial.
Universidad Complutense de Madrid.

April 2001

[1]Departamento de Economía Financiera y Contabilidad 1 (Economía Financiera y Actuarial). Facultad de Ciencias Económicas y Empresariales. Universidad Complutense de Madrid. Campus de Somosaguas. Edificio 5ºCurso. Pozuelo de Alarcón. 28223 Spain. TF: 34 91 3942568. Fax: 34 91 3942570. E-mail: aheras@ccee.ucm.es

Abstract

This paper is devoted to the explanation of a new methodology in bonus malus system design, capable of taking into account very well known theoretical conditions like fairness and …nancial equilibrium of the portfolio, in addition to market conditions that could …t the resulting scale of premiums into competitive commercial settings. This is done through the resolution of a classical Bayesian decision problem, by means of minimization of the absolute error instead of the classical quadratic error. It is at this stage that we apply Goal Programming methods, which are linear thanks to the equivalence between the minimization of the absolute error and the minimization of the sum of some deviation variables which have a natural interpretation as rating errors. We show in an example how does the new methodology work. All the linear programs have been solved using the simplex method.

Keywords: Goal Programming, Simplex method, Bonus-malus system, Bayes scale, Rating error, Bayesian decision.

Contents

1 Introduction

This paper is devoted to the problem of bonus-malus systems design combining efficiency properties with a certain flexibility to fit the requirements demanded by both the insurance companies and market conditions.

It is well known that a priori rating techniques cannot eliminate the risk heterogeneity into the policyholders classes, due to the fact that some of the most important risk factors are unobservable. This fact forces many insurance companies to adopt bonus-malus rating systems, in order to adjust the premium to the policyholders claims experience.

The design of optimal or efficient bonus-malus premium scales for a given set of transition rules has been addressed in the literature as early as in 1963 (see Pesonen (1963)), proposing that the premium for a given bonus-malus class should be the expected claim amount per year of an infinitely old policy in that class. In Norberg (1976), it was proved that Pesonen's premiums minimize the expected squared rating error for a randomly chosen infinitely old policy. The premium scale obtained this way is called the Bayes Scale.

Although reasonable from a theoretical point of view, the Bayes Scale may give rise to some drawbacks when applied to practical problems. For instance, the premium levels may not form a monotonous sequence; the difference between the premiums associated with two consecutive classes may be too large or too short, as it may also be the difference between the premiums of the two extreme classes; it can strongly penalize the members of a certain class... . In short, the Bayes Scale may be inadequate because of its lack of flexibility in order to perform reasonable characteristics of real world bonus-malus scales.

Although there exist in the literature several references addressing these questions (see for example Borgan, Hoem and Norberg (1981), Sundt (1984), Lemaire (1985, 1995)), we develop in this paper a new methodology that combines simplicity and flexibility. Under certain hypothesis, an alternative scale for a bonus-malus system can be defined as the optimal solution of a certain linear program, using a multiobjective technique known as Goal Programming. The same technique has been applied very recently in actuarial mathematics (see Vilar (2000)) to avoid the obtention of negative masses in the discretization process of distributions. To our knowledge, these are the only applications of goal programming into the domain of actuarial mathematics. A brief résumé of this optimization technique can be found in the last reference, though a deeper and complete introduction to the topic of goal programming methods can be found for instance in Romero (1991, 1993).

2 Definition of a bonus malus system

Consider a group of policies which is homogeneous with respect to some observable risk characteristics. Nevertheless, there remain risk differentials within the group, due to unobservable factors. As it is usual in the literature, we assume that the risk characteristics of each policy are resumed in the value of a certain parameter ¤, and that the claim numbers from different years are conditionally independent and identically distributed given the risk parameter of the policy. We also assume that the individual claim amounts are independent of the claim numbers and the risk parameter, and mutually independent and identically distributed. Such claim numbers and amounts are also independent of the choice of the bonus-malus system, that is, bonus hunger and moral hazard are not taken into account.

Following several authors (see for example Lemaire (1985,1995), Walhin and Paris (1999)), we identify the value of the risk parameter of a policy with his mean claim frequency. Such mean claim frequency is assumed to be stationary in time, i.e. not time dependent. In this case, taking the mean individual claim cost as one monetary unit, our objective is to calculate a pure premium for the insured as close as possible to the (unknown) true value of his parameter. We will try to perform this objective by means of a bonus-malus system. Of course, such a system will be based on the number of claims and not on their amount. In fact, almost all the real bonus-malus systems around the world are exclusively based on the number of claims (see Lemaire (1985, p. 129)).

Finally, we assume that the risk parameter ¤ is a random variable with known cumulative function $U(\cdot)$. Such a distribution is not a subjective distribution in the pure Bayesian sense, it has a frequency interpretation as different policies will have different values of their risk parameters.

Following Lemaire (1995, p. 6), we say that an insurance company uses a bonus-malus system when the following conditions hold:

- There exists a finite number of classes $(C_1, \ldots, C_n)$ such that each policy stays in one class throughout each insurance period (usually a year).
- The premium for each policy depends only on the class where it stays.
- The class for a given period is determined by the class in the preceding period and the number of claims reported in that period (Markovian Condition).

Every bonus-malus system is determined by three elements:

- The initial class, where the new policies are assigned.
- The premium scale $(P_1, \ldots, P_n)$, where P_i is the premium for policies in the class C_i.
- The transition rules, that is, the rules that establish the conditions under which a policy in class C_i is transferred to class C_j in the next period.

Focusing the last point, such rules are usually defined by means of transformations T_k such that $T_k(i) = j$ when policyholders in class C_i reporting k claims are transferred to class C_j in the next period. Transformations T_k are usually described by means of matrices,

$$T_k = \left(t_{ij}^k \right)$$

where

$$\begin{aligned} t_{ij}^k &= 1 \text{ if } T_k(i) = j \\ t_{ij}^k &= 0 \text{ if } T_k(i) \neq j \end{aligned}$$

The conditional transition probability from C_i to C_j in one period, given that $\Lambda = \lambda$, can be calculated as

$$p_{ij}(\lambda) = \sum_{k=0}^{\infty} p_k(\lambda) t_{ij}^k$$

where $p_k(\lambda)$ is the conditional probability of reporting k claims in one period given that $\Lambda = \lambda$, that is,

$$p_k(\lambda) = \Pr[N = k \mid \Lambda = \lambda]$$

The conditional transition matrix, given that $\Lambda = \lambda$, is defined as

$$P(\lambda) = (p_{ij}(\lambda))$$

These definitions allow us to look at the bonus-malus system as a Markov chain. This chain is homogeneous, since we have assumed that each claim frequency λ is stationary in time. The previously defined transition matrix $P(\lambda)$ will be the transition matrix of the corresponding Markov chain.

If we also assume that the chain is ergodic and without cycles, then it is well known (see for instance Grimmett and Stirzaker (1992)) that there exists a stationary (conditional) probability distribution $(\pi_1(\lambda); \ldots; \pi_n(\lambda))$, where $\pi_i(\lambda)$ is defined as the limit value (when the number of periods $\to \infty$) of the conditional probability that a policy belongs to the class C_i, given that $\Lambda = \lambda$.

It can be shown that the stationary probability distribution coincides with the left eigenvector associated with the eigenvalue 1 of the corresponding transition matrix, and whose components add the unit.

It is also possible to define the stationary (unconditional) probability distribution $(\pi_1; \ldots; \pi_n)$ for an arbitrary policy as the mean value of the stationary conditional probability distributions $(\pi_1(\lambda); \ldots \pi_n(\lambda))$. That is,

$$\pi_i = \int \pi_i(\lambda) dU(\lambda) \qquad (1)$$

It is clear that the probabilities π_i and $\pi_i(\lambda)$ can also be interpreted as the fraction of arbitrary policies and policies conditioned to $\Lambda = \lambda$, respectively, that belong to class C_i when stationarity is attained.

It should be clear that the knowledge of the stationary distributions could be very useful in order to design a bonus-malus system, because it informs us about the long term distribution of the policies.

3 The Bayes Scale

There are three problems related to the construction of a bonus-malus system:

- How to choose the number of classes and the transition rules.
- How to choose the initial class.
- How to choose the premium associated with every class.

This paper focuses mainly in the third problem, though some comments will be also made about the choice of the number of bonus-malus classes.

The first problem still constitutes an open problem: it is not possible in general to find the optimal number of classes and the optimal set of transition rules (see Lemaire (1995)), although it is possible to conclude that a certain set of classes and rules is better than another set.

The selection of the initial class cannot be based on the stationary distribution, since the last one does not depend upon such a choice. However, this initial class has a certain influence into the time the system needs to approach stationarity.

The standard solution to the calculation of the scale of premiums is based on the stationary distribution. As we mentioned in the introduction, Norberg (1976) has proposed (following an idea explained in Pesonen (1963)) to define the premium associated with a certain class as the expected yearly loss of an infinitely old policy in that class, that is, of a policy that stays in the class once the system has reached stationarity:

$$P_i = E[\theta \mid \text{Policy} \in C_i \text{ after infinite periods}]$$

that is,

$$P_i = \frac{1}{\pi_i} \int \theta \pi_i(\theta) dU(\theta)$$

Norberg (1976) showed that such a scale (known as the Bayes Scale) minimizes the expected squared rating error, defined as the expected squared difference between the mean claim frequency and the premium actually paid, for an infinitely old randomly chosen policy. This expected squared rating error is also useful in order to compare the Bayes Scales associated with different transition rules.

It is important to remark that Norberg's result can be obtained as the optimal solution of a certain Bayesian Decision Problem.

In general decision problems it is well known (see, for instance, DeGroot (1970)) that, if the decision maker's preferences over the possible consequences of his decisions are consistent with certain axioms of rational behavior, then it is possible to define a function over those consequences (called the utility of the consequences) such that one feasible decision will be preferred to another if, and only if, the expected utility of the possible consequences is larger for the first decision than it is for the second. In decision problems it is usual to specify the negative of the utility, instead of the utility, and to call it the loss. Then the decision maker should choose as optimal the decision that minimizes the expected loss of his consequences.

Specifically, the decision maker must proceed as follows: first, he must define a consequence for every feasible decision and every possible realization of the random parameters; second, he must specify a numerical loss associated with every consequence; finally, he calculates the expected loss of every feasible decision, and he chooses as optimal the decision with minimal expected loss.

In order to apply this general framework to our particular problem, let us consider an arbitrary policy in the stationary state. If we formulate a Bayesian decision problem in which the random states are the parameter θ associated with the policy and the class C_i to which the policy belongs, the feasible decisions are the possible scales $(P_1, \ldots, P_n)$ and the loss associated with every state (λ, C_i) and every feasible decision $(P_1, \ldots, P_n)$ is a quadratic loss $(P_i - \lambda)^2$, then the optimal decision is the one that minimizes the expected loss

$$\int \sum_{i=1} (P_i - \lambda)^2 \pi_i(\lambda) dU(\lambda)$$

which coincides with the previously defined expected squared rating error. The optimal decision is therefore to adopt the Bayes Scale.

An equivalent expression for the expected squared rating error would be

$$E(P_I - \Lambda)^2$$

where I denotes the class as a random variable with conditional probabilities $\pi_i(\lambda)$ given $\Lambda = \lambda$.

As we commented in the Introduction, the Norberg model based on Bayes scales constitutes the standard mathematical model for the construction of an optimal bonus-malus. Indeed, the use of the quadratic loss function implies that the expected premiums will be equal to the expected claims, that is, the system is financially balanced. This is a very attractive property in premium calculation as it guarantees an expected equilibrium between claims and premiums. Nevertheless, the Bayes scale suffers from several drawbacks. For example, in Sundt (1984) it is remarked that Bayes scales do not necessarily form monotonous sequences, and a solution is proposed to restrict the class of admissible scales. Moreover, in Borgan, Hoem and Norberg (1981) the basic model is modified in order to include young policies that have not had enough time to approach stationarity.

But we think that there are some other drawbacks not yet discussed in the literature about Bayes scales. Let us briefly comment some of them:

- The use of quadratic loss functions implies the equal valuation of over-achievements and underachievements around the true value of the parameter λ. Nevertheless, it could be interesting to distinguish between the policyholders' and insurer's extraordinary earnings, respectively.

- The formula for the expected squared rating error equally weights the errors in all the bonus-malus classes. But we think that the weights could depend of the value of Λ: it seems clear that an error $(P_i - \lambda)^2 = 1$ is less important when $\lambda = 10$ than when $\lambda = \frac{1}{2}$, for instance.

- And it is also unclear how to obtain a smoothed bonus-malus system if the original one resulted too harsh to the policyholders and therefore they were tempted to quit the insurance company. The same can be said about the inclusion in the bonus-malus system of certain desirable features such as upper (or lower) limits for the differences between two consecutive premiums, or for the difference between the two extreme premiums, etc.

In summary, we think that another drawback of the model is its lack of flexibility in order to incorporate several reasonable properties. This is due to the major difficulty of solving a quadratic program with a great number of constraints. Moreover, if we succeed in solving the constrained quadratic program, the optimal solution does not verify in general the desirable properties previously mentioned, such as the financial equilibrium. Let us illustrate this fact with an easy example:

Let us suppose three possible values of the random parameter Θ ($\lambda_1 = 0.5$ (with probability $\frac{1}{3}$), $\lambda_2 = 1$ (with probability $\frac{1}{3}$) and $\lambda_3 = 1.5$ (with probability $\frac{1}{3}$)), and three bonus-malus classes (C_1, C_2 and C_3). The transition rules imply the following conditional stationary probability distributions:

- If $\Theta = \lambda_1$, $\pi_1(\lambda_1) = \frac{3}{5}$, $\pi_2(\lambda_1) = \frac{1}{5}$, $\pi_3(\lambda_1) = \frac{1}{5}$.
- If $\Theta = \lambda_2$, $\pi_1(\lambda_2) = \frac{1}{3}$, $\pi_2(\lambda_2) = \frac{1}{2}$, $\pi_3(\lambda_2) = \frac{1}{6}$.
- If $\Theta = \lambda_3$, $\pi_1(\lambda_3) = \frac{1}{4}$, $\pi_2(\lambda_3) = \frac{1}{4}$, $\pi_3(\lambda_3) = \frac{1}{2}$.

The unconditional stationary probability distribution is

$$\pi_1 = \left(\frac{3}{5} + \frac{1}{3} + \frac{1}{4}\right) \times \frac{1}{3} = \frac{71}{60} \times \frac{1}{3}$$

$$\pi_2 = \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{4}\right) \times \frac{1}{3} = \frac{19}{20} \times \frac{1}{3}$$

$$\pi_3 = \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{2}\right) \times \frac{1}{3} = \frac{13}{15} \times \frac{1}{3}$$

The Bayes scale is defined as the premiums $(P_1; P_2; P_3)$ that minimize the value of the following objective function:

$$\begin{aligned}
&(P_1 - \lambda_1)^2 \frac{3}{5} \times \frac{1}{3} + (P_1 - \lambda_2)^2 \frac{1}{3} \times \frac{1}{3} + (P_1 - \lambda_3)^2 \frac{1}{4} \times \frac{1}{3} + \\
&+ (P_2 - \lambda_1)^2 \frac{1}{5} \times \frac{1}{3} + + (P_2 - \lambda_2)^2 \frac{1}{2} \times \frac{1}{3} + (P_2 - \lambda_3)^2 \frac{1}{4} \times \frac{1}{3} + \\
&+ (P_3 - \lambda_1)^2 \frac{1}{5} \times \frac{1}{3} + (P_3 - \lambda_2)^2 \frac{1}{6} \times \frac{1}{3} + + (P_3 - \lambda_3)^2 \frac{1}{2} \times \frac{1}{3}
\end{aligned}$$

The optimal solution of this program is

$$P_1 = \frac{121}{142} \simeq .85211; P_2 = \frac{39}{38} \simeq 1.0263; P_3 = \frac{61}{52} \simeq 1.1731$$

We know that Bayes scales give rise to financially balanced bonus-malus systems. It is easy to check the financial equilibrium in this example: the expected claims are $E(\Theta) = 1$, while the expected premiums are

$$P_1\pi_1 + P_2\pi_2 + P_3\pi_3 = 1$$

But let us imagine that this system becomes inadequate for practical purposes, perhaps because the policyholders prefer larger differences between the two extreme premiums. In such a case marketing reasons could recommend the construction of a bonus-malus system such that the good drivers pay, for example, half as much as bad drivers. That is, in mathematical terms, such that $P_3 = 2P_1$. If we add this constraint to our objective function, we obtain a constrained quadratic program. The optimal solution is now,

$$P_1 = 0.6541; P_2 = 1.0263; P_3 = 1.3082$$

Unfortunately, we have lost the financial equilibrium property: in fact, the expected premium takes now the value of 0.96, and the company will get ruined in the long run.

The introduction of additional constraints in the quadratic program is not a good idea, since we lose, in general, the financial equilibrium property, which is the most important property of Bayes scales. Moreover, the resolution of the constrained quadratic program becomes difficult in many real world problems with a great number of classes and constraints. An alternative procedure could be to change the transition rules and/or the number of classes. But again in real world bonus-malus systems it may be difficult to find the transition rules giving scales with the appropriate properties.

In the next section we build an alternative bonus-malus scale obtained as an optimal solution of a linear program. In such a program it is easy to introduce the desirable properties of the bonus-malus system, related for instance to fairness, commercial requirements and financial equilibrium, by means of additional linear constraints.

4 An alternative model for the construction of an optimal bonus malus system.

4.1 Theoretical settings

Pursuing the objective of solving all the problems previously mentioned, we suggest a change in the formulation of the Bayesian problem that gave rise to the bonus-malus system.

The Bayesian decision problem defined in the previous section is not the only possible formulation in order to design an optimal bonus-malus system. An arbitrary policy that reaches the stationary state does not necessarily stay forever in a same class. It is the probability $\pi_i(\lambda)$ of (temporarily) belonging to class C_i that remains constant for a policy with parameter λ in the stationary state. Such policy can change the class according to these probabilities, and therefore the mean value of the premiums paid by that policyholders will be $\sum_{i=1}^{n} P_i \pi_i(\lambda)$.

Consequently, we propose a new Bayesian decision problem in which the feasible decisions are again the scales $(P_1, \ldots, P_n)$, the random states are the values of θ and the loss associated with every feasible decision $(P_1, \ldots, P_n)$ and every realization λ of the random parameter θ is not the squared rating error, but the absolute value of such a rating error

$$\left| \sum_{i=1}^{n} P_i \pi_i(\lambda) - \lambda \right|$$

In this case, the optimal decision will be a scale $(P_1, \ldots, P_n)$ that minimizes

$$\int \left| \sum_{i=1}^{n} P_i \pi_i(\lambda) - \lambda \right| dU(\lambda) \tag{2}$$

This expression is, in general, rather difficult to solve. Nevertheless, if we assume a discrete distribution for the parameter θ (taking the values $\lambda_1, \ldots, \lambda_m$ with probabilities $q_1, \ldots, q_m$, respectively), then it becomes equivalent to both:

$$\sum_{j=1}^{m} \left| \sum_{i=1}^{n} P_i \pi_i(\lambda_j) - \lambda_j \right| q_j$$

and the linear program

$$\min \sum_{j=1}^{m} \left(y_j^+ + y_j^- \right) q_j; \quad s.t.: \left\{ \begin{array}{l} P_1 \pi_1(\lambda_1) + \ldots + P_n \pi_n(\lambda_1) + y_1^- - y_1^+ = \lambda_1 \\ \vdots \\ P_1 \pi_1(\lambda_m) + \ldots + P_n \pi_n(\lambda_m) + y_m^- - y_m^+ = \lambda_m \\ y_j^+, y_j^- \geq 0; \; j = 1, \ldots, m \end{array} \right. \tag{3}$$

where the optimal values of the new variables y_j^+, y_j^- (let us denote them as y_j^{*+}, y_j^{*-}) represent the positive and negative rating error, respectively, for a policy with parameter λ_j. This fact, as well as the equivalence between both previous programs, are consequences of the fact that the optimal solution of the linear program verifies $y_j^{*+} y_j^{*-} = 0, \forall j$ (see for instance Sawaragi, Nakayama and Tanino (1985, p. 253)), that is, only one of them can be non null, and therefore

$$\forall j, \left| \sum_{i=1}^{n} P_i \pi_i(\lambda_j) - \lambda_j \right| = y_j^{*+} + y_j^{*-}$$

It is important to remark that program (3) is a particular case of the multiobjective technique known as Goal Programming. This fact has an easy interpretation: by means of Goal Programming the decision maker tries to find the values of the decision variables such that certain objective functions take values as close as possible to a set of targets previously defined. In our problem such targets are $\lambda_1, \ldots, \lambda_m$, the possible values of the parameter θ, and we try to find the values of the decision variables $P_1, \ldots, P_n$ in order to approximate, for every policy with parameter λ_j, the mean value of the premiums paid by that policyholder ($\sum_{i=1}^{n} P_i \pi_i(\lambda_j)$) and his real mean claim frequency λ_j.

Goal Programming methodology has become one of the most important tools for solving multicriteria optimization problems, perhaps because it is an easy way to handle technique with a high flexibility in fitting the characteristics of real world problems. We see below that this flexibility allows to incorporate in our bonus-malus system all the reasonable features mentioned in the previous section.

For instance, if we define the objective function of the linear program as

$$\min \sum_{j=1}^{m} \left(\omega_j^+ y_j^+ + \omega_j^- y_j^- \right) q_j$$

then it would be possible to assign different weights (even depending on the different discrete values of $*$) to positive and negative rating errors.

It is also possible to establish a relation between the optimal solution of our linear program and the degree of financial equilibrium: in fact, this is measured by the expression

$$\sum_{i=1} P_i \pi_i - \sum_{j=1}^{m} \lambda_j q_j \qquad (4)$$

where $\pi_i = \sum_{j=1}^{m} \pi_i(\lambda_j) q_j$ is now a discretization of (1). Positive values of (4) denote a gain for the insurance company, while negative ones denote a loss (i.e. a gain for the policyholders). Multiplying each constraint of the linear program by the corresponding probability and adding all these constraints, we obtain a third equivalent expression for the degree of financial equilibrium:

$$\sum_{j=1}^{m} \left(y_j^{*+} - y_j^{*-} \right) q_j \qquad (5)$$

where $y_j^{*+}; y_j^{*-}$ are the optimal deviation variables of our linear program.

On the other hand, the linear formulation of the decision problem allows to easily incorporate any characteristic that could be considered necessary. For instance, making null any of the previous expressions (4) or (5), allow us the introduction of the financial equilibrium condition in our linear program, by means of a new linear constraint.

The nonnegativity of the premiums should be introduced as

$$P_i \ge 0; \forall i$$

Linear constraints as

$$P_{i+1} - P_i \ge d$$

introduce a lower limit $d > 0$ for the distance between two consecutive premiums. Similarly, the constraints

$$0 \le P_n - P_1 \le D$$

introduce an upper limit $D > 0$ for the distance between the two extreme premiums.

As in the case of Bayes scales, the premiums obtained as optimal solutions of our linear program do not necessarily form a monotonous sequence. In order to get that monotony property we only need to include some additional constraints:

$$P_1 \leq P_2 \leq \dots \leq P_n$$

It is also possible to modify our program in the sense of Borgan, Hoem and Norberg (1981), allowing the existence of policies that have not yet approached stationarity.

Finally, we must remember that the previous results find a premium scale only for some given classes and transition rules. As in the case of Bayes scales, alternative sets of different classes and transition rules should be evaluated and compared according to the optimal value of the objective function.

As an easy example of the methodology, we can apply it to the construction of the bonus-malus discussed at the end of section 3. Remember that we calculated the Bayes scale, and we supposed that it was unsatisfactory because the range of the scale was too narrow. Of course, in such a case one should try to define new classes and/or change the transition rules, in order to get a new Bayes scale with the desired properties. Such methodology may be successful in many cases, but also may be inadequate when the system has a great number of classes and complicated transition rules. In that case, it may be reasonable to introduce the desired properties as linear constraints using our Goal Programming methodology. In this example, the linear program is

$$\min \left(y_1^+ + y_1^- + y_2^+ + y_2^- + y_3^+ + y_3^- \right) \frac{1}{3}; \quad s.t. \begin{cases} P_1\frac{3}{5} + P_2\frac{1}{5} + P_3\frac{1}{5} + y_1^- - y_1^+ = 0.5 \\ P_1\frac{1}{3} + P_2\frac{1}{2} + P_3\frac{1}{6} + y_2^- - y_2^+ = 1 \\ P_1\frac{1}{4} + P_2\frac{1}{4} + P_3\frac{1}{2} + y_3^- - y_3^+ = 1.5 \\ P_i \geq 0; \forall i \\ y_j^+; y_j^- \geq 0; \forall j \\ P_3 - 2P_1 = 0 \\ P_1\frac{71}{60}\frac{1}{3} + P_2\frac{19}{20}\frac{1}{3} + P_3\frac{13}{15}\frac{1}{3} = 1 \end{cases}$$

The last constraint establishes the financial equilibrium condition. The optimal solution of this linear program is easy to find by means of the simplex method:

$$P_1 = 0.6666; P_2 = 1.1114; P_3 = 1.3332$$

In the following sections we are going to apply the methodology step by step in a more precise way.

4.2 Exemplification of the new methodology

In this section we will proceed to the application of the new methodology of bonus-malus system design to a theoretical portfolio of policies. Along the exemplification we will define and discuss the following points:

- The density function that models the portfolio heterogeneity, and its discretizations.
- The transition rules and the number of bonus malus-classes.
- The feasible set of the linear program.
- The objective function and the linear program that furnishes the scale of premiums.
- The behavior of the solution.

4.2.1 The density function $u(\lambda)$

We will consider a portfolio with claims distributed by means of a Poisson distribution mixed with a Gauss Inverse. This model states that given $\Lambda = \lambda$; the claim numbers are conditionally independent and identically Poisson distributed with parameter λ; and Λ has an inverse Gaussian distribution with density function:

$$u(\lambda) = \sqrt{\frac{\theta}{2\pi\lambda^3}}\, e^{-\frac{\theta(\lambda-\mu)^2}{2(\lambda\mu^2)}}; \qquad \lambda > 0.$$

In our example we arbitrarily set the two parameters to be $\mu = 0.5$ and $\theta = 3.5$, obtaining the density over a time unit period represented graphically in figure 1.

FIGURE 1

In a real application this density function should be obtained as the result of a classical fitting process of some mixed Poisson distribution to the number of claims data over a certain period.

Once the number of claims model has been set out, our method requires the discretization of the density function. At this stage it is worth noting that the function $u(\lambda)$ is an endogenous element describing the quality of the selected portfolio (its heterogeneity), while any discretization will be just a representation of that function. For example, we must arbitrarily choose the mesh and the class markers among an infinity of possible Λ values. Though we expect that the discretized version tends towards the real function when this mesh vanishes, we cannot a priori tell anything about the effect of changing the representation on the final result (i.e. on the bonus malus scale). In other words, if we are interested in measuring any possible sensitivity of the method with respect to different discretizations of the density function, we will have to investigate it using several discretizations of $u(\lambda)$. Given the shape of this one, we have chosen three different values of the mesh $h > 0$ in the aim of recreating a gradually closer representation of the density function. These are $h = 0.3;\ 0.15;\ 0.075$; with supports containing $5; 10; 20$ values of Λ respectively:

With regard to the discretization method, we proceed in the three cases to the arithmetization of $u(\lambda)$ by applying linear goal programming as explained in Vilar (2000). The only difference is that the first mass has been placed at the end of the first subinterval instead of the origin. As a result we get an arithmetic probability function with first and second moments very near to the original ones (the non exactness is due to the mass transfer operated in the first subinterval).

We next summarize the probability functions obtained for the three cases, that will be used in the sequel:

$h = 0.3$: The probabilities are

$$q_1 = .4655835; \quad q_2 = .4088527; \quad q_3 = .1209949; \quad q_4 = .002537622;$$
$$q_5 = .0020313.$$

placed over the points

$$\lambda_1 = 0.3; \quad \lambda_2 = 0.6; \quad \lambda_3 = 0.9; \quad \lambda_4 = 1.2; \quad \lambda_5 = 1.5.$$

For instance λ_1 is to be interpreted as representing all the policyholders with Poisson parameters comprised between 0 and 0.6; λ_2 stands for the ones whose parameters lies between 0.6 and 0.9. The last point λ_5 stands for the ones having $\lambda \geq 1.5$. In raw, we can interpret that this portfolio consists in 46% of "good" policies (that is, they have Poisson parameter comprised between 0 and 0.6), 41% of "less good" policies (with Poisson parameter comprised between 0.6 and 0.9), etc...

$h = 0.15$: The probabilities are

$$\begin{aligned} &q_1 = .03384627; \quad q_2 = .1923699; \quad q_3 = .4448879; \quad q_4 = .1462023; \\ &q_5 = .1364640; \quad q_6 = .02036232; \quad q_7 = .02039220; \quad q_8 = .002342893; \\ &q_9 = .002511476; \quad q_{10} = .0006207 \end{aligned} \tag{6}$$

placed over the points

$$\begin{aligned} &\lambda_1 = 0.15; \quad \lambda_2 = 0.3; \quad \lambda_3 = 0.45; \quad \lambda_4 = 0.6; \quad \lambda_5 = 0.75; \\ &\lambda_6 = 0.9; \quad \lambda_7 = 1.05; \quad \lambda_8 = 1.2; \quad \lambda_9 = 1.35; \quad \lambda_{10} = 1.5. \end{aligned}$$

The same comments as in the first case could be made.

$h = 0.075$: The probabilities are

$$\begin{aligned} &q_1 = .00008576107; \quad q_2 = .0004790433; \quad q_3 = .06647720; \quad q_4 = .09503722; \\ &q_5 = .2356366; \quad q_6 = .1228080; \quad q_7 = .1937991; \quad q_8 = .07100643; \\ &q_9 = .09909049; \quad q_{10} = .03105860; \quad q_{11} = .04144804; \quad q_{12} = .01201900; \\ &q_{13} = .01571538; \quad q_{14} = .004368267; \quad q_{15} = .005650766; \quad q_{16} = .001533364; \\ &q_{17} = .001970548; \quad q_{18} = .0005272460; \quad q_{19} = .0006751511; \quad q_{20} = .0006138; \end{aligned}$$

placed over the points

$$\begin{aligned}
\lambda_1 &= 0.075; \quad \lambda_2 = 0.15; \quad \lambda_3 = 0.225; \quad \lambda_4 = 0.3; \quad \lambda_5 = 0.375; \\
\lambda_6 &= 0.45; \quad \lambda_7 = 0.525; \quad \lambda_8 = 0.6; \quad \lambda_9 = 0.675; \quad \lambda_{10} = 0.75; \\
\lambda_{11} &= 0.825; \quad \lambda_{12} = 0.9; \quad \lambda_{13} = 0.975; \quad \lambda_{14} = 1.05; \quad \lambda_{15} = 1.125; \\
\lambda_{16} &= 1.2; \quad \lambda_{17} = 1.275; \quad \lambda_{18} = 1.35; \quad \lambda_{19} = 1.425; \quad \lambda_{20} = 1.5.
\end{aligned}$$

From hereafter when referring in general to the probability masses of the arithmetized density function we will note $(q_j)_{j=1}^{m}$.

4.2.2 The transition rules and the number of bonus malus classes.

The transition rules are a design element very important indeed, and its exogenous character is also clear, as they are chosen by the designer. Another element open to the designer's choice is the number n of bonus malus classes. As we want to investigate the influence of a growing number of classes into the resulting solutions, our transition rules will consist always in the same idea applied as many times as n.

Having in mind that a typical policy of the portfolio has claim frequency $\lambda = 0.5$ (i.e. 5 claims each ten years), the idea for defining the rules is the following: if a policy has one or two claims during the year, it will stay in the same class; it will climb one class for three or more claims; and it will descend one class for zero claims. These rules must be adapted for the lowest (cheapest) and highest bonus malus class: in the first case the policy remains in the same class if it has $0, 1$, 2 claims and goes to the superior class for 3 or more claims; in the second case it remains in the upper bonus malus class for 1 or more claims and descend to the previous bonus malus class for 0 claims. When writing down the transition rules, we will use the notation

$$i \xrightarrow{\{k_1, \ldots, k_n\}} j$$

meaning that in case a policy had $k_1, \ldots, k_n$ claims during the time period, it would be transferred from class i to class j. For instance for the case $n = 5$ the rules are the following:

$$\begin{array}{lll}
1 \xrightarrow{\{0,1,2\}} 1; & 1 \xrightarrow{\{3,4,\ldots\}} 2 & \\
2 \xrightarrow{\{0\}} 1; & 2 \xrightarrow{\{1,2\}} 2; & 2 \xrightarrow{\{3,4,\ldots\}} 3 \\
3 \xrightarrow{\{0\}} 2; & 3 \xrightarrow{\{1,2\}} 3; & 3 \xrightarrow{\{3,4,\ldots\}} 4 \\
4 \xrightarrow{\{0\}} 3; & 4 \xrightarrow{\{1,2\}} 4; & 4 \xrightarrow{\{3,4,\ldots\}} 5 \\
5 \xrightarrow{\{0\}} 4; & 5 \xrightarrow{\{1,2,\ldots\}} 5 &
\end{array}$$

While for $n = 3$ we get:

$$
\begin{array}{lll}
1 \xrightarrow{\{0,1,2\}} 1; & 1 \xrightarrow{\{3,4,\dots\}} 2 & \\
2 \xrightarrow{\{0\}} 1; & 2 \xrightarrow{\{1,2\}} 2; & 2 \xrightarrow{\{3,4,\dots\}} 3 \\
3 \xrightarrow{\{0\}} 2; & 3 \xrightarrow{\{1,2,\dots\}} 3 &
\end{array}
$$

We will let n vary from 3 to 10 bonus malus classes, and then look to the resulting scales of premiums and rating errors.

In all these cases we will have to obtain the corresponding transition matrix $P(\lambda)$, and calculate the stationary distribution $\pi(\lambda)$. Thanks to the last we will be able later to write down the fairness and financial equilibrium constraints. As an example we give here the results corresponding to $n = 5$. Writing down only the non zero coefficients of the transition matrix:

$$
\begin{aligned}
&p_{11}(\lambda) = e^{-\lambda} + \lambda e^{-\lambda} + \tfrac{1}{2}\lambda^2 e^{-\lambda} \\
&p_{12}(\lambda) = p_{23}(\lambda) = p_{34}(\lambda) = p_{45}(\lambda) = 1 - e^{-\lambda} - \lambda e^{-\lambda} - \tfrac{1}{2}\lambda^2 e^{-\lambda} \\
&p_{21}(\lambda) = p_{32}(\lambda) = p_{43}(\lambda) = p_{54}(\lambda) = e^{-\lambda} \\
&p_{22}(\lambda) = p_{33}(\lambda) = p_{44}(\lambda) = \lambda e^{-\lambda} + \tfrac{1}{2}\lambda^2 e^{-\lambda} \\
&p_{55}(\lambda) = 1 - e^{-\lambda}
\end{aligned}
$$

It is worth using a symbolic calculator assistant to get the stationary distributions. In the case $n = 5$, a fine use of it will give us the five coordinates of the left eigenvector associated to the unit eigenvalue of $P(\lambda)$. We write it L^1-normalized, where $\|\pi\|_1$ stands for the sum of the five coordinates of the eigenvector:

$$
\begin{aligned}
\pi_1(\lambda) &= \frac{1}{\|\pi\|_1} \\
\pi_2(\lambda) &= -\frac{1}{2\|\pi\|_1 e^{-\lambda}}(-2 + 2e^{-\lambda} + 2\lambda e^{-\lambda} + \lambda^2 e^{-\lambda}) \\
\pi_3(\lambda) &= \frac{1}{4\|\pi\|_1 e^{-2\lambda}}(-2 + 2e^{-\lambda} + 2\lambda e^{-\lambda} + \lambda^2 e^{-\lambda})^2 \\
\pi_4(\lambda) &= -\frac{1}{8\|\pi\|_1 e^{-3\lambda}}(-2 + 2e^{-\lambda} + 2\lambda e^{-\lambda} + \lambda^2 e^{-\lambda})^3 \\
\pi_5(\lambda) &= \frac{1}{16\|\pi\|_1 e^{-4\lambda}}(-2 + 2e^{-\lambda} + 2\lambda e^{-\lambda} + \lambda^2 e^{-\lambda})^4
\end{aligned}
$$

For the sake of brevity we do not write down either $P(\lambda)$ or $\pi(\lambda)$ in the other seven cases.

4.2.3 The feasible set

The feasible set is defined by means of three types of restrictions, corresponding to different properties that we want to see verified by the solutions.

Fairness of the bonus malus system A first set of m equality constraints, defines the fairness or equity of the bonus malus-system. They state that a policy belonging to class j; should pay through its path over the bonus malus classes in the long run, its marked value λ_j:

$$j = 1, \dots, m: \sum_{i=1} P_i \pi_i(\lambda_j) + y_j^- - y_j^+ = \lambda_j$$

This set of constraints should be always included in the definition of any feasible set as it stands for a good property that should be verified by any scale of premiums.

Financial equilibrium The last observation also applies to the second set, which in fact consists in taking (4) as an equality that ensures the financial equilibrium of the portfolio, that is to say, when there is no gain for the insurance company or the policyholders in the long run. Remember this constraint was $\sum_{i=1}^{n} P_i \pi_i = \sum_{j=1}^{m} \lambda_j q_j$; where π_i was to be interpreted as a discretization (1). In other words the expected premiums have to equal the expected claims.

Market constraints Here we want to translate into the feasible set some market conditions. Thus these constraints will depend upon some estimation of the policyholders preferences, the market characteristics (the scales of premiums used by the other insurance companies, for instance), the subjective appreciation of the bonus malus designer in tying to catch the former and the later, and also the targets of the insurance company which wants to introduce the new bonus malus system.

We also recall that we are going to construct and solve not just one linear program but a family of them in order to extract some conclusions. For instance, we wrote that we were going to let the number n of bonus malus classes growing from 3 to 10. As we have to use the same type of feasible sets in all these resolutions, our definition of the market constraints should be therefore enough general to be particularized in all these cases.

Having this in mind, we have proceeded as follows. Our constraints are always defined in such a way that they seek for the cheapest (lowest) bonus malus class a maximal bonus of 60% of the premium $P_{Central}$ corresponding to a central bonus malus class (say the third class if we had previously chosen five or six bonus malus-classes), and a maximal malus in the more expensive (highest) class of 100% of $P_{Central}$. Thus we obtain the following two constraints:

$$P_1 \geq 0.6 P_{Central}; \quad P_n \leq 2 P_{Central}$$

This is to guarantee that the cheapest premium is significant for the insurance company and represents an appreciable global bonus level for the policyholder, while the most expensive one does not exceed some kind of policyholder's tolerance. We could also think that these inequalities try to translate some kind of usual fork between the extreme premiums of bonus malus systems found in the market. As an additional comment, we could see the central class as the initial class for the new policies entering the portfolio. Moreover, we want the bonus between two consecutive classes P_i and P_{i+1}; to be the larger the better (so they will be significantly appreciated by present and future policyholders), so we have imposed a minimum amount for it in terms of percentage (10%) of the cheapest premium P_i :

$$P_{i+1} \geq 1.1\, P_i \,; \quad i = 1, \ldots, n-1$$

We have not written nonnegativity constraints in the hope that these will be fulfilled by the solution in a natural way. In fact, this is the case as it will be seen in the solutions. Monotony conditions are implicitly assumed through the last $n-1$ inequalities. As an example, the case $n = 5$ gives us the following constraints:

$$P_1 \geq 0.6\, P_3; \quad P_5 \leq 2\, P_3; \quad P_{i+1} \geq 1.1\, P_i \quad (i = 1, \ldots, 4)$$

Another example with $n = 9$:

$$P_1 \geq 0.6\, P_5; \quad P_9 \leq 2\, P_5; \quad P_{i+1} \geq 1.1\, P_i \quad (i = 1, \ldots, 8)$$

4.2.4 Objective functions and linear programs

As told earlier in this paper, our linear programs seek the minimization of the sum of the rating errors. Alternatively we can call them under and overachievements (y_j^-,y_j^+ respectively), or simply deviations.

In our objective function are contained another kind of variables that can be substituted by the bonus malus designer at his own will, the weights $\omega_i^+; \omega_i^-$. These could be quite practical because taking different weights could produce new optimums -scales of premiums- to be submitted to the choice of the bonus malus designer. This is a well known technique in mathematical programming. Nevertheless it is not the case in the present example, as it will be seen: when setting the financial equilibrium constraint, the method produces the same optimal scale of premiums and rating errors for different values of the weights (at least for the values we have substituted!).

Thus taking all the weights equal ($\omega_j^+ = \omega_j^- = 1$) we obtain the following linear program:

$$\min \sum_{j=1}^{m} \left(y_j^+ + y_j^- \right) q_j \,; \quad s.t.: \begin{cases} \sum_{i=1}^{n} P_i\, \pi_i(\lambda_j) + y_j^- - y_j^+ = \lambda_j \,; \quad j = 1, \ldots, m \\ \sum_{i=1}^{n} P_i\, \pi_i = \sum_{j=1}^{m} \lambda_j\, q_j \\ P_1 \geq 0.6\, P_{Central}; \quad P_n \leq 2\, P_{Central} \\ P_{i+1} \geq 1.1\, P_i \,; \quad i = 1, \ldots, n-1 \end{cases} \tag{7}$$

Recall this is a general form that is going to be solved in the particular cases $m = 5, 10, 20$ and $n = 3, 4, \ldots, 10$. For these n-values the central classes have been chosen as $Central = 2, 3, 3, 3, 3, 4, 5, 5$, respectively.

4.2.5 Resolution for a growing number of discretization classes

We have solved (7) for the cases $n = 5$ and $m = 5, 10, 20$. The results have been reported in tables 1 and 2.

TABLE 1

TABLE 2

The first one contains the scales of premiums obtained for each m, the respective bonus/malus values between two consecutive classes, and finally the values of the objective function (deviation from equity) and the financial balance (always 0). The second one reports the rating errors obtained for each discretization class in the three cases $h = 0.3, 0.15, 0.075$. Having a look to tables 1 and 2, two important facts must be noted.

- The scales of premiums are almost the same in the three cases. This is particularly true in the last two (corresponding to the meshes $h = 0.15, 0.075$). Therefore it seems in this example as if the method were not sensitive to changes in the discretization of $u(\theta)$. We conclude that our method has succeeded in calculating the scale of premiums associated to the density function given all the others elements defining the bonus malus system.

- Looking to the optimal values of the objective function, we see that there is a light improvement between the cases $m = 5$ and $m = 10$. Afterwards the value remains almost the same, in a neighborhood of 0.13.

We have plotted, in the case $m = 20$, the points with abscissas the class marker θ_j and ordinate the respective non null rating error, multiplying this last by (-1) if it is $y_j^{-} > 0$. The resulting figure 2 illustrates the shape of the rating errors.

FIGURE 2

Founded on these observations, we will from hereafter work our calculations using the arithmetization of $u(\theta)$ with mesh $h = 0.15$ and 10 points in its support (see (6)).

Discussing deeper about our scale of premiums, we can see (looking to the case $h = 0.15$) that its range, which goes from 0.4892 to 1.6309, is in accord with the claims number distribution selected at the beginning (see figure 1). On the other hand the minimal difference of 10% of the lower class premium, that we asked to be fulfilled between two consecutive bonus malus classes, is verified for classes second and third, and also fourth and fifth. This is not the case for classes first and second, with a difference of 51.5%, and classes third and fourth with a difference of 81.8%.

The rating error is to be interpreted as the mean value that a policyholder belonging to that discretization class will pay in excess or in defect in the long run. For example, if we take the discretization class $\lambda_3 = 0.45$, every member of this class will pay in mean an excess of 0.043602254 monetary units. This is near to an excess of 9.6% over their mean λ_3 or, alternatively, a 4.3% over the mean individual claim cost in the portfolio. An extreme case is represented by policyholders with Poisson parameter belonging to the highest class ($\lambda_{10} = 1.5$), who are going to pay in mean a defect of approximately 37.3% over their mean λ_{10}, which is the same as saying a 55.97% over the mean individual claim cost in the portfolio. It is to notice that our bonus malus system is advantageous for the worst policies, as can be seen in table 2. We insist in the important fact that this scale of premiums satisfies all the previous requirements that were translated into the feasible set, including the financial equilibrium of the portfolio. Finally we recall that in this example, changing the weights in the objective function does not change the optimal scale of premiums or the rating errors.

4.2.6 Resolution for a growing number of bonus malus classes

In the precedent section the choice $n = 5$ was arbitrarily made. Now we want to distinguish between bonus malus systems that only differentiate among them because of the different numbers of bonus malus classes. The aim is to be able to answer what would be the optimal number of bonus malus classes if the others elements of the bonus malus system were given.

With this aim we have proceeded to the resolution of the linear program (7) in the cases $m = 10$, $n = 3, \ldots, 10$. All the weights were set to be equal to one because there is no gain in varying them (at least for the values we tried!), as was told earlier. The results have been reported in tables 3 and 4.

TABLE 3

TABLE 4

Looking to these tables we can note the following facts.

- The range of the scale of premiums is almost invariant for a growing number of bonus malus classes. For $n = 3$ it is $[0.485, 1.617]$, while for $n = 10$ it becomes $[0.494, 1.648]$. This stability in the range is certainly produced by the type of market constraints introduced in the definition of the feasible set.

- If we have a look to the optimal values of the objective function, we will see that the minimal value 0.1309 is attained for $n = 3$, though the eight optimal values are neighbors contained in the interval $[0.1309, 0.1399]$. Strictly interpreting these values, we shall conclude that we will have to choose the number of bonus malus classes to be $n = 3$: But we carefully think that the differences among the eight optimal values are so short that other considerations could possibly prevail in the final choice for n: For instance, in Spain it is a well known fact in automobile insurance that an excess of malus in the higher classes of any scale could drive the policyholders to quit the insurance company for another one, as there does not exist a collective data base recording the claim history of the policyholders. This simple and real fact could justify the preference of the bonus malus designer for a bigger value of n; for instance $n = 5$; that introduces the same level of malus (remember this was set to be the 100% of a central premium) in a more gradual way. Perhaps the preservation of the total number of policyholders (the financial equilibrium being guaranteed) could compensate the little loss in equity of the bonus malus system finally adopted.

- Observing the rating errors reported in table 4, we can see the same pattern through the variation of $n = 3; \ldots; 10$: classes of discretization $j = 1; 2; 3$ are paying mean excesses over their true parameter, while classes $j = 4; 5; 6; 7; 8; 9; 10$ are paying mean defects, either in the long run.

In summary we would like to insist that our method has been able to furnish, in this example, important elements to adopt a decision about the best value of n; solving the problem from a strictly mathematical point of view.

4.2.7 Comments about the financial equilibrium constraint.

Along the discussion, the financial balance has been modeled by means of an equality constraint, reflecting the fact that no total mean gain can be made by either sides in the long run. Another possibility would be to investigate the optimums found when solving a linear program where the equality constraint is substituted by the inequality

$$\sum_{i=1} P_i \pi_i \geq \sum_{j=1} \lambda_j q_j; \tag{8}$$

which makes possible a favorable financial balance for the insurance company. In order to explore this possibility we have built linear programs identical to (7) except in the financial equilibrium constraint is substituted by the inequality (8).

Solving this new program in the cases $n = 5$; and $m = 5; 10; 20$, all the weights being equal to one, we have noted the following facts (see table 5).

TABLE 5

Firstly, the solution is dependent of the discretization of the density function, and secondly, when we take a smaller mesh the optimum tends to be the same than in the preceding case of program (7): this is easily seen looking to the last row of table 5 and comparing with the results reported in table 1.

We have tried to change the weights in order to find other optimal scales of premiums. For example the substitution $\omega_j^+ = 1/10$; $\omega_j^- = 9/10$ $\forall j = 1, \ldots, m$; makes this technical trick succeed. The meaning of these weights is that we are giving nine times more importance to the underachievements y_j^- than to the overachievements y_j^+; thus the former will be minimized more intensively than the later, simplex algorithm running (this is a kind of penalty strategy). As a result, we can expect the new scale of premiums to be higher than the one calculated already. This is in fact the case, as is reflected in table 6

TABLE 6

Here we can make the same remarks as in the last case: the scale is not independent of the discretization of the density function, though the optimum seems to become quite stable for values of the mesh below $h = 0.15$. The range of the scale of premiums is now $[0.71; 2.38]$. This optimum is different because it corresponds to a financial balance equal to 0.2326 worth to the insurance company, as was seek. The reason is that the bad policies are now paying less in defect, though the good ones now pay a bigger excess than in the precedent optimum. This is easily seen comparing the two tables number 7 and 2.

TABLE 7

5 Conclusions

This paper focuses in the resolution of the problem consisting in the calculation of a scale of premiums given the claim number distribution, the transition rules and the number of bonus malus classes. We begin showing how it is possible to interpret this problem as a Bayesian decision problem. This new point of view opens the resolution to settings properly suited for designing purposes. This is because it gives us the possibility of modelling all the characteristics to be verified by the resulting scale of premiums by means of constraints included in a mathematical program. Nevertheless some serious drawbacks would arise when expressing this program following the usual tendency found in the actuarial literature, as it is to minimize the expected squared rating error. Firstly there is a difficulty in solving a quadratic program with many restrictions, and secondly, even if it was solvable this technique would not in general preserve the financial equilibrium of the portfolio (as shown in the example given at the end of section 3). This last has a deeper nature as it tells us that the optimums would not be acceptable from an actuarial point of view.

The Bayesian decision problem defined firstly is not the only possible formulation in order to design an optimal bonus-malus system, thanks that it is possible to give another interpretation for the stationary distribution. An arbitrarily policy that approaches stationarity does not necessarily stay forever in the same class. It is the probability $\pi_i(\lambda)$ of (temporarily) belonging to bonus malus class C_i that remains constant for a policy with parameter $\Lambda = \lambda$ in the stationary state. Such policy can change the class according to these probabilities, and therefore the mean value of the premiums paid by that policyholders will be $\sum_{i=1}^{n} P_i \pi_i(\lambda)$. Therefore, it make sense to set a Bayesian decision problem where the optimal decision is the one that minimizes the expected absolute error between λ and $\sum_{i=1}^{n} P_i \pi_i(\lambda)$.

The following step consists in applying Goal Programming methods, which are linear thanks to the equivalence between the minimization of the absolute error and the minimization of the sum of some deviation variables which have a natural interpretation as rating errors.

As it is shown in the exemplification of section 4.2, the design of a bonus malus system following our method would consist in adding linear constraints into the definition of the feasible set. These constraints model the fairness (or equity) of the system, the financial equilibrium, the commercial settings, the monotony of the scale of premiums, and also the nonnegativity of the rating errors. Then minimizing the sum of the rating errors over the discretization classes of the density function, drives us to an optimal scale of premiums (in the sense of an optimal Bayesian decision minimizing the expected absolute error) satisfying all the precedent requirements, particularly the financial equilibrium of the portfolio.

We have seen that, at least in our example, our method is robust with respect to the discretization of $u(\lambda)$, and also with respect to the weights of the objective function. We have also seen that it can give us some hints about the choice in the number of bonus malus classes, given a general frame for the transition rules.

When changing the financial equilibrium by a financial balance worth to the insurance company, it does not drive us to new optimums except when we take different the weights of the objective function in order to penalize the underachievements (i.e. negative deviations). This gives way to a scale of premiums quite advantageous for the insurance company.

We have to stress that the rating errors y_j^-, y_j^+ are such a valuable information furnished by our method, as they tell us the mean defect or excess for a policy of Poisson parameter λ to be paid in the long run (transition rules working), if the scale of premiums is the optimal one.

For all these reasons, we may think that our methodology could be able to furnish solutions that

- Answer the third main problem of section 3 (how to determine the premium associated with every bonus malus class), respecting the well established good properties (as equity of the system or financial balance of the portfolio)

- while preserving real world conditions expressed along the paper as the market conditions or commercial settings.

References

Borgan O., Hoem J.M., Norberg R., 1981. A nonasimptotic criterion for the evaluation of automobile bonus systems. Scandinavian Actuarial Journal, 165-178.

DeGroot, M., 1970. Optimal statistical decisions. McGraw-Hill, New York.

Grimmett, Stirzaker, 1992. Probability and Random Processes. Second edition. Oxford University Press. Oxford.

Lemaire, J., 1985. Automobile insurance: actuarial models. Kluwer. Boston.

Lemaire, J., 1995. Bonus-Malus systems in automobile insurance. Kluwer. Boston.

Norberg, R, 1976. A credibility theory for automobile bonus systems. Scandinavian Actuarial Journal, 92-107.

Pesonen, M., 1963. A numerical method of …nding a suitable bonus scale. Astin Bulletin 2, 102-108.

Romero,C., 1991. Handbook of Critical Issues in Goal Programming. Pergamon Press, Oxford.

Romero,C., 1993. Teoría de la Decisión Multicriterio: Conceptos, Técnicas y Aplicaciones. Alianza Universidad Textos, Madrid.

Sawaragi Y., Nakayama H., Tanino T.,1985. Theory of multiobjective optimization. Academic Press

Sundt, B., 1984. An introduction to non-life insurance mathematics. Verlag Versicherungswirtschaft, Karlsruhe.

Vilar, J.L., 2000. Arithmetization of distributions and linear goal programming. Insurance: Mathematics and Economics 27, 113-122.

Walhin J.F., Paris J., 1999. Using mixed Poisson processes in connection with bonus-malus systems. Astin Bulletin, Vol. 29, nº.1, 81-99.