Smooth negligibility and subdifferential calculus in Banach spaces, with applications

Daniel Azagra

Doctoral dissertation directed by Jesús A. Jaramillo

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO

FACULTAD DE CIENCIAS MATEMÁTICAS

UNIVERSIDAD COMPLUTENSE DE MADRID

October, 1997

Contents

Introduction

This doctoral dissertation may be classified within the framework of the theory of differentiability in Banach spaces, but the main results presented herein have also nice applications in other branches of mathematics, such as Infinite-Dimensional Differential Topology, Ordinary Differential Equations in Banach spaces, and Hamilton-Jacobi Equations in infinite dimensions. To tell the truth, none of the potential applications of the results I wanted to prove was known to me when starting this work.

Even at the risk of boring the reader I would like to recount the story of the birth of this thesis. Let it serve as a special tribute to all the people who helped and encouraged me to face the basic problems in the origin of the present work.

As I was studying the theory of subdifferentiability in Banach spaces, Juan Ferrera drew my attention to the following question: is Rolle's theorem true in infinite dimensions? Juan showed to me a nice counterexample he and Juan Bès had found, and this was the beginning of my thinking about the possible generalizations of Rolle's theorem for infinite-dimensional Banach spaces and for non-differentiable functions. It was my adviser Jesús A. Jaramillo that hinted to me the following *approximate version* of Rolle's theorem in a Banach space: if a differentiable function oscillates less than a positive number 2ε in the boundary of the unit ball, there should exist a point in the interior of the ball at which the differential of the function should be less than ε (in norm). After some struggle this conjecture was proved to be true, and with the help of Javier Gómez Gil we got a shorter and more elegant proof (though maybe less intuitive) than the original one–the two proofs have been included in the second chapter of this thesis. Next, in collaboration with Robert Deville we obtained subdifferential versions of that approximate Rolle's theorem, as well as a new mean value inequality for subdifferential functions which only requires a bound for one but not all of the subgradients of the function at each point.

Almost at the same time Jesús showed to me a paper by Shkarin [63] in which Rolle's theorem was proved to fail in infinite-dimensional superreflexive Banach spaces by means of the construction of a so-called deleting diffeomorphism, that is to say, a diffeomorphism between the space and the space minus one of its subsets (a point in this case). It was then that Jesús introduced me to the work of Czesław Bessaga and Tadek Dobrowolski on deleting diffeomorphism, for which I am very grateful to him, since this was the most exciting discovery I have ever made as a post-graduate student. I immediately began to think about the possible generalization of Bessaga's theorem [7] on the C^∞ topological equivalence of the Hilbert space and its sphere, which has since haunted my imagination. The spaces whose spheres possess a natural differential structure are those which have Fréchet differentiable norms (resp. C^p smooth norms). So, the following problem arises naturally: does every infinite-dimensional Banach space with a C^p smooth norm admit a C^p diffeomorphism between its unit sphere and one of its closed hyperplanes?

The key to the proof of Bessaga's astonishing result was the construction of a diffeomorphism between the Hilbert space H and $H\setminus\{0\}$ that is the identity outside a ball, which was possible thanks to the existence of a C^∞ smooth non-complete norm in H. Tadek Dobrowolski [35] developed Bessaga's non-complete norm technique and showed that every infinite-dimensional Banach space X having a C^p smooth non-complete norm is C^p diffeomorphic to $X \setminus K$, where K is any compact subset of X. In particular this was true for all Banach spaces which are linearly injectable into some $c_0(\Gamma)$, since those spaces possess C^∞ non-complete norms. Thus, regarding the generalization of Bessaga and Dobrowolski's results to every infinite-dimensional Banach space having a C^p smooth norm, with $p \in \mathbf{N} \cup \{\infty\}$, it is natural to ask: does every infinite-dimensional Banach space with a C^p smooth equivalent norm have a C^p smooth non-complete norm too?

Surprisingly enough, the latter seems to be a difficult open question. For a long time I desperately tried and failed to give a positive answer to this problem. When I had already decided to give up thinking of this matter, it occurred to me that one might face the problem of generalizing Bessaga's theorem in another way: by trying to change the non-complete norm for some other *non-complete* object in Bessaga's negligibility scheme. What *non-complete* function might one choose? A smooth non-complete norm in a Banach space can be described as the Minkowski functional of a smooth symmetric convex body which contains no rays and yet is not bounded. In an infinite-dimensional *reflexive* Banach space such a convex body does exist. But in the non-reflexive case it is not clear at all how one could build a smooth non-complete norm out of a smooth equivalent norm. Nevertheless, making use of James's theorem, it is not difficult to construct a smooth *asymmetric* convex body which contains no rays and yet is not bounded. If we take the Minkowski functional of this body we obtain what one could call an *asymmetric non-complete smooth norm*. Then one can try to replace the non-complete norm with this kind of asymmetric non-complete norm in Bessaga's negligibility scheme. This happened to be a successful approach, although it demanded some additional changes; in particular it required using a specific *fixed point lemma* for real functions instead of Banach's contraction principle.

In this way I proved that for every infinite-dimensional Banach space X having a C^p smooth equivalent norm there exists a diffeomorphism from X onto $X \setminus \{0\}$ which restricts to the identity outside a ball, and I deduced that for such a Banach space its unit sphere is C^p diffeomorphic to each of its closed hyperplanes. Then I showed this result to Tadek Dobrowolski, who encouraged me to keep on studying this kind of problems and told me some of the potential applications of smooth negligibility, such as Garay's phenomena for ODE's. He soon realized that this *asymmetric* approach could be generalized so as to construct diffeomorphisms extracting compact sets from a Banach space with a smooth norm. In a fruitful collaboration we both developed these ideas and proved the following results: first, if an infinite-dimensional Banach space X has a (not necessarily equivalent) C^p smooth norm then X is C^p diffeomorphic to $X \setminus K$, where K is any compact subset of X; second, if X has a C^p smooth (resp. real-analytic) seminorm whose set of zeros is a subspace F of infinite codimension then there exists a C^p diffeomorphism (resp. a real-analytic diffeomorphism) between X and $X \setminus F$. Consequently, every infinite-dimensional Banach space X with a (not necessarily equivalent) real-analytic norm is real-analytic topologically equivalent to $X \setminus \{0\}$. As a result we obtained a classification of the smooth convex bodies of an arbitrary Banach space. In particular we proved that every smooth convex body containing no rays in an infinite-dimensional Banach space is diffeomorphic to a closed half-space. There are many applications of these results. We will refer to them later on.

Leaving my personal research experience aside, I believe that it is not so an unforgivable crime to blend smooth negligibility with subdifferential calculus in this essay. I know this may sound like a rather artificial combination, but there is an important link between them both: convex bodies. On the one hand, the theory of subdifferentiability of functions in Banach spaces can be viewed as an attempt to generalize the classic subdifferential of Convex Analysis. It applies in particular to convex functions, and speaking of convex functions is almost the same as dealing with convex bodies. On the other hand, the existence of a smooth convex body U containing no rays in an infinite-dimensional Banach space X has a great impact on the differentiability properties of the space and in particular forces X to be diffeomorphic to $X \setminus \{0\}$ by means of a diffeomorphism which restricts to the identity outside U, as we will show in this work. Moreover, the results on deleting diffeomorphism that can be deduced from the existence of non-trivial smooth convex bodies in a Banach space yield in turn a complete classification of the smooth convex bodies of all Banach spaces. Thus, not only are convex bodies the common threads running through the topics covered herein, but, in a subtler sense, the main characters of this dissertation too.

Before going into a detailed explanation of the contents and main results of this thesis, I will try to give an overview of the most important developments in the areas in which our work can be set.

As far as I know, what one could call negligibility theory in infinite-dimensional Banach spaces started in 1951 when Victor L. Klee [56] proved that if X is either a non-reflexive Banach space or an infinite-dimensional L^p space and K is a compact subset of X then there exists a homeomorphism between X and $X \setminus K$ which restricts to the identity outside a neighbourhood of K. Moreover, Klee showed that the removal of a compact set from the space may happen at the end of an isotopy. His work was motivated by that of Tychonoffs's [67] and Kahutani's [54]. From Tychonoff's fixed-point theorem it follows that, in the *weak* topology, the unit ball B of the Hilbert space H must have the fixed-point property; that is to say, every weakly continuous map from B into itself has at least one fixed point. In the *norm* topology, however, things are quite different. S. Kakutani [54] constructed a homeomorphism without fixed points from B onto itself. Making use of this fact he showed that the unit sphere S of H is contractible and it is a deformation retract of B. Kakutani raised several question related to this results. Are any two of H, B and S homeomorphic? Does B admit a periodic homeomorphism without fixed points? What is the situation in general Banach spaces? Several authors dealt with these problems and gave partial answers to them. J. Dugundji [38] proved that the unit ball of a normed linear space has the fixed point property only if the space is finite-dimensional. P. A. Smith [64] had proved that each prime-period homeomorphism of the Euclidean space $\mathbb{R}^n$ must have a fixed point and asked [40] (p. 259) whether Hilbert space H admits a two-periodic diffeomorphism without fixed points. O. H. Keller [55] showed that the infinite-dimensional compact convex subsets of H are mutually homeomorphic. W. A. Blankinship [15] proved that if K is a relatively compact subset of H then $H \setminus K$ is contractible.

In his fundamental work [56], by using his pioneering results on negligibility, Victor L. Klee answered the questions of Kakutani and Smith, sharpened the theorems of Keller and Blankinship, and gave a complete topological classification of the convex bodies of Hilbert space H. In particular he showed that H is homeomorphic to its unit sphere S. He also proved that, for each $n \geq 2$, H admits a self-homeomorphism of pure period n without fixed points.

Klee's results on topological negligibility of sets and classification of convex bodies were extended to general normed linear spaces by Corson and Klee [18], and by Bessaga and Klee [10, 11]. Klee's original construction of deleting homeomorphisms and isotopies was of a geometrical character, which made them quite difficult to handle in an analytical manner. This geometrical approach of Klee's was rediscovered and simplified by K. Goebel and J. Wośko in [48], where a recipe for a construction of homeomorphisms removing convex bodies from non-reflexive Banach spaces is given. It was C. Bessaga [7, 8, 12] that suggested a beautiful simple analytical method of removing sets in normed spaces which has come to be known as *non-complete-norm technique of deleting sets*. Bessaga proved that if $(X, \|\cdot\|)$ is a normed space which has a continuous *non-complete norm* ϱ and A is a subset of X which is complete with respect to ϱ (for instance a compact subset of X), then X and $X \setminus A$ are homeomorphic. Recall that if $(X, \|\cdot\|)$ is a Banach space and $\varrho : X \longrightarrow [0, \infty)$ is a continuous norm in X, the norm ϱ is said to be *non-complete* provided the normed space (X, ϱ) is not complete. This is equivalent to say that the set $\{x \in X \mid \varrho(x) \leq 1\}$ is a symmetric convex body which contains no rays and yet is not bounded in $(X, \|\cdot\|)$.

At this point it is natural to ask whether all those results on deleting homeomorphisms and topological classification of convex bodies can be sharpened so as to get *diffeomorphisms* instead of mere homeomorphisms. For instance, if X is an infinite-dimensional Banach space with a C^p smooth norm, is X diffeomorphic to $X \setminus \{0\}$? Is the unit sphere of X diffeomorphic to each closed hyperplane of X? Can we get a complete classification of the smooth convex bodies of a Banach space like in the topological case?

At first glance these seem to be more delicate questions. In the theory of Banach spaces there are many objects which are homeomorphic but can never be diffeomorphic. Recall, for instance, that every infinite-dimensional *separable* Banach space is homeomorphic to Hilbert space H, but in general is not diffeomorphic to H, since a diffeomorphism induces a linear isomorphism between the tangent spaces. For the same reason, according to Gowers and Maurey's work on hereditarily indecomposable spaces [49, 50], a Banach space can be homeomorphic but not diffeomorphic to each of its closed hyperplanes.

It is not clear at all whether Klee's geometrical construction of deleting homeomorphisms may be strengthened so as to obtain *diffeomorphisms* removing compact subsets from a Banach space. In contrast, Bessaga's non-complete norm technique did prove to be flexible enough to get such deleting diffeomorphisms for a large class of Banach spaces, namely, that of all spaces having smooth non-complete norms. In 1966, making use of this technique of his, Bessaga showed [7] that every infinite-dimensional Hilbert space H is C^∞ diffeomorphic to $H \setminus \{0\}$ by means of a diffeomorphim which restricts to the identity outside a ball, and deduced that H is diffeomorphic to its unit sphere.

Tadek Dobrowolski [35] developed Bessaga's non-complete norm technique in the smooth case and proved that, if X is an infinite-dimensional Banach space having a non-complete C^p smooth norm, and K is a compact subset of X, then X is C^p diffeomorphic to $X \setminus K$. This is true in particular for every infinite-dimensional Banach space which is linearly injectable into some $c_0(\Gamma)$. He also used these results to give a classification of the smooth convex bodies of every WCG Banach space. More recently [36] he showed that every infinite-dimensional Hilbert space is not only C^∞ diffeomorphic to its unit sphere, but also real-analytically diffeomorphic to it.

Bessaga's theorem on the C^∞ topological equivalence of Hilbert space H and $H \setminus \{0\}$ played a fundamental rôle in the important work of D. Burghelea and N. Kuiper on Hilbert manifolds [16] which, together with other results by J. Eells, D. Elworthy and N. Moulis [39, 58], led to the proof that homotopy equivalent Hilbert manifolds are C^∞ diffeomorphic, among other outstanding results.

As we said above, when one wants to generalize Bessaga and Dobrowolski's results to the class of Banach spaces having C^p smooth norms, one faces the following problem. Does an infinite-dimensional Banach space with an equivalent C^p smooth norm admit a *non-complete* C^p smooth norm as well? This seems to be a difficult question and remains open.

Without showing the existence of smooth non-complete norms, we have managed to prove a number of results on smooth negligibility that extend those of Bessaga's and Dobrowolski's to the class of all Banach spaces having C^p smooth norms or seminorms. In the first chapter we prove that if X is an infinite-dimensional Banach space having a (not necessarily equivalent) C^p smooth norm ϱ (with $p \in \mathbb{N} \cup \{\infty\}$), then there exists a C^p diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\varrho(x) \geq 1$. From this we deduce a full generalization of Bessaga's theorem [7]: if $(X, \|\cdot\|)$ is a Banach space with an equivalent C^p smooth norm $\|\cdot\|$ then the unit sphere of X, $S_X = \{x \in X : \|x\| = 1\}$, is C^p diffeomorphic to each closed hyperplane of X. This chapter is intended to serve as a sort of introduction to the negligibility scheme that will be thoroughly developed in chapter 4. The main results of chapter 1 have been published in [2]. The negligibility method we employ here is, however, more general than that of [2], since it holds for both reflexive and non-reflexive Banach spaces.

In chapter 4 we strengthen this negligibility scheme so as to obtain diffeomorphisms removing compacta and cylinder over compacta from Banach spaces having smooth norms or seminorms. In the first section we show that, if X is an infinite-dimensional Banach space having a (not necessarily equivalent) C^p smooth norm ϱ and K is a compact subset of X, there exists a C^p smooth diffeomorphism φ between X and $X \setminus K$. Furthermore, for each open ϱ-ball B containing K, we can additionally require that φ be the identity outside B.

In section 4.2 it is shown that, if X is a Banach space having a C^p smooth seminorm ϱ whose set of zeros $F = \varrho^{-1}(0)$ is a subspace of infinite codimension of X, and A is a subset of X of the form $A = \pi^{-1}(K)$, where K is a compact subset of X/F and $\pi : X \longrightarrow X/F$ is the natural projection, then X is C^p diffeomorphic to $X \setminus A$. Such sets A are called cylinders over compacta. Moreover, assuming that A is contained in an open cylinder $C = \{x \in X \mid \varrho(x) < r\}$, we prove that there exists a C^p diffeomorphism φ from X onto $X \setminus A$ with the property that φ is the identity outside C. In particular, X is C^p diffeomorphic to $X \setminus F$ by means of a diffeomorphism which restricts to the identity outside a ϱ-cylinder.

Section 4.3 concerns real-analytic negligibility of points and subspaces of infinite-codimension. We show that if X is a Banach space having a real-analytic seminorm ϱ whose set of zeros $F = \varrho^{-1}(0)$ is a subspace of X such that the quotient space X/F is infinite-dimensional, then X and $X \setminus F$ are real-analytically diffeomorphic.

In section 4.4 we note that our results somehow characterize what one could call *convex* negligibility of points. Namely, for a Banach space X the following statements are equivalent: (i) There exists a C^p smooth convex body in X which contains no rays (that is to say, the space X has a–not necessarily equivalent–C^p smooth norm); and (ii) There exists a C^p diffeomorphism φ from X onto $X \setminus \{0\}$ whose support is a C^p smooth convex body containing no rays emanating from the origin. Recall that the support of a map $\varphi : X \longrightarrow X$ is defined as the closure of the complement of the set of fixed points of φ. The statements remain equivalent if we change the words *containing no rays* for *bounded*. We also link this characterization of *convex* smooth negligibility to the failure of Rolle's theorem in infinite-dimensional Banach spaces.

Finally, in section 4.5 we prove that if X is an infinite-dimensional Banach space having an equivalent Fréchet differentiable norm then the removal of a compact set from X may happen at the end of a C^1 smooth isotopy. Making use of this fact we give some results on negligibility of points in Banach manifolds. Namely, we show that if $\mathcal{M}$ is a Banach manifold of class C^1 with boundary $\partial\mathcal{M}$, modelled on an infinite-dimensional Banach space X which has an equivalent Fréchet differentiable norm, and V is an open neighbourhood of a point x_0 in $\partial\mathcal{M}$, then there exists a diffeomorphism from the pair $(\mathcal{M}, \partial\mathcal{M})$ onto $(\mathcal{M} \setminus \{x_0\}, \partial\mathcal{M} \setminus \{x_0\})$ with support in V. We also prove that if $\mathcal{M}$ is a Banach manifold of class C^1 modelled on an infinite-dimensional Banach space with an equivalent Fréchet smooth norm, and U is an open neighbourhood of a point $x_0 \in \mathcal{M}$ then there exists a C^1 smooth isotopy deleting x_0 from $\mathcal{M}$ with support in U.

In chapter 5 we give the announced classification of the smooth convex bodies of a Banach space. Recall that a convex body (that is to say, a closed convex subset with a non-empty interior) in a Banach space is said to be a C^p smooth body provided it is a C^p submanifold with 1-codimensional boundary. Given a convex body U of a Banach space X we can always assume that the origin is an interior point of U, and we can define the characteristic cone of U as the set $ccU = \{x \in X \mid \forall r > 0 \;\; rx \in U\}$. If U_1, U_2 are C^p convex bodies in X we will say that U_1 and U_2 are C^p relatively diffeomorphic provided there exists a C^p self-diffeomorphism of X such that $\varphi(U_1) = U_2$. For a C^p convex body U in a Banach space X we show that: (a) If ccU is a linear subspace of finite codimension (say $X = ccU \oplus Z$, where Z is finite-dimensional), then U is relatively diffeomorphic to $ccU + B$, where B is an Euclidean ball in Z; and (b) If ccU is not a linear subspace or ccU is a linear subspace such that the quotient space X/ccU is infinite-dimensional, then U is C^p relatively diffeomorphic to a closed half-space. In particular, if U does not contain any rays, then U is relatively diffeomorphic to a closed half-space, and hence X and $X \setminus U$ are diffeomorphic. All these results are shown in the first two sections of chapter 5. The last section presents a partial classification of the smooth starlike bodies of every WCG Banach space.

In chapter 6 we give a brief sample of other applications of smooth negligibility, pointing out how the results of chapter 4 enlarge the class of spaces within which those applications are valid. Section 6.1 concerns Garay's phenomena for ordinary differential equations in Banach spaces. B. M. Garay [45, 46] studied the topological properties of cross sections of solution funnels to ODEs in infinite-dimensional Banach spaces. Making use of some results on smooth negligibility he showed that, for several classes of Banach spaces, including Hilbert space, every compact set can be represented as a cross section of a solution funnel to some ODE. Now, using our results of chapter 4, this theorem of Garay's can be extended to the class of all infinite-dimensional Banach spaces having C^p smooth norms, with $p \in \mathbb{N} \cup \{\infty\}$.

The starting point of section 6.2 is a theorem of Klee's concerning the existence of periodic homeomorphisms without fixed points in Hilbert space H. As we said above Klee [56] proved that, for each $n \geq 2$, H admits a self-homeomorphism of pure period n without fixed points. In many Banach spaces this result can be sharpened so as to obtain self-diffeomorphisms of arbitrary period without fixed points. In fact, this smooth version of Klee's theorem can be viewed as a corollary of new results on free actions of the n-torus on Banach spaces. For Banach spaces of the form $X = Y \oplus Z$, where Z is a separable infinite-dimensional space which is isomorphic to its square, and for each positive integer n, we prove that there exists a real-analytic free action of the n-torus T^n on X.

Now we will turn our attention from smooth negligibility to the other main topic of our dissertation: subdifferential calculus. Let X be a Banach space and $\mathcal{U}$ be an open subset of X. A function $f:\mathcal{U}\longrightarrow\mathbb{R}$ is said to be Fréchet subdifferentiable at a point $x\in\mathcal{U}$ provided there exists $p\in X^*$ such that

$$\liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}\geq 0,$$

and the subdifferential set of f at the point x is defined by

$$D^-f(x)=\{p\in X^*\mid \liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}\geq 0\}.$$

In the same way, f is said to be Fréchet superdifferentiable at x whenever there exists $p\in X^*$ such that

$$\limsup_{h\to 0}\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}\leq 0,$$

and the superdifferential set of f at x is defined by

$$D^+f(x)=\{p\in X^*\mid \limsup_{h\to 0}\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}\leq 0\}.$$

The function f is said to be Gâteaux subdifferentiable at x provided there exists $p\in X^*$ such that for every $h\in X\setminus\{0\}$

$$\liminf_{t\to 0}\frac{f(x+th)-f(x)-\langle p,th\rangle}{\|th\|}\geq 0,$$

and the Gâteaux subdifferential set of f at the point x is defined by

$$D^-_Gf(x)=\{p\in X^*\mid \forall h\in S_X, \liminf_{t\to 0}\frac{f(x+th)-f(x)-\langle p,th\rangle}{\|th\|}\geq 0\}.$$

Gâteaux superdifferentiability is defined in a similar way. A function f is said to be (Fréchet or Gâteaux) subdifferentiable (resp. superdifferentiable) on a set $\mathcal{U}$ provided that it is subdifferentiable (resp. superdifferentiable) at each point x in $\mathcal{U}$. A function f is (Fréchet or Gâteaux) differentiable at x if and only if it is both subdifferentiable and superdifferentiable at x, and in this case we have $\{df(x)\}=D^-f(x)=D^+f(x)$. On the other hand it is clear that $D^-f(x)\subset D^-_Gf(x)$, so that every Fréchet subdifferentiable function is also Gâteaux subdifferentiable.

This notion of subdifferential generalizes that of Convex Analysis. Recall that if f is a convex function, the classic subdifferential of f at a point x is defined by $\partial f(x)=\{p\in X^*\mid \langle p,y-x\rangle\leq f(y)-f(x)\ \ \forall y\in X\}$. In this case $\partial f(x)=D^-f(x)$ holds for every x in the domain of f. On the other hand it is well known that every continuous convex function $f:D\longrightarrow\mathbb{R}$ satisfies $\partial f(x)\neq\emptyset$ for every $x\in D$. Hence, every continuous convex function is Fréchet subdifferentiable everywhere in its domain, and all the results concerning subdifferentiability of general functions apply to convex functions.

Apart from being a useful generalization of the theory of subdifferentiability of convex functions, the notion of subdifferentiability we are dealing with plays a fundamental rôle in the study of Hamilton-Jacobi equations. Not only is this concept necessary to understand the notion of *viscosity solution* (introduced by M. G. Crandall and P. L. Lions, see [19, 20, 21, 22, 23, 24, 25, 26, 27, 28]). From many results concerning subdifferentials one can also deduce relatively easy proofs of the existence, uniqueness and regularity of viscosity solutions to Hamilton-Jacobi equations. See, for instance, [30, 31, 42, 32].

In the first section of chapter 3 we present the basic definitions and facts concerning subdifferential calculus which are needed to understand and prove the results of sections 3.2 and 3.3. In section 3.2 we give a subdifferential mean value inequality which holds in every Banach space and presents some advantages with respect to other subdifferential mean value theorems. In the literature there are several mean value theorems known for subdifferentiable functions. As one of the most relevant we may cite that of Robert Deville [30]. A common feature of all the known subdifferential mean value theorems is that they demand a bound for all the lower subgradients of the considered function at each point. Here we give a subdifferential mean value inequality for Gâteaux subdifferentiable continuous functions f which only requires a bound for one but not necessarily all of the subgradients of f at every point of its domain. That is, if $\mathcal{U}$ is a convex open subset of X and $f:\mathcal{U}\longrightarrow\mathbb{R}$ is a continuous function such that for every $x\in\mathcal{U}$ there exists $p\in D^-_G f(x)$ such that $\|p\|\leq M$, then

$$|f(x)-f(y)|\leq M\|x-y\|$$

for all $x,y\in\mathcal{U}$. From this we may deduce that if a continuous function $f:\mathcal{U}\longrightarrow\mathbb{R}$ satisfies $0\in D^-f(x)$ for all $x\in\mathcal{U}$ then f is necessarily constant. This corollary cannot be deduced from other subdifferential mean value inequalities like theorem [30] or that in [1]. Moreover we show that if $f:\mathcal{U}\longrightarrow\mathbb{R}$ is a continuous function, $x,y\in\mathcal{U}$ and $M\geq 0$ are so that for every $t\in[0,1]$ there exists $p\in D^-_G f(tx+(1-t)y)$ with $\|p\|\leq M$, then $|f(x)-f(y)|\leq M\|x-y\|$.

Before stating the main results of the last section of chapter 3, we must say something about the contents of chapter 2. Rolle's theorem in finite-dimensional spaces ensures that, for every bounded connected open subset $\mathcal{U}$ of $\mathbb{R}^n$ and every continuous function $f:\overline{\mathcal{U}}\longrightarrow\mathbb{R}$ such that f is differentiable in U and constant on $\partial\mathcal{U}$, there exists a point in $\mathcal{U}$ at which the differential of f vanishes. It was S. A. Shkarin [63] that first showed the failure of Rolle's theorem in a large class of infinite-dimensional Banach spaces, including all superreflexive and all non-reflexive Banach spaces with equivalent Fréchet differentiable norms (although he did not study the reflexive but non-superreflexive case). Other explicit examples were found in c_0 and ℓ_2 by J. Ferrera and J. Bès [13], and independently by J. Ferrer [44]. In the first section of chapter 2 we conjecture that Rolle's theorem fails in an infinite-dimensional Banach space X if and only if X has a C^1 smooth bump function, and, by relating the failure of Rolle's theorem to the existence of deleting diffeomorphisms, we prove this conjecture to be true within the class of all infinite-dimensional Banach spaces having (not necessarily equivalent) Fréchet differentiable norms.

Despite the failure of an exact Rolle's theorem in infinite-dimensional Banach spaces, we see in section 2.2 that an interesting approximate version of Rolle's theorem remains true in all Banach spaces. By an approximate Rolle's theorem we mean that if a differentiable function oscillates between $-\varepsilon$ and ε on the boundary of the unit ball then there exists a point in the interior of the ball at which the differential of the function has norm less than or equal to ε. More generally we prove the following. Let $\mathcal{U}$ be a bounded connected open set in a Banach space X, and let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be continuous and bounded, Gâteaux differentiable in $\mathcal{U}$. Let $R > 0$ and $x_0 \in \mathcal{U}$ be such that $\mathrm{dist}(x_0, \partial\mathcal{U}) = R$. Suppose that $f(\partial\mathcal{U}) \subset [-\varepsilon, \varepsilon]$. Then there exists an $x_\varepsilon \in \mathcal{U}$ such that $\|df(x_\varepsilon)\| \leq \varepsilon/R$.

At this point it is natural to try to extend the approximate Rolle's theorem to the setting of subdifferentiable functions. This is what we do in the last section of chapter 3, where we give both Fréchet and Gâteaux subdifferential versions of the approximate Rolle's theorem which hold within the class of all Banach spaces having a Fréchet (respectively Gâteaux) differentiable Lipschitz bump function. We see that if a Gâteaux subdifferentiable function oscillates between $-\varepsilon$ and ε on the boundary of the unit ball then there exists a point x in the interior of the ball and there exists $p \in D^- f(x)$ (resp. $p \in D^-_G f(x)$) such that $\|p\| \leq 2\varepsilon$. In fact, for a Banach space $(X, \|\cdot\|)$ having a Fréchet differentiable Lipschitz bump function, we give a stronger result which does not require our function f to be subdifferentiable. Namely, if B_X is the unit ball of X and S_X is its boundary, we prove that every bounded *continuous* function $f : B_X \longrightarrow \mathbb{R}$ such that f oscillates between $-\varepsilon$ and ε on S_X satisfies $\inf\{\|p\| : p \in D^- f(x) \cup D^+ f(x), \|x\| < 1\} \leq 2\varepsilon$.

ACKNOWLEDGEMENTS

It is doubtful whether this work would have been written without the help and enthusiasm of the following people, to whom I am deeply grateful.

Jesús A. Jaramillo, director of this work, for introducing me to this kind of problems, for many valuable conversations and suggestions, and for his constant supplying me with more and more interesting papers.

Tadek Dobrowolski, for helping and encouraging me at any time, and for giving me the opportunity of working with him on smooth negligibility.

Robert Deville, for his valuable suggestions and a fruitful collaboration on subdifferential Rolle's and mean-value inequality theorems.

Javier Gómez Gil, for his valuable help with the approximate Rolle's theorem.

Gilles Godefroy, for his kind and valuable suggestions and several stimulating conversations.

Czesław Bessaga, for his kind and valuable suggestions about some applications of our results (in particular for providing me with theorem 6.8).

Juan Ferrera, for introducing me to the problem of Rolle's theorem in infinite dimensions.

José Luis González Llavona, for including me in the project DGICYT PB 93/0452 and allowing me to attend many interesting congress and seminars.

I must also thank all my colleagues in the Departamento de Análisis de la Facultad de Matemáticas de la UCM for their support. Special thanks are due to José Luis Gámez for his kind assistance with computers.

Chapter 1

Diffeomorphisms between spheres and hyperplanes in Banach spaces

In this chapter we will show that every infinite-dimensional Banach space with a smooth equivalent norm admits a diffeomorphism between its unit sphere and each of its closed hyperplanes. This result provides a full generalization of the famous theorem of Bessaga's on the C^∞ topological equivalence of the Hilbert space and its unit sphere. Recall that a norm in a Banach space X is said to be Fréchet differentiable (resp. C^p smooth, with $p \in \mathbb{N} \cup \{\infty\}$) if it is so in $X \setminus \{0\}$.

Theorem 1.1 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a C^p smooth norm $\|\cdot\|$, and let S_X be its unit sphere. Then, for every closed hyperplane H in X, there exists a C^p diffeomorphism between S_X and H.*

The key to the proof of theorem 1 is the following

Theorem 1.2 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a (not necessarily equivalent) C^p smooth norm ϱ. Then, for each $\varepsilon > 0$ there exists a C^p diffeomorphism $\varphi = \varphi_\varepsilon$ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\varrho(x) \geq \varepsilon$.*

In order to prove this result we will modify Bessaga's non-complete-norm negligibility scheme (see [7, 12]), changing the non-complete norm for a different kind of *non-complete* asymmetric convex function, and using the following *fixed point lemma* instead of Banach's contraction principle.

Lemma 1.3 *Let $F : (0, \infty) \longrightarrow [0, \infty)$ be a continuous function such that, for every $\beta \geq \alpha > 0$,*

$$F(\beta) - F(\alpha) \leq \frac{1}{2}(\beta - \alpha), \quad \textit{and} \quad \limsup_{t \to 0^+} F(t) > 0.$$

Then there exists a unique $\alpha > 0$ such that $F(\alpha) = \alpha$.

Proof. Note that $\lim_{\beta\to\infty}[F(\beta)-\beta] \leq \lim_{\beta\to\infty}[F(1)+\frac{1}{2}(\beta-1)-\beta] = -\infty$, while $\limsup_{\beta\to 0^+}[F(\beta)-\beta] > 0$. Then, from Bolzano's theorem we get an $\alpha > 0$ such that $F(\alpha) = \alpha$. Moreover, the first condition in the statement implies that the function defined by $\beta \longrightarrow F(\beta)-\beta$ is strictly decreasing, which yields the uniqueness of this α.

The following lemma shows that for every Banach space with a C^p smooth norm there exists a kind of smooth *asymmetric non-complete distance* which will act as a smooth non-complete norm in its absence.

Lemma 1.4 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a (not necessarily equivalent) C^p smooth norm ϱ. Then there exists a continuous functional $\omega : X \longrightarrow [0,\infty)$ which is C^p smooth on $X\setminus\{0\}$ and satisfies the following properties:*

1. *$\omega(x+y) \leq \omega(x)+\omega(y)$, and, consequently, $\omega(x)-\omega(y) \leq \omega(x-y)$, for every $x, y \in X$;*
2. *$\omega(rx) = r\omega(x)$ for every $x \in X$, and $r \geq 0$;*
3. *$\omega(x) = 0$ if and only if $x = 0$;*
4. *$\omega(\sum_{k=1}^{\infty} z_k) \leq \sum_{k=1}^{\infty}\omega(z_k)$ for every convergent series $\sum_{k=1}^{\infty} z_k$; and*
5. *for every $\varepsilon > 0$, there exists a sequence of vectors (y_k) such that*

$$\omega(y_k) \leq \frac{\varepsilon}{4^{k+1}}$$

   for every $k \in \mathbb{N}$, as well as

$$\liminf_{n\to\infty} \omega(y - \sum_{j=1}^{n} y_j) > 0$$

   for every $y \in X$.

Notice that ω need not be a norm in X; in general, $\omega(x) \neq \omega(-x)$.

Proof. We will consider three cases.

Case I: The norm ϱ is complete and the space X is non-reflexive.

The norm ϱ is continuous with respect to $\|\cdot\|$ (because it is C^p smooth in X), and complete. Hence, according to the open mapping theorem, ϱ is a C^p smooth equivalent norm in X, and we can assume that $\varrho = \|\cdot\|$. Since X is not reflexive, according to James's theorem [52], there exists a continuous linear functional $T : X \longrightarrow \mathbb{R}$ such that T does not attain its norm. We may assume $\|T\| = 1$, so that $\sup\{T(x) : \|x\| = 1\} = 1$, and yet $T(x) < \|x\|$ for every $x \neq 0$. Let us define $\omega : X \longrightarrow [0,\infty)$ by

$$\omega(x) = \|x\| - T(x).$$

Note that $\omega(x)=0$ if and only if $x=0$, $\omega(x+y)\le\omega(x)+\omega(y)$ for every $x,y\in X$, and $\omega(rx)=r\omega(x)$ for each $r>0$, although ω is not a norm in X because $\omega(x)\neq\omega(-x)$ in general. The property $\omega(z+y)\le\omega(z)+\omega(y)$ implies that $\omega(x)-\omega(y)\le\omega(x-y)$, as well as $\omega(\sum_{k=1}^{\infty} z_k)\le\sum_{k=1}^{\infty}\omega(z_k)$ for every convergent series $\sum_{k=1}^{\infty} z_k$. Then ω satisfies properties 1–4, and it only remains to check that ω satisfies property 5. For a given $\varepsilon>0$, since $\sup\{T(x):\|x\|=1\}=1$, there exists a sequence (y_k) such that $\|y_k\|=1$ and $\omega(y_k)=\|y_k\|-T(y_k)\le\frac{\varepsilon}{4^{k+1}}$ for every $k\in\mathbb{N}$. Let us see that, for such a sequence (y_k),

$$\liminf_{n\to\infty}\omega(y-\sum_{j=1}^{n} y_j)>0$$

holds for every $y\in X$. We have that

$$\omega(y-\sum_{j=1}^{n} y_j)=\|y-\sum_{j=1}^{n} y_j\|-T(y-\sum_{j=1}^{n} y_j)$$
$$\ge -T(y-\sum_{j=1}^{n} y_j)=-T(y)+\sum_{j=1}^{n} T(y_j),$$

and since $T(y_k)\to 1$ as $k\to\infty$, it is clear that $\sum_{j=1}^{\infty} T(y_j)=\infty$, and hence

$$\lim_{n\to\infty}\omega(y-\sum_{j=1}^{n} y_j)=\infty.$$

Case II: The norm ϱ is complete and the space X is reflexive.

We will reduce us to case III by showing that every infinite-dimensional reflexive space has a non-complete C^∞ smooth norm ω. Indeed, for every reflexive space X there exists a linear injection $J: X\longrightarrow c_0(\Gamma)$ for some (infinite) set Γ (see, e.g., [33], chapter VI, p. 246). It is also well known that for an infinite set Γ, the space $c_0(\Gamma)$ is c_0-saturated, that is, every infinite-dimensional closed subspace of $c_0(\Gamma)$ has a closed subspace which is isomorphic to c_0. This clearly implies that $c_0(\Gamma)$ contains no closed infinite-dimensional reflexive subspaces. Therefore $J(X)$ is not a closed subspace of $c_0(\Gamma)$. On the other hand, the space $c_0(\Gamma)$ has an equivalent C^∞ smooth norm g ([33], chapter V, theorem 1.5). Then we can define a C^∞ smooth norm ω in X by $\omega(x)=g(J(x))$, and the norm ω happens to be non-complete because the subspace $J(X)$ is not closed in $c_0(\Gamma)$.

Case III: The norm ϱ is non-complete.

Define $\omega=\varrho$. As ϱ is a C^p smooth norm, it is clear that ω satisfies conditions 1–4. Let us see that ω also satisfies condition 5. Since the norm ω is non-complete, for every $\varepsilon>0$ we can find a sequence (y_k) in X such that $\omega(y_k)\le\frac{\varepsilon}{4^{k+1}}$ for each k, and a point $\hat{y}$ in the completion of (X,ω), denoted by $(\hat{X},\hat{\omega})$, such that $\hat{y}\notin X$, and

$\lim_{n\to\infty}\hat{\omega}(\hat{y}-\sum_{k=1}^{n}y_k)=0$. Moreover,

$$\lim_{n\to\infty}\omega(y-\sum_{j=1}^{n}y_j)=\lim_{n\to\infty}\hat{\omega}(y-\sum_{j=1}^{n}y_j)=\hat{\omega}(y-\hat{y})>0,$$

because $\hat{y}\in\hat{X}\setminus X$ and $y\in X$. In particular, $\liminf_{n\to\infty}\omega(y-\sum_{j=1}^{n}y_j)>0$.

Using the properties of the functional ω we can construct a deleting path as follows.

Lemma 1.5 *Let $(X,\|\cdot\|)$ be a Banach space, and let ω be a functional satisfying conditions 1, 2, and 5 of lemma 1.4. Then, for every $\varepsilon>0$, there exists a C^∞ path $p=p_\varepsilon:(0,\infty)\longrightarrow X$ such that*

1. *$\omega(p(\alpha)-p(\beta))\leq\frac{1}{2}(\beta-\alpha)$ if $\beta\geq\alpha>0$;*
2. *$\limsup_{t\to0^+}\omega(y-p(t))>0$ for every $y\in X$; and*
3. *$p(t)=0$ if $t\geq\varepsilon$.*

Proof. Let $\gamma:[0,\infty)\longrightarrow[0,1]$ be a non-increasing C^∞ function such that $\gamma=1$ in $[0,\varepsilon/2]$, $\gamma=0$ in $[\varepsilon,\infty)$ and $\sup\{|\gamma'(t)|:t\in[0,\infty)\}\leq4/\varepsilon$. For our ε, choose a sequence of vectors (y_k) which satisfies condition 5 of lemma 1.4. Define a required path $p:(0,\infty)\longrightarrow X$ by the following formula

$$p(t)=\sum_{k=1}^{\infty}\gamma(2^{k-1}t)y_k.$$

It is clear that p is a well-defined C^∞ path, and p satisfies condition 3.

If $\beta\geq\alpha$ then $\gamma(2^{k-1}\alpha)-\gamma(2^{k-1}\beta)\geq0$ because γ is non-increasing, and also $\gamma(2^{k-1}\alpha)-\gamma(2^{k-1}\beta)\leq\frac{4}{\varepsilon}|2^{k-1}\alpha-2^{k-1}\beta|$ because $\sup\{|\gamma'(t)|\mid t\in[0,\infty)\}\leq4/\varepsilon$. Taking this into account and using the properties of ω listed in lemma 1.4, we may estimate as follows

$$\begin{aligned}
&\omega(p(\alpha)-p(\beta))=\omega(\sum_{k=1}^{\infty}(\gamma(2^{k-1}\alpha)-\gamma(2^{k-1}\beta))y_k)\\
&\leq\sum_{k=1}^{\infty}\omega((\gamma(2^{k-1}\alpha)-\gamma(2^{k-1}\beta))y_k)=\sum_{k=1}^{\infty}(\gamma(2^{k-1}\alpha)-\gamma(2^{k-1}\beta))\omega(y_k)\\
&\leq\sum_{k=1}^{\infty}\frac{4}{\varepsilon}|2^{k-1}\alpha-2^{k-1}\beta|\omega(y_k)=\sum_{k=1}^{\infty}\frac{2^{k+1}\omega(y_k)}{\varepsilon}|\beta-\alpha|\\
&\leq\sum_{k=1}^{\infty}\frac{2^{k+1}}{\varepsilon}\frac{\varepsilon}{4^{k+1}}|\beta-\alpha|=\frac{1}{2}(\beta-\alpha)
\end{aligned}$$

for every $\beta\geq\alpha$. Hence, the first condition is fulfilled.

Let us see that p also satisfies the second condition. Using condition 5 of lemma 1.4, we have

$$\begin{aligned}
&\limsup_{t\to 0^+} \omega(y - p(t)) \geq \limsup_{n\to\infty} \omega(y - p(\frac{\varepsilon}{2^n})) \geq \liminf_{n\to\infty} \omega(y - p(\frac{\varepsilon}{2^n})) \\
&= \liminf_{n\to\infty} \omega(y - \sum_{k=1}^{\infty} \gamma(\varepsilon 2^{k-1-n}) y_k) = \liminf_{n\to\infty} \omega(y - \sum_{k=1}^{n} \gamma(\varepsilon 2^{k-1-n}) y_k) \\
&= \liminf_{n\to\infty} \omega(y - \sum_{k=1}^{n} y_k) > 0,
\end{aligned}$$

so that $\limsup_{t\to 0^+} \omega(y - p(t)) > 0$, for every $y \in X$.

Now we are ready to prove theorem 1.2. Let us take a C^p smooth functional ω from lemma 1.4, and for a given $\varepsilon > 0$, pick a C^∞ path $p = p_\varepsilon$ from lemma 1.5. For every $x \in X \setminus \{0\}$, define

$$\psi(x) = x + p(\omega(x)).$$

We will see that $\psi : X \setminus \{0\} \longrightarrow X$ is a C^p diffeomorphism, but before giving a formal argument to prove this fact, let us say a few words about the way in which the mapping ψ works. For each $r > 0$, $x \in X$, consider the *asymmetric spheres* $S_\omega(x, r) = \{y \in X \mid \omega(y - x) = r\}$, the *asymmetric open balls* $B_\omega(x, r) = \{y \in X \mid \omega(y - x) < r\}$, and the *asymmetric closed balls* $\overline{B}_\omega(x, r) = \{y \in X \mid \omega(y - x) \leq r\}$. It is obvious that ψ is merely a translation when it is restricted to a *sphere* $S_\omega(0, r)$; in particular ψ is injective when restricted to $S_\omega(0, r)$. Moreover, ψ maps diffeomorphically the set $S_\omega(0, r)$ onto $S_\omega(p(r), r)$. Then, in order to see that ψ is one-to-one, it would be enough to prove that the *spheres* $S_\omega(p(r), r)$ have the following property: if $0 < r < s$ then $S_\omega(p(r), r) \subset B_\omega(p(s), s)$. That is, we would have to show that the balls $\overline{B}_\omega(p(s), s)$ form a descending (as s goes to 0) tower of sets. Using the convexity properties of the functional ω, this is not difficult to check. Indeed, if $0 < r < s$ and $\omega(y - p(r)) \leq r$ then

$$\begin{aligned}
&\omega(y - p(s)) = \omega(y - p(r) + p(r) - p(s)) \leq \omega(y - p(r)) + \omega(p(r) - p(s)) \\
&\leq \omega(y - p(r)) + \frac{1}{2}(s - r) \leq r + \frac{1}{2}(s - r) < s,
\end{aligned}$$

so that $\overline{B}_\omega(p(r), r) \subset B_\omega(p(s), s)$, and ψ is one-to-one. Moreover, the descending tower $\overline{B}_\omega(p(s), s)$ has an empty intersection,

$$\bigcap_{s>0} \overline{B}_\omega(p(s), s) = \emptyset,$$

because otherwise we would have some $y \in X$ such that $0 \leq \omega(y - p(s)) \leq s$ for all $s > 0$, and therefore $\lim_{t\to 0^+} \omega(y - p(t)) = 0$, which contradicts condition 2 of lemma 1.5. This implies that the *spheres* $\{S_\omega(p(s), s)\}_{s>0}$ cover the whole of the space X, and hence the mapping ψ is a surjection. We have just seen that the map ψ is a bijection from $X \setminus \overline{B}_\omega(0,s)$ onto $X \setminus \overline{B}_\omega(p(s),s)$ for every $s>0$, and since $\bigcap_{r>0} \overline{B}_\omega(0,r) = \{0\}$ while $\bigcap_{r>0} \overline{B}_\omega(p(r),r) = \emptyset$, we may conclude that ψ is a bijection from $X \setminus \{0\}$ onto X. So, it is geometrically clear that ψ must be a diffeomorphism between $X \setminus \{0\}$ and X, with the nice additional property that ψ restricts to the identity on $\{x \in X \mid \omega(x) \geq \varepsilon\}$.

Now let us give an analytical proof of this fact. Let y be an arbitrary vector in X, and let $F_y : (0,\infty) \longrightarrow [0,\infty)$ be defined by $F_y(\alpha) = \omega(y - p(\alpha))$ for $\alpha > 0$. Let us see that $F_y(\alpha)$ satisfies the conditions of lemma 1.3. Using the properties of the functional ω and condition 1 of lemma 1.5, we have that

$$\begin{aligned} &F_y(\beta) - F_y(\alpha) = \omega(y - p(\beta)) - \omega(y - p(\alpha)) \leq \omega\big(y - p(\beta) - (y - p(\alpha))\big) \\ &= \omega(p(\alpha) - p(\beta)) \leq \frac{1}{2}(\beta - \alpha) \end{aligned}$$

for every $\beta \geq \alpha$. Hence, the first condition of lemma is fulfilled.

On the other hand, condition 2 of lemma 1.5 reads

$$\limsup_{t\to 0^+} F_y(t) = \limsup_{t \to 0^+} \omega(y - p(t)) > 0,$$

so that F_y also satisfies the second condition.

Then, applying lemma 1.3, we deduce that the equation $F_y(\alpha) = \alpha$ has a unique solution. This means that for each $y \in X$, a number $\alpha(y) > 0$ with the property

$$\omega\big(y - p(\alpha(y))\big) = \alpha(y),$$

is uniquely determined. This implies that the mapping

$$\psi(x) = x + p(\omega(x))$$

is one-to-one from $X \setminus \{0\}$ onto X, whose inverse satisfies

$$\psi^{-1}(y) = y - p(\alpha(y)).$$

Indeed, if $\psi(x) = \psi(z) = y$ then $\omega(y - p(\omega(x))) = \omega(x)$ and also $\omega(y - p(\omega(z))) = \omega(z)$, so that $\omega(x) = \omega(z) = \alpha(y) > 0$ by the uniqueness of $\alpha(y)$, and therefore $x = y - p(\alpha(y)) = z$. Moreover, for each $y \in X$, since $\psi(y - p(\alpha(y))) = y - p(\alpha(y)) + p(\omega(y - p(\alpha(y)))) = y - p(\alpha(y)) + p(\alpha(y))$, the point $x = y - p(\alpha(y))$ satisfies $\psi(x) = y$, and also $x \neq 0$ (because $\omega(x) = \alpha(y) > 0$ and $\omega^{-1}(0) = \{0\}$).

As ω is C^p smooth on $X \setminus \{0\}$ and p is C^p smooth, so is ψ. Let us define $\Phi : X \times (0,\infty) \longrightarrow \mathbb{R}$ by

$$\Phi(y,\alpha) = \alpha - \omega(y - p(\alpha)).$$

Since for every $y \in X$ we have $y - p(\alpha(y)) \neq 0$, the mapping Φ is differentiable on a neighbourhood of every point $(y_0, \alpha(y_0))$ in $X \times (0,\infty)$. On the other hand, since

$F_y(\beta) - F_y(\alpha) \leq \frac{1}{2}(\beta - \alpha)$ for $\beta \geq \alpha > 0$, it is clear that $F'_y(\alpha) \leq \frac{1}{2}$ for every α in a neighbourhood of $\alpha(y)$, and therefore

$$\frac{\partial \Phi(y,\alpha)}{\partial \alpha} = 1 - F'_y(\alpha) \geq 1 - 1/2 > 0.$$

Thus, using the implicit function theorem we obtain that the mapping $y \to \alpha(y)$ is of class C^p and therefore $\psi : X \setminus \{0\} \longrightarrow X$ is a C^p diffeomorphism. Moreover, it is obvious that $\psi(x) = x$ whenever $\omega(x) \geq \varepsilon$. So, for every $\varepsilon > 0$ we have constructed a C^p diffeomorphism $\psi_\varepsilon : X \setminus \{0\} \longrightarrow X$ such that ψ_ε is the identity outside the set $\{x \in X \mid \omega(x) \leq \varepsilon\}$.

In order to conclude the proof of theorem 1.2, we only need to compose ψ_ε with a C^p diffeomorphism $g : X \longrightarrow X$ transforming the set $\{x \in X : \varrho(x) \leq \varepsilon\}$ onto $\{x \in X : \omega(x) \leq \varepsilon\}$. The existence of such a diffeomorphism is ensured by the following lemma 1.6. So, define $\varphi = \varphi_\varepsilon = g^{-1} \circ \psi_\varepsilon^{-1} \circ g$. It is clear that φ is a C^p diffeomorphism from X onto $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\varrho(x) \geq \varepsilon$.

Let us formally state the result which we have just used in the final part of the preceding proof (and which we will use again later on). First, recall that a convex body U (that is, a closed convex subset with a non-empty interior) in a Banach space X is said to be a C^p body provided U is a C^p submanifold with one-codimensional boundary ∂U. For the sake of simplicity we will assume that $0 \in intU$, and we will write $ccU = \{x \in X \mid \forall r > 0 \;\; rx \in U\}$, which stands for the characteristic cone of U. If U_1, U_2 are C^p convex bodies in a Banach space X, we will say that U_1 and U_2 are C^p relatively diffeomorphic provided there exists a C^p diffeomorphism $\varphi : X \longrightarrow X$ such that $\varphi(U_1) = U_2$.

For a convex body U in a Banach space X we define the Minkowski functional of U, $q_U : X \longrightarrow [0, \infty)$, by

$$q_U(x) = \inf\{\lambda > 0 \mid \frac{1}{\lambda} x \in U\}.$$

It is easily seen that for every convex body U its Minkowski functional q_U is a Lipschitz function which satisfies $q_U(x+y) \leq q_U(x) + q_U(y)$ and $q_U(rx) = rq_U(x)$ for every $r \geq 0$; $x, y \in X$. Note also that

$$ccU = \{x \in X \mid q_U(x) = 0\}, \quad \text{and} \quad U = \{x \in X \mid q_U(x) \leq 1\}.$$

Moreover, a standard use of the implicit function theorem shows that if U is a C^p smooth convex body then the functional q_U is of class C^p on the set $X \setminus ccU = X \setminus q_U^{-1}(0)$.

Lemma 1.6 *Let X be a Banach space, and let U_1, U_2 be C^p smooth convex bodies such that the origin is an interior point of both U_1 and U_2, and $ccU_1 = ccU_2$. Then there exists a C^p diffeomorphism $g : X \longrightarrow X$ such that $g(U_1) = U_2$, $g(0) = 0$, and $g(\partial U_1) = \partial U_2$, where ∂U_j stands for the boundary of U_j. Moreover, $g(x) = \mu(x)x$, where $\mu : X \longrightarrow [0, \infty)$, and hence g preserves the rays emanating from the origin.*

Proof. First of all let us see that the statement is true if we make the additional assumption that $U_1 \subseteq U_2$. So, let us suppose that U and V are convex bodies such that the origin is an interior point of both U and V, $ccU = ccV$, and $U \subseteq V$ (so that $q_V(x) \leq q_U(x)$ for every x), and see that there exists a C^p diffeomorphism $g : X \longrightarrow X$ such that $g(U) = V$, $g(0) = 0$, and $g(\partial U) = \partial V$.

Let $\lambda(t)$ be a non-decreasing real function of class C^∞ defined for $t > 0$, such that $\lambda(t) = 0$ for $t \leq 1/2$ and $\lambda(t) = 1$ for $t \geq 1$. Let

$$g(x) = \left[\lambda(q_U(x))\frac{q_U(x)}{q_V(x)} + 1 - \lambda(q_U(x))\right]x$$

for $x \notin ccV$, and $g(x) = x$ whenever $q_V(x) = 0$. It is clear that g is a C^p smooth mapping. Let $y \notin ccV$ be an arbitrary vector of X and put

$$G_y(t) = \left[\lambda(tq_U(y))\frac{q_U(y)}{q_V(y)} + 1 - \lambda(tq_U(y))\right]t$$

for $t > 0$. Note that $G_y(t)$ is strictly increasing and satisfies $\lim_{t\to 0^+} G_y(t) = 0$, and $\lim_{t\to\infty} G_y(t) = \infty$. This implies that for every $y \in X \setminus ccV$ a number $t(y) > 0$ such that $G_y(t(y)) = 1$ is uniquely determined, which means that g is a one-to-one mapping from $X \setminus ccV$ onto $X \setminus ccV$, with $g^{-1}(y) = t(y)y$. It is also clear that g fixes all the points in ccV, so that g is a bijection from X onto X. Let us define $\Phi : (X \setminus ccV) \times (0, \infty) \longrightarrow \mathbb{R}$ by

$$\Phi(y,t) = \left[\lambda(tq_U(y))\frac{q_U(y)}{q_V(y)} + 1 - \lambda(tq_U(y))\right]t.$$

Taking into account that $q_V(x) \leq q_U(x)$ and λ is non-decreasing, one can easily check that $\frac{\partial\Phi}{\partial t}(y,t) \geq 1 > 0$. Then, using the implicit function theorem we obtain that $y \longrightarrow t(y)$ is a C^p smooth function on $X \setminus ccV$, and therefore so is g^{-1}. On the other hand, from the definition above it is clear that the map g restricts to the identity on a neighbourhood of the subspace ccV, and hence both g and g^{-1} are C^p smooth on the whole of X. Thus, g is a C^p diffeomorphism from X onto X, and it is obvious that g transforms the body $U = \{x \in X \mid q_U(x) \leq 1\}$ onto $V = \{x \in X \mid q_V(x) \leq 1\}$, and its boundary $\partial U = \{x \in X \mid q_U(x) = 1\}$ onto $\partial V = \{x \in X \mid q_V(x) = 1\}$.

Now let us consider the general case. Let $U = \{x \in X \mid q_{U_1}(x) + q_{U_2}(x) \leq 1\}$, which is a C^p smooth convex body satisfying $ccU = ccU_j$ and $U \subseteq U_j$, for $j = 1, 2$. From the first part of the proof we know that there exist self-diffeomorphisms of X, g_1 and g_2, such that $g_j(U) = U_j$ and $g_j(\partial U) = \partial U_j$, $j = 1, 2$. Then, if we put $g = g_2 \circ g_1^{-1}$, we get a self-diffeomorphism of X transforming U_1 onto U_2 and ∂U_1 onto ∂U_2.

Let us complete the proof of theorem 1.1. We will do nothing but adapt the ideas of Bessaga [7] to the more general setting of a differentiable C^p norm $\|\cdot\|$ whose sphere might contain segments and consequently the usual stereographic projection might not be well defined for the whole sphere.

Let us choose a point $x_0 \in S_X$ and see first that $S_X \setminus \{x_0\}$ is diffeomorphic to any hyperplane H in X. Put $x^* = d\|\cdot\|(x_0)$, $Z = \ker x^*$, and consider the decomposition $X = [x_0] \oplus Z = \mathbb{R} \times Z$. Take a C^∞ convex body U on the plane $\mathbb{R}^2$ such that the set $\{(t,s) : t^2 + s^2 = 1, t \geq 0\} \cup \{(-1,s) : |s| \leq 1/2\}$ is contained in ∂U, the boundary of U. Consider the Minkowski functional of U, $q_U(t,s) = \inf\{\lambda > 0 : (t,s) \in \lambda U\}$, which is C^∞ smooth away from $(0,0)$. Define $p(t,z) = q_U(t, \|z\|)$ for every $(t,z) \in \mathbb{R} \times Z$. It is quite clear that p is a C^p function away from the ray $\{\lambda x_0 : \lambda > 0\}$ (and p is C^1 smooth on $X \setminus \{0\}$). Now consider the convex body $V = \{(t,z) \in X : p(t,z) \leq 1\}$ and its boundary ∂V. The proof of lemma 1.6 shows that the sets $\partial V \setminus \{x_0\}$ and $S_X \setminus \{x_0\}$ are C^p diffeomorphic (whereas ∂V and S_X are C^1 diffeomorphic). Note that for every $z \in Z$ the ray joining z to x_0 intersects the set ∂V at a unique point. This means that the stereographic projection $\pi : \partial V \setminus \{x_0\} \longrightarrow Z_{-1}$ (where $Z_{-1} = \{x \in X : x^*(x) = -1\}$ is the tangent hyperplane to ∂V at $-x_0$), defined by means of the rays emanating from x_0, is a well defined one-to-one mapping from $\partial V \setminus \{x_0\}$ onto Z_{-1}, and it is easy to check that π is a C^p diffeomorphism between $\partial V \setminus \{x_0\}$ and Z_{-1}. Since any two closed hyperplanes in X are isomorphic this proves that $\partial V \setminus \{x_0\}$ is C^p diffeomorphic to each hyperplane H in X, and hence so is $S_X \setminus \{x_0\}$.

Thus, to complete the proof of theorem 1.1 it only remains to show that $S_X \setminus \{x_0\}$ and S_X are C^p diffeomorphic, which we can do by choosing a suitable atlas for S_X and using theorem 1.2. Let us recall that $x^* = d\|\cdot\|(x_0)$ and $Z = \ker x^*$. Define $D_1 = \{x \in S_X : x^*(x) > -1/2\}$ and $D_2 = \{x \in S_X : x^*(x) < 1/2\}$, and let $\pi_1 : D_1 \longrightarrow Z$ be the stereographic projection defined by means of the rays coming from $-x_0$, and $\pi_2 : D_2 \longrightarrow Z$ the stereographic projection defined by means of the rays emanating from x_0. Note that, although the sphere S_X might contain segments, these stereographic projections are well defined because they have been restricted to D_1 and D_2, sets which cannot contain a segment passing through $-x_0$ and x_0 respectively. Let $G_1 = \{x \in D_1 : x^*(x) > 1/2\}$ and consider $\pi_1(G_1) \subseteq Z$. Since $\pi_1(G_1)$ is an open set in Z containing 0, there exists $\varepsilon > 0$ such that $\{z \in Z : \|z\| \leq \varepsilon\} \subseteq \pi_1(G_1)$. Now, from theorem 1.2 we get a diffeomorphism $\varphi : Z \longrightarrow Z \setminus \{0\}$ such that $\varphi(z) = z$ whenever $\|z\| \geq \varepsilon$. Finally, define $g : S_X \longrightarrow S_X \setminus \{x_0\}$ by

$$g(x) = \begin{cases} x & \text{if } x \in D_2 \\ \pi_1^{-1}(\varphi(\pi_1(x))) & \text{if } x \in D_1 \end{cases}$$

It is easy to check that g is a C^p diffeomorphism from S_X onto $S_X \setminus \{x_0\}$. This concludes the proof of theorem 1.1.

It should be noted that the deleting diffeomorphisms obtained in 1.2 are arbitrarily closed to the identity. More precisely,

Theorem 1.7 *Let X be an infinite-dimensional Banach space having an equivalent C^p smooth norm $\|\cdot\|$. Then, for every $\varepsilon > 0$ there exists a C^p diffeomorphism $\varphi_\varepsilon : X \longrightarrow X \setminus \{0\}$ so that $\|\varphi_\varepsilon(x) - x\| \leq \varepsilon$ for all $x \in X$.*

Proof. It is enough to take a C^p diffeomorphism $\varphi = \varphi_\varepsilon : X \longrightarrow X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq \varepsilon/2$. With this choice, if $\|x\| \geq \varepsilon/2$ then $\|\varphi(x) - x\| = 0$, while, if $\|x\| \leq \varepsilon/2$ we have $\|\varphi(x)\| \leq \varepsilon/2$ too, and therefore $\|\varphi(x) - x\| \leq \varepsilon$.

Remark 1.8 Let $(X, \|.\|)$ be an infinite-dimensional Banach space having a (not necessarily complete) Fréchet differentiable norm ϱ. It is natural to consider the unit sphere $S_\varrho = \{x \in X : \varrho(x) = 1\}$ and ask whether S_ϱ is diffeomorphic to each closed hyperplane H in X. One can easily show that this is the case, using theorem 1.2 as in the proof of theorem 1.1.

Before closing this chapter let us observe that in general one cannot obtain a diffeomorphism between the whole of a Banach space X having a differentiable norm $\|\cdot\|$ and its unit sphere $S_X = \{x \in X \mid \|x\| = 1\}$. This is an almost trivial consequence of the existence of the so-called hereditarily indecomposable Banach spaces.

Remark 1.9 Let X be the Banach space constructed by W. T. Gowers and B. Maurey in [50]. X is reflexive and is another counterexample to the problem of the hyperplane, apart from that in [49]. That is, X is not isomorphic to any of its closed hyperplanes. Being reflexive, X has an equivalent Fréchet differentiable norm $\|.\|$ (see [66], or [33] for instance). Then its unit sphere $S = \{x \in X : \|x\| = 1\}$ is diffeomorphic to each closed hyperplane H in X, but S is not diffeomorphic to X. Indeed, if there exists a diffeomorphism $f : X \longrightarrow S$ then, for each point $x \in X$, the differential of f at x, $df(x)$, induces a linear isomorphism between the tangent spaces to X and S at the points x and $f(x)$ respectively; that is, $df(x)$ establishes a linear isomorphism between X and one of its closed hyperplanes H, which is quite impossible. This example suggests that the natural generalization of Bessaga's theorem [7] is that for every Banach space having a differentiable norm, its unit sphere is diffeomorphic to each closed hyperplane, rather than being diffeomorphic to the whole space.

Chapter 2

Rolle's theorem in infinite dimensional Banach spaces

2.1 The failure of Rolle's theorem in infinite dimensional Banach spaces

Rolle's theorem in finite-dimensional spaces ensures that, for every bounded connected open subset $\mathcal{U}$ of $\mathbb{R}^n$ and every continuous function $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ such that f is differentiable in $\mathcal{U}$ and constant on $\partial\mathcal{U}$, there exists a point in $\mathcal{U}$ at which the differential of f vanishes. Unfortunately, Rolle's theorem does not remain valid in infinite dimensions. It was S. A. Shkarin [63] that first showed the failure of Rolle's theorem in infinite-dimensional superreflexive Banach spaces and non-reflexive Banach spaces with equivalent Fréchet smooth norms.

In the preceding chapter we showed that every infinite-dimensional Banach space X having an equivalent C^p smooth norm admits a diffeomorphism between X and $X \setminus \{0\}$ with bounded support, that is, being the identity outside a ball centered at the origin. Making use of this fact it is quite easy to show that Rolle's theorem fails for a large class of infinite-dimensional Banach spaces, namely, that of all infinite-dimensional Banach spaces with equivalent Fréchet differentiable norms. Indeed, for such a space $(X, \|\cdot\|)$, pick a C^1 diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ so that $\varphi(x) = x$ whenever $\|x\| \geq 1/2$ and define $f : B \longrightarrow [0, 1]$ by

$$f(x) = 1 - \|\varphi(x)\|$$

for every x in the unit ball $B = \{z \in X : \|x\| \leq 1\}$. Obviously f is a C^1 smooth bounded function on B satisfying $f \equiv 0$ on the unit sphere, and it is easy to see that $f'(x) \neq 0$ for every $x \in X$. This counterexample is essentially the same as that given by Shkarin [63] for superreflexive Banach spaces.

On the other hand, Rolle's theorem trivially holds in all non-Asplund Banach spaces, because of the harmonic behaviour of differentiable maps in such spaces: if X is a non-Asplund space, $\mathcal{U}$ is a bounded connected open subset in X, and we have a continuous bounded function $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ which is Fréchet differentiable in $\mathcal{U}$ and $f \equiv 0$ on $\partial\mathcal{U}$, then necessarily $f \equiv 0$ on $\mathcal{U}$ (see [33], chapter III, page 97).

Thus, in many infinite-dimensional Banach spaces, Rolle's theorem either fails or it is trivial, depending on the smoothness of the space. In this setting, it does not seem too risky to conjecture that Rolle's theorem holds in an infinite-dimensional Banach space if and only if our space does not have a C^1 bump function. We will prove this conjecture to be true within the class of those Banach spaces X having (not necessarily equivalent) Fréchet differentiable norms. This is quite a general condition, as it is satisfied by every WCD Banach space, every space which can be injected into some $c_0(\Gamma)$, and even by every space injectable into some $C(K)$, where K is a scattered compact with $K^{(\omega_1)} = \emptyset$. Obviously, every Banach space which is linearly injectable into a Banach space having an equivalent Fréchet differentiable norm will satisfy this condition. Nevertheless, there exist Banach spaces which do not have any differentiable (equivalent or non-equivalent) norm. For instance, the space m_0 defined on page 76 of [33] (see also p. 89) does not possess any Gâteaux differentiable norm.

This conjecture is closely related to the question posed in [35] whether for every Banach space X having a C^1 bump function there exists a C^1 diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ such that φ is the identity outside a ball centered at 0. Next we give an affirmative answer to this question within the class of all Banach spaces having (not necessarily equivalent) Fréchet differentiable norms.

Proposition 2.1 *For every infinite-dimensional Banach space X having a (not necessarily equivalent) Fréchet differentiable norm, the following are equivalent.*

1. *X has a C^1 bump function.*
2. *There exists a C^1 diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ such that φ is the identity outside a ball centered at 0.*

Proof. If $\varphi : X \longrightarrow X \setminus \{0\}$ is a C^1 diffeomorphism such that $\varphi(x) = x$ whenever $\|x\| \geq r$ for some $r > 0$, then, by taking $T \in X^*$ such that $T(\varphi(0)) \neq 0$ and defining $f(x) = T(\varphi(x) - x)$ we obtain a C^1 bump function f such that $f(0) \neq 0$ and $f(x) = 0$ if $\|x\| \geq r$, which proves that (2) implies (1).

Now suppose that X has a C^1 bump function. Proposition 5.1 of [33], chapter II, gives us a function Q on X such that Q is C^1 smooth on $X \setminus \{0\}$, $Q(tx) = |t|Q(x)$ for $x \in X$ and $t \in \mathbb{R}$, and there are constants $a > 0$ and $b > 0$ such that $a\|x\| \leq Q(x) \leq b\|x\|$ for $x \in X$. Let $\lambda : (0, \infty) \longrightarrow (0, \infty)$ be a non-decreasing C^∞ function such that $\lambda(t) = 0$ for $t \leq 1/2$ and $\lambda(t) = 1$ for $t \geq 1$. Let ϱ be a (not necessarily equivalent) Fréchet differentiable norm in X. We may assume that $\varrho(x) \leq Q(x)$ for all $x \in X$. Define

$$H(x) = [\lambda(Q(x))\frac{Q(x)}{\varrho(x)} + 1 - \lambda(Q(x))]x,$$

for $x \neq 0$, and $H(0) = 0$. It is quite clear that H is a one-to-one mapping from X onto X transforming the set $\{x \in X : Q(x) \leq 1\}$ onto $\{x \in X : \varrho(x) \leq 1\}$, and H

is C^1. Using the implicit function theorem as in the proof of theorem 1.6 we obtain that H^{-1} is also C^1. According to theorem 1.2, there exists a diffeomorphism $\psi : X \longrightarrow X \setminus \{0\}$ being the identity outside the (not necessarily bounded) ball $\{x \in X \mid \varrho(x) \leq 1\}$. By composing this diffeomorphism with H, we get a C^1 diffeomorphism between X and $X \setminus \{0\}$ that is the identity outside the set $\{x \in X : Q(x) \leq 1\}$, and hence outside an equivalent ball centered at 0.

We can now use this result to prove our conjecture within the aforementioned class of Banach spaces.

Theorem 2.2 *If an infinite-dimensional Banach space X has a (not necessarily equivalent) Fréchet differentiable norm, the following are equivalent:*

1. *X has a C^1 smooth bump function.*
2. *There exist a bounded connected open subset $\mathcal{U}$ and a continuous bounded function $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ such that f is $C^1(\mathcal{U})$, $f \equiv 0$ on $\partial\mathcal{U}$ and yet $df(x) \neq 0$ for all $x \in \mathcal{U}$; that is, Rolle's theorem fails in X.*
3. *There exist a C^1 bounded function $f : X \longrightarrow \mathbb{R}$ and an open connected bounded subset $\mathcal{U}$ in X such that $f \equiv 0$ on $X \setminus \mathcal{U}$ and yet $df(x) \neq 0$ for all $x \in \mathcal{U}$.*

Proof. It is obvious that (3) implies (2) and it is easy to see that (2) implies (1). Let us prove that (1) implies (3). Let Q be the *starlike* functional employed in the proof of 2.1. From proposition 2.1 we get a C^1 diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$. Let $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ be an even C^∞ function such that $\theta(0) = 1$, $\theta'(t) < 0$ for all $t \in (0, 1)$, and $\theta(t) = 0$ for all $t \geq 1$. We define $f : X \longrightarrow \mathbb{R}$ by $f = \theta \circ Q \circ \varphi$. Since f is the composition of the C^1 smooth functions $\varphi : X \longrightarrow X \setminus \{0\}$, $Q : X \setminus \{0\} \longrightarrow \mathbb{R}$, and θ, it follows that f is of class C^1, and f is bounded because so is θ. Moreover, we have $f(x) = 0$ if $Q(\varphi(x)) \geq 1$. However, $f'(x) \neq 0$ for all x such that $Q(\varphi(x)) < 1$. Indeed,

$$f'(x)(y) = \theta'(Q(\varphi(x)))dQ(\varphi(x))(\varphi'(x)(y)) \neq 0$$

for some $y \in X$, because $\varphi'(x)$ is a linear isomorphism, $dQ(z) \neq 0$ for all $z \in X \setminus \{0\}$ and $\theta'(Q(\varphi(x))) < 0$ whenever $Q(\varphi(x)) < 1$. So, if we take $\mathcal{U} = \{x \in X : Q(\varphi(x)) < 1\}$, (1) implies (3) is proved.

2.2 An approximate Rolle's theorem

Despite the failure of an exact Rolle's theorem in infinite-dimensional Banach spaces, we will see that an interesting approximate version of Rolle's theorem remains true in all Banach spaces. By an approximate Rolle's theorem we mean that if a differentiable function oscillates between $-\varepsilon$ and ε on the boundary of the unit ball then there exists a point in the interior of the ball at which the differential of the function is less than or equal to ε (in norm). More generally we will prove the following. Let $\mathcal{U}$ be a bounded connected open set in a Banach space X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be continuous and bounded, Gâteaux differentiable in $\mathcal{U}$. Let $R > 0$ and $x_0 \in \mathcal{U}$ be such that $\operatorname{dist}(x_0, \partial\mathcal{U}) = R$. Suppose that $f(\partial\mathcal{U}) \subset [-\varepsilon, \varepsilon]$. Then there exists an $x_\varepsilon \in \mathcal{U}$ such that $\|df(x_\varepsilon)\| \leq \varepsilon/R$.

We will present two proofs of this result, the first one being longer but more intuitive. Those readers who are keen on short and elegant proofs should skip a few pages and go on reading on page 33. Hereafter $B(x_0, R)$ stands for the closed ball of centre x_0 and radius R, and $S(x_0, R)$ denotes its boundary.

Theorem 2.3 (Approximate Rolle's theorem) *Let X be a Banach space and $R, \varepsilon > 0$. Let $f : B(0, R) \longrightarrow \mathbb{R}$ be a continuous bounded function, and suppose that f is Gâteaux differentiable in $\operatorname{int}B(0, R)$ and $f(S(0, R)) \subseteq [-\varepsilon, +\varepsilon]$. Then there exists an $x_\varepsilon \in \operatorname{int}B(0, R)$ such that $\|df(x_\varepsilon)\| \leq \varepsilon/R$.*

The idea of the first proof of this result is as simple as follows. Suppose that the result is not true. Then, for every point x_0 in the interior of the ball we can find a short path starting at x_0 along which the function f increases more than ε/R times its length. Next, using Zorn's lemma we obtain a path, starting at the origin and reaching the sphere, along which f increases more than ε/R times its length (and hence more than ε). In a similar manner we get another path, starting at the origin and reaching the sphere, along which the function f decreases more than ε. So we obtain two points in the sphere at which the function f takes values whose difference exceeds 2ε. And this contradicts the fact that f oscillates between $-\varepsilon$ and ε on the sphere.

In this first proof we will need the following result of Measure Theory, which can be obtained by combining 6.3.10 and E.11 of [17].

Theorem 2.4 *Let X be a Banach space and $F : [a, b] \longrightarrow X$ be a continuous function such that:*

1. *F is differentiable at all except countably many of the points in $[a, b]$, and*
2. *F' is Bochner integrable.*

Then $F(t) = F(a) + \int_a^t F'(s)ds$ holds at each $t \in [a, b]$.

Now let us start the proof of the approximate Rolle's theorem. Suppose that $\|df(x)\| > \varepsilon/R$ for all $x \in \operatorname{int}B(0, R)$, and we will get a contradiction.

If $\|df(0)\| > \varepsilon/R$, there exists $h_0 \in S_X$ such that $df(0)(h_0) > \varepsilon/R$. Since f is Gâteaux differentiable at 0 there exists $\delta_0 \in (0, R)$ such that

$$\frac{|\ f(th_0) - f(0) - df(0)(th_0)\ |}{t} < \varepsilon' = df(0)(h_0) - \frac{\varepsilon}{R}$$

for all $t \in (0, \delta_0]$. By taking $t = \delta_0$ we get

$$f(\delta_0 h_0) > f(0) + \frac{\varepsilon}{R}\delta_0 \tag{1}$$

that is, on the segment $[0, y_0]$, where $y_0 = \delta_0 h_0$, f increases more than ε/R times the length of this segment.

Let us define Ω as the set of those paths $\alpha : [0, t_\alpha] \longrightarrow B(0, R)$ satisfying the following conditions:

(i) α is continuous, and it is differentiable at all except countably many of the points in $[0, t_\alpha]$;

(ii) $\alpha(0) = y_0$, and $\|\alpha'(t)\| = 1$ for all $t \in [0, t_\alpha]$ where α is differentiable, that is to say, α travels along its trace at a constant speed;

(iii) α' is Bochner integrable; and

(iv) $f(\alpha(t_\alpha)) \geq t_\alpha \varepsilon/R + f(y_0)$, that is, along the path α, the function f increases more than ε/R times the length of α.

We define the following ordering in Ω

$$\alpha \leq \beta \text{ if and only if } t_\alpha \leq t_\beta \text{ and } \alpha(t) = \beta(t) \text{ for all } t \in [0, t_\alpha],$$

so that $(\Omega, \leq)$ is a partially ordered set.

Roughly speaking Ω consists of all the continuous and differentiable paths in $B(0, R)$ starting at y_0, along which f increases more than ε/R times its length. Let us note that $\text{length}(\alpha) = t_\alpha$ for all $\alpha \in \Omega$. The paths of Ω are ordered by the inclusion of their traces. It is clear that $\Omega \neq \emptyset$.

Let us see now that every totally ordered subset (chain) in $(\Omega, \leq)$ has an upper bound. Let $\{\alpha_i\}_{i \in I} \neq \emptyset$ be a chain in $(\Omega, \leq)$, and define

$$r = \sup\{t_{\alpha_i} \mid i \in I\}.$$

Clearly we have $r < \infty$ because f is bounded on $B(0, R)$. Choose $(i_n) \subset I$ such that $t_{\alpha_{i_n}}$ is an increasing sequence converging to r. It will be convenient to denote $\alpha_{i_n} = \alpha_n$ whenever there is no ambiguity. Let us define $x_n = \alpha_n(t_{\alpha_n})$ for each $n \in \mathbb{N}$.

Let us check that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Indeed, as $\alpha_n, \alpha_{n+p} \in \{\alpha_i\}_{i \in I}$ and $t_{\alpha_n} < t_{\alpha_{n+p}}$, we have $\alpha_n \leq \alpha_{n+p}$, so that $\alpha_{n+p} = \alpha_n$ on $[0, t_{\alpha_n}]$, and therefore

$$\begin{aligned}
&\|x_{n+p} - x_n\| = \|\alpha_{n+p}(t_{\alpha_{n+p}}) - \alpha_n(t_{\alpha_n})\| \\
&= \|\alpha_{n+p}(t_{\alpha_{n+p}}) - \alpha_{n+p}(t_{\alpha_n})\| = \| \int_{t_{\alpha_n}}^{t_{\alpha_{n+p}}} \alpha'_{n+p}(s) ds\| \\
&\leq \int_{t_{\alpha_n}}^{t_{\alpha_{n+p}}} \|\alpha'_{n+p}(s)\| ds = |t_{\alpha_{n+p}} - t_{\alpha_n}|,
\end{aligned}$$

which proves that (x_n) is a Cauchy sequence because so is (t_{α_n}). It should be noted that here we have used theorem 2.4.

Let $y = \lim_n x_n \in B(0, R)$. Now we define $\alpha : [0, r] \longrightarrow B(0, R)$ by $\alpha(t) = \alpha_n(t)$ if $t \in [0, t_{\alpha_n}]$ for some n, and $\alpha(t) = y$ if $t = r$. It is easy to check that this definition does not depend on the choice of the sequence (t_{α_n}). Besides, α is well defined because $\alpha_m = \alpha_n$ on $[0, t_{\alpha_n}]$ if $m \geq n$.

From the definition it is obvious that α is continuous in $[0, r)$; let us see that α is also continuous at r. Let $(t_k) \subset [0, r)$ be such that $t_k \longrightarrow r$. For each $k \in \mathbb{N}$ we can choose $n_k \in \mathbb{N}$ such that $r > t_{\alpha_{n_k}} > t_k$ and $(n_k) \nearrow \infty$. Then

$$\begin{aligned}
&\|\alpha(t_k) - y\| \leq \|\alpha(t_k) - \alpha(t_{\alpha_{n_k}})\| + \|\alpha(t_{\alpha_{n_k}}) - y\| \\
&= \|\alpha_{n_k}(t_k) - \alpha_{n_k}(t_{\alpha_{n_k}})\| + \|\alpha_{n_k}(t_{\alpha_{n_k}}) - y\| \\
&= \| \int_{t_k}^{t_{\alpha_{n_k}}} \alpha'_{n_k}(s) ds\| + \|x_{n_k} - y\| \\
&\leq \int_{t_k}^{t_{\alpha_{n_k}}} \|\alpha'_{n_k}(s)\| ds + \|x_{n_k} - y\| \\
&= |t_{\alpha_{n_k}} - t_k| + \|x_{n_k} - y\|,
\end{aligned}$$

so that $\lim_k \alpha(t_k) = y$.

It is easy to check that α is differentiable at all except countably many of the points in $[0, r]$ and α' is Bochner integrable, with $\|\alpha'\| = 1$. Moreover, $\alpha(0) = y_0$, and $f(\alpha(r)) \geq r\varepsilon/R + f(y_0)$, since

$$\begin{aligned}
f(\alpha(r)) &= f(y) = f\left(\lim_n x_n\right) = \lim_n f(x_n) \\
&= \lim_n f(\alpha_n(t_{\alpha_n})) \geq \lim_n [t_{\alpha_n}\varepsilon/R + f(y_0)] \\
&= r\varepsilon/R + f(y_0).
\end{aligned}$$

Therefore, $\alpha \in \Omega$. On the other hand, one can easily show that $\alpha \geq \alpha_i$ for all $i \in I$, so that α is an upper bound for the chain $\{\alpha_i\}$ in $(\Omega, \leq)$.

Hence, from Zorn's lemma, we can deduce that there exists a maximal element β in $(\Omega, \leq)$.

Let us put $z = \beta(t_\beta)$. It must be $z \in S(0, R)$. Indeed, if z were in $\mathrm{int}B(0, R)$, since f is Gâteaux differentiable in $\mathrm{int}B(0, R)$ and $\|df(z)\| > \varepsilon/R$, there would exist $h \in S_X$ and $\delta > 0$ such that $f(z + \delta h) > f(z) + \frac{\varepsilon}{R}\delta \geq f(y_0) + \frac{\varepsilon}{R}(t_\beta + \delta)$, so that, defining $\gamma : [0, t_\beta + \delta] \longrightarrow B(0, R)$ by

$$\gamma(t) = \begin{cases} \beta(t) & \text{if } t \in [0, t_\beta] \\ z + (t - t_\beta)h & \text{if } t \in [t_\beta, t_\beta + \delta], \end{cases}$$

we would get $\gamma \in \Omega$ and $\gamma \geq \beta$, but $\gamma \neq \beta$, which contradicts the maximality of β.

Therefore $\beta \in \Omega$ joins y_0 to $z \in S(0, R)$, which implies

$$\begin{aligned}
&\mathrm{length}(\beta) = t_\beta = \int_0^{t_\beta} \|\beta'(s)\| ds \geq \| \int_0^{t_\beta} \beta'(s) ds\| = \|\beta(t_\beta) - \beta(0)\| \\
&= \|z - y_0\| \geq \mathrm{dist}(y_0, S(0, R)) = R - \|y_0\| = R - \delta_0,
\end{aligned}$$

and then we have $f(z) \geq f(y_0) + \frac{\varepsilon}{R}t_\beta \geq f(y_0) + \frac{\varepsilon}{R}(R - \delta_0)$, that is

$$f(z) \geq f(y_0) + \frac{\varepsilon}{R}(R - \delta_0). \tag{2}$$

By combining (1) and (2) we get

$$f(z) > f(0) + \varepsilon. \tag{3}$$

Similar reasoning on the set Ω' consisting of all the paths $\alpha : [0, t_\alpha] \longrightarrow B(0, R)$ satisfying:

(i) α is continuous, and it is differentiable at all except countably many of the points in $[0, t_\alpha]$;

(ii) $\alpha(0) = 0$, and $\|\alpha'(t)\| = 1$ for all $t \in [0, t_\alpha]$ where α is differentiable;

(iii) α' is Bochner integrable; and

(iv) $f(\alpha(t_\alpha)) \leq -t_\alpha \varepsilon / R + f(0)$ (i.e., along the path α the function f decreases more than ε/R times the length of α),

equipped with the ordering

$$\alpha \leq \beta \text{ if and only if } t_\alpha \leq t_\beta \text{ and } \alpha = \beta \text{ on } [0, t_\alpha],$$

shows that there exists $z' \in S(0, R)$ such that

$$f(z') \leq -\varepsilon + f(0). \tag{4}$$

Finally, by combining (3) and (4) we get

$$f(z) - f(z') > 2\varepsilon\,, \text{ with } z, z' \in S(0, R),$$

which contradicts the fact that $f\left(S(0, R)\right) \subset [-\varepsilon, \varepsilon]$. Thus, theorem 2.3 is proved.

Looking at the proof of the preceding result, it is clear that the same method works to prove the following.

Theorem 2.5 *Let $\mathcal{U}$ be a bounded connected open set in a Banach space X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be continuous and bounded, Gâteaux differentiable in $\mathcal{U}$. Let $R > 0$ and $x_0 \in \mathcal{U}$ be such that dist$(x_0, \partial\mathcal{U}) = R$. Suppose that $f(\partial\mathcal{U}) \subset [-\varepsilon, \varepsilon]$. Then there exists an $x_\varepsilon \in \mathcal{U}$ such that $\|df(x_\varepsilon)\| \leq \varepsilon/R$.*

Next we will give a shorter but maybe less intuitive proof of the approximate Rolle's theorem, based on Ekeland's Variational Principle. The version of this variational principle that we will use is as follows.

Theorem 2.6 (Ekeland's Variational Principle) *Let X be a Banach space and $f : X \longrightarrow [-\infty, \infty]$ be a proper lower semicontinuous function which is bounded below. Let $\varepsilon > 0$ and $x_0 \in X$ such that $f(x_0) < \inf\{f(x) : x \in X\} + \varepsilon$. Then for each λ with $0 < \lambda < 1$ there exists a point $z \in Dom(f)$ such that:*

(i) $\lambda\|z - x_0\| \leq f(x_0) - f(z)$

(ii) $\|z - x_0\| < \varepsilon/\lambda$

(iii) $\lambda\|x - z\| + f(x) > f(z)$ *whenever* $x \neq z$.

For a proof of this result, see lemma 3.13 in [59], or [41].

In order to prove the approximate Rolle's theorem we need the following lemmas, which are themselves interesting.

Lemma 2.7 *Let X be a Banach space and $\mathcal{U}$ be a bounded connected open subset of X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be a bounded continuous function such that:*

1. *f is Gâteaux differentiable in $\mathcal{U}$*
2. $\inf f(\overline{\mathcal{U}}) < \inf f(\partial\mathcal{U})$ *or* $\sup f(\overline{\mathcal{U}}) > \sup f(\partial\mathcal{U})$.

Then, for every $\alpha > 0$ there exists $x \in \mathcal{U}$ such that $\|df(x)\| \leq \alpha$.

Proof. We may suppose $\inf f(\overline{\mathcal{U}}) < \inf f(\partial\mathcal{U})$. Let us choose $x_0 \in \mathcal{U}$ such that $f(x_0) < \inf f(\partial\mathcal{U})$, and let α, λ be so that $0 < \alpha < \inf f(\partial\mathcal{U}) - f(x_0)$ and $0 < \lambda < \alpha/R$, where $R = \sup\{\|x_0 - x\| : x \in \overline{\mathcal{U}}\} + 1$. From Ekeland's Variational Principle it follows that there exists $x_1 \in \overline{\mathcal{U}}$ such that

$$f(x_1) < f(x) + \lambda\|x - x_1\| \tag{1}$$

for all $x \neq x_1$. In particular

$$f(x_1) \leq f(x_0) + \lambda\|x_0 - x_1\| \leq f(x_0) + \lambda R < \inf f(\partial\mathcal{U})$$

and therefore $x_1 \in \mathcal{U}$. On the other hand, inequality (1) implies that for every h such that $\|h\| = 1$,

$$df(x_1)(h) = \lim_{t \to 0^+} \frac{f(x_1 + th) - f(x_1)}{t} \geq -\lambda,$$

which proves $\|df(x_1)\| \leq \lambda < \alpha$.

Lemma 2.8 *Let X be a Banach space and $\mathcal{U}$ be a bounded connected open subset of X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be a bounded continuous function such that:*

1. *f is Gâteaux differentiable in $\mathcal{U}$*
2. *$f(\overline{\mathcal{U}}) \subseteq [a, b]$, where $a < b$.*

Then, for every $x_0 \in \mathcal{U}$ and $R > 0$ so that $\mathrm{int}B(x_0, R) \subseteq \mathcal{U}$, there exists $x_1 \in \mathrm{int}B(x_0, R)$ such that $\|df(x_1)\| \leq (b - a)/2R$.

Proof. We may suppose that $[a, b] = [-\varepsilon, \varepsilon]$. Two cases will be considered.

Case I: $f(x_0) \neq 0$. We may suppose $f(x_0) < 0$ (the case $f(x_0) > 0$ is analogous). From Ekeland's Variational Principle it follows that there exists $x_1 \in \overline{\mathcal{U}}$ such that

1. $\|x_0 - x_1\| \leq \frac{f(x_0)+\varepsilon}{\varepsilon/R} < R$, and
2. $f(x_1) < f(x) + \frac{\varepsilon}{R}\|x - x_1\|$ for all $x \neq x_1$.

From (1) we get $x_1 \in \mathcal{U}$, and (2) implies that, for every h with $\|h\| = 1$,

$$df(x_1)(h) = \lim_{t \to 0^+} \frac{f(x_1 + th) - f(x_1)}{t} \geq -\varepsilon/R,$$

which proves $\|df(x_1)\| \leq \varepsilon/R$.

Case II: $f(x_0) = 0$. We may suppose $\|df(x_0)\| > \varepsilon/R$, since otherwise we would have finished. If $\|df(x_0)\| > \varepsilon/R$ there exists h with $\|h\| = 1$ such that $df(x_0)(h) < -\varepsilon/R$ and therefore there exists $\delta > 0$ such that $f(x_0 + \delta h)/\delta < -\varepsilon/R$. Applying Ekeland's Variational Principle again we obtain $x_1 \in \overline{\mathcal{U}}$ such that:

1. $\|x_1 - (x_0 + \delta h)\| \leq \frac{f(x_0+\delta h)+\varepsilon}{\varepsilon/R} < \frac{-\varepsilon\delta/R+\varepsilon}{\varepsilon/R} = R - \delta$ and
2. $f(x_1) < f(x) + \frac{\varepsilon}{R}\|x - x_1\|$ for all $x \neq x_1$.

From (1) it follows that $\|x_1 - x_0\| \leq \|x_1 - (x_0 + \delta h)\| + \delta < R$, so that $x_1 \in \mathrm{int}B(x_0, R) \subseteq \mathcal{U}$, and (2) implies $\|df(x_1)\| \leq \varepsilon/R$.

Now the general version of the approximate Rolle's theorem is immediately deduced as a consequence of lemmas 2.7 and 2.8.

Theorem 2.9 (Approximate Rolle's theorem) *Let X be a Banach space and $\mathcal{U}$ be a bounded connected open subset of X. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be a bounded continuous function. Suppose that f is Gâteaux differentiable in $\mathcal{U}$ and $f(\partial\mathcal{U}) \subseteq [a, b]$, with $a < b$. Then, for every $R > 0$ and $x_0 \in \mathcal{U}$ such that $dist(x_0, \partial\mathcal{U}) = R$, there exists an $x_1 \in \mathcal{U}$ such that*

$$\|df(x_1)\| \leq \frac{b-a}{2R}.$$

Before closing this chapter let us state two results which are easy consequences of Ekeland's Variational Principle and can also be obtained as immediate corollaries of the approximate Rolle's theorem.

Corollary 2.10 *Let $\mathcal{U}$ be a bounded connected open subset of a Banach space X. Let $f : \overline{\mathcal{U}} :\longrightarrow \mathbb{R}$ be continuous, bounded, and Gâteaux differentiable in $\mathcal{U}$. Suppose that f is constant on $\partial\mathcal{U}$. Then,*

$$\inf_{x \in \mathcal{U}} \|f'(x)\| = 0.$$

Corollary 2.11 *Let X be a Banach space and $f : X \longrightarrow \mathbb{R}$ be continuous, Gâteaux differentiable and bounded on X. Then,*

$$\inf_{x \in X} \|f'(x)\| = 0.$$

Proof. Let $M > \|f\|_\infty$. Taking $b = -a = M > 0$ and $\mathcal{U} = B(0, n)$ for each $n \in \mathbb{N}$ in theorem 2.3 we obtain a sequence $x_n \in B(0, n)$ such that $\|f'(x_n)\| \leq M/n$ for all $n \in \mathbb{N}$, which implies $\inf_{x \in X} \|f'(x)\| = 0$.

Chapter 3

Some subdifferential calculus

3.1 Preliminaries

Definition 3.1 Let X be a Banach space, $D \subseteq X$ an open set, $f : D \longrightarrow \mathbb{R} \cup \{\infty\}$, $x_0 \in D(f) = \{x \in D : f(x) < \infty\}$. The Fréchet subdifferential set of f at x_0 is defined as

$$D^- f(x_0) = \{p \in X^* \mid \liminf_{h \to 0} \frac{f(x_0 + h) - f(x_0) - \langle p, h \rangle}{\|h\|} \geq 0\},$$

and the Fréchet superdifferential set of f at x_0 as

$$D^+ f(x_0) = \{p \in X^* \mid \limsup_{h \to 0} \frac{f(x_0 + h) - f(x_0) - \langle p, h \rangle}{\|h\|} \leq 0\}.$$

The function f is said to be Fréchet subdifferentiable at x_0 provided $D^- f(x_0) \neq \emptyset$, and f is Fréchet superdifferentiable at x_0 whenever $D^+ f(x_0) \neq \emptyset$. A function f is said to be Fréchet subdifferentiable (resp. superdifferentiable) on a set $\mathcal{U}$ provided it is Fréchet subdifferentiable (resp. superdifferentiable) at every point x in $\mathcal{U}$.

It is clear that a function f is Fréchet subdifferentiable at x_0 if and only if $-f$ is Fréchet superdifferentiable at x_0, and, in this case, $D^+(-f)(x_0) = -D^- f(x_0)$.

Remark 3.2 *A function f is Fréchet differentiable at x_0 if and only if f is both Fréchet subdifferentiable and superdifferentiable at x_0. In this case, $\{df(x_0)\} = D^- f(x_0) = D^+ f(x_0)$.*

Proof. It is clear that every function which is Fréchet differentiable at a point x_0 is both Fréchet subdifferentiable and superdifferentiable at x_0.

Conversely, let us see that if $D^- f(x_0) \neq \emptyset \neq D^+ f(x_0)$ then f is Fréchet differentiable at x_0, and $\{df(x_0)\} = D^- f(x_0) = D^+ f(x_0)$. Indeed, let $p \in D^- f(x_0), q \in D^+ f(x_0)$, then we have

$$\liminf_{h \to 0} \frac{f(x_0 + h) - f(x_0) - \langle p, h \rangle}{\|h\|} \geq 0$$

and

$$\limsup_{h\to 0} \frac{f(x_0+h)-f(x_0)-\langle q,h\rangle}{\|h\|} \leq 0,$$

that is,

$$\liminf_{h\to 0} \frac{-f(x_0+h)+f(x_0)+\langle q,h\rangle}{\|h\|} \geq 0,$$

and by summing these inequalities we get

$$\begin{aligned}
-\|q-p\| &= \liminf_{h\to 0} \frac{\langle q-p,h\rangle}{\|h\|} \\
&= \liminf_{h\to 0} \frac{f(x_0+h)-f(x_0)-\langle p,h\rangle - f(x_0+h)+f(x_0)+\langle q,h\rangle}{\|h\|} \\
&\geq \liminf_{h\to 0} \frac{f(x_0+h)-f(x_0)-\langle p,h\rangle}{\|h\|} \\
&+ \liminf_{h\to 0} \frac{-f(x_0+h)+f(x_0)+\langle q,h\rangle}{\|h\|} \geq 0+0=0,
\end{aligned}$$

and therefore $\|q-p\|=0$. Hence $q=p$, and

$$\begin{aligned}
0 &\leq \liminf_{h\to 0} \frac{f(x_0+h)-f(x_0)-\langle p,h\rangle}{\|h\|} \\
&\leq \limsup_{h\to 0} \frac{f(x_0+h)-f(x_0)-\langle p,h\rangle}{\|h\|} \leq 0,
\end{aligned}$$

so that

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-\langle p,h\rangle}{\|h\|} = 0,$$

and f is Fréchet differentiable, with $df(x_0)=p=q$. This argument also proves that $\{df(x_0)\} = D^- f(x_0) = D^+ f(x_0)$.

It should be noted that the subdifferential introduced in the preceding definition generalizes the subdifferential of the classical convex analysis. Recall that if f is a convex function, the classic subdifferential of f at a point x is defined by $\partial f(x) = \{p\in X^* \mid \langle p, y-x\rangle \leq f(y)-f(x) \ \ \forall y\in X\}$.

Remark 3.3 *Let D be an open convex set of a Banach space X, and let $f : D \subseteq X \longrightarrow \mathbb{R}\cup\{\infty\}$ be a convex function. Then $\partial f(x) = D^- f(x)$ holds for every $x\in D$.*

Proof. From the definition it is obvious that $\partial f(x) \subseteq D^- f(x)$. Let us see that $D^- f(x) \subseteq \partial f(x)$. Let $p\in D^- f(x)$. Let $y\in X$, and define $h=y-x$; we may suppose $y\neq x$. For each $t\in(0,1)$, since f is convex, we have

$$\frac{f\left(x+th\right)-f(x)}{t} \leq f(x+h)-f(x).$$

Indeed, $f(x+th) = f(t(x+h)+(1-t)x) \le tf(x+h)+(1-t)f(x)$, and therefore $f(x+th)-f(x) \le t\left(f(x+h)-f(x)\right)$. Then,

$$\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|} \ge \frac{f(x+th)-f(x)-\langle p,th\rangle}{t\|h\|}$$

for every $t \in (0,1)$, and hence

$$\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|} \ge \liminf_{t\to 0^+} \frac{f(x+th)-f(x)-\langle p,th\rangle}{t\|h\|}$$
$$\ge \liminf_{\vartheta\to 0} \frac{f(x+\vartheta)-f(x)-\langle p,\vartheta\rangle}{\|\vartheta\|} \ge 0$$

because $p \in D^- f(x)$. Therefore, $0 \le f(x+h)-f(x)-\langle p,h\rangle$, that is,

$$f(y)-f(x) \ge \langle p, y-x\rangle.$$

Since this holds for each y we may conclude that $p \in \partial f(x)$.

It is well known (see e.g. [59]) that every continuous convex function $f : D \longrightarrow \mathbb{R}$ satisfies $\partial f(x) \neq \emptyset$ for every $x \in D$. Therefore, from the preceding remark it follows that every continuous convex function is Fréchet subdifferentiable everywhere in its domain.

Definition 3.4 Let X be a Banach space, $D \subseteq X$ an open set, $f : D \longrightarrow \mathbb{R}\cup\{\infty\}$, $x_0 \in D(f) = \{x \in D : f(x) < \infty\}$. The Gâteaux subdifferential set of f at x_0 is defined as

$$D_G^- f(x_0) = \{p \in X^* \mid \forall h \in S_X \ \ \liminf_{t\to 0} \frac{f(x_0+th)-f(x_0)-t\langle p,h\rangle}{|t|} \ge 0\},$$

and the Gâteaux superdifferential set of f at x_0 as

$$D_G^+ f(x_0) = \{p \in X^* \mid \forall h \in S_X \ \ \limsup_{t\to 0} \frac{f(x_0+th)-f(x_0)-t\langle p,h\rangle}{|t|} \le 0\}.$$

The function f is said to be Gâteaux subdifferentiable at x_0 provided $D_G^- f(x_0) \neq \emptyset$, and f is Fréchet superdifferentiable at x_0 whenever $D_G^+ f(x_0) \neq \emptyset$. A function f is said to be Gâteaux subdifferentiable (resp. superdifferentiable) on a set $\mathcal{U}$ provided it is Gâteaux subdifferentiable (resp. superdifferentiable) at each point x in $\mathcal{U}$.

As before, it is easily seen that f is Gâteaux subdifferentiable and Gâteaux superdifferentiable at x_0 if and only if f is Gâteaux differentiable at x_0, and in this case $D_G^- f(x_0) = D_G^+ f(x_0) = \{d_G f(x_0)\}$. Note that $D^- f(x) \subseteq D_G^- f(x)$, that is, every Fréchet subdifferentiable function is also Gâteaux subdifferentiable. On the other hand, remark 3.3 holds for Gâteaux differentiable functions. Hence, if f is a convex function then $\partial f(x) = D_G^- f(x) = D^- f(x)$, and these sets are non-empty whenever f is continuous.

Now we will study the basic properties of subdifferentiable and superdifferentiable functions. First of all, as one might hope, every Fréchet subdifferentiable function is lower semicontinuous.

Proposition 3.5 *Let $f : D \subseteq X \longrightarrow \mathbb{R} \cup \{\infty\}$ be Fréchet subdifferentiable at x_0. Then f is lower semicontinuous (l.s.c.) at x_0. In the same way, if f is Fréchet superdifferentiable at x_0 then f is upper semicontinuous (u.s.c.) at x_0.*

Proof. Let $\varepsilon > 0$. Since

$$\liminf_{h \to 0} \frac{f(x_0 + h) - f(x_0) - \langle p, h \rangle}{\|h\|} \geq 0,$$

there exists $\delta > 0$ such that if $0 < \|h\| \leq \delta$ then

$$\frac{f(x_0 + h) - f(x_0) - \langle p, h \rangle}{\|h\|} \geq -\varepsilon,$$

so that

$$f(x_0 + h) \geq f(x_0) + \langle p, h \rangle - \varepsilon \|h\|$$

whenever $0 \leq \|h\| \leq \delta$, and therefore

$$\begin{aligned} \liminf_{x \to x_0} f(x) &= \liminf_{h \to 0} f(x_0 + h) \\ &\geq \liminf_{h \to 0} \left[f(x_0) + \langle p, h \rangle - \varepsilon \|h\| \right] = f(x_0), \end{aligned}$$

that is, $\liminf_{x \to x_0} f(x) \geq f(x_0)$, which means that f is l.s.c. at x_0.

In general, a subdifferentiable function need not be continuous. For instance, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 \text{ if } x \in [0, 1] \\ 1 \text{ otherwise} \end{cases}$$

is Fréchet subdifferentiable everywhere in $\mathbb{R}$, and yet f is not continuous at 0 and 1.

Now let us see that the sum of subdifferentiable functions is itself subdifferentiable.

Proposition 3.6 *Let $f, g : D \subseteq X \longrightarrow \mathbb{R} \cup \{\infty\}$ be Fréchet subdifferentiable functions at x (resp. Gâteaux subdifferentiable). Then $f + g$ is Fréchet subdifferentiable at x (resp. Gâteaux subdifferentiable), and*

$$D^- f(x) + D^- g(x) \subseteq D^-(f + g)(x).$$

Proof. Let $p \in D^- f(x)$, $q \in D^- g(x)$. Then

$$\begin{aligned} &\liminf_{h \to 0} \frac{(f + g)(x + h) - (f + g)(x) - \langle p + q, h \rangle}{\|h\|} = \\ &= \liminf_{h \to 0} \frac{f(x + h) - f(x) - \langle p, h \rangle + g(x + h) - g(x) - \langle q, h \rangle}{\|h\|} \geq \\ &\geq \liminf_{h \to 0} \frac{f(x + h) - f(x) - \langle p, h \rangle}{\|h\|} + \liminf_{h \to 0} \frac{g(x + h) - g(x) - \langle q, h \rangle}{\|h\|} \\ &\geq 0 + 0 = 0, \end{aligned}$$

so that $p + q \in D^-(f + g)(x)$.

Obviously, a similar result holds for superdifferentiable functions. The preceding proposition has a sort of approximate converse due to El Mahjoub El Haddad and Robert Deville. This result will be very useful to deduce a subdifferential version of the approximate Rolle's theorem. The proof of this formula for the subdifferential of the sum can be found in [42].

Theorem 3.7 (Formula for the subdifferential of the sum) *Let X be a Banach space having a Lipschitz C^1 smooth bump function. Let $f, g : X \longrightarrow \mathbb{R}$ be such that f is lower semicontinuous and g is uniformly continuous. Then, for any given $x_0 \in X, p \in D^-(f + g)(x_0)$ and $\varepsilon > 0$, there exist $x_1, x_2 \in X, p_1 \in D^- f(x_1)$ and $p_2 \in D^- g(x_2)$ such that:*

(i) $\|x_1 - x_0\| < \varepsilon$ *and* $\|x_2 - x_0\| < \varepsilon$.

(ii) $| f(x_1) - f(x_0) | < \varepsilon$ *and* $| g(x_2) - g(x_0) | < \varepsilon$.

(iii) $\|p_1 + p_2 - p\| < \varepsilon$.

It is clear that the subdifferentiability of a function f does not imply that the function $-f$ is also subdifferentiable (unless f is differentiable). Thus, in principle it is not possible to obtain results on the subdifferentiability of the product, difference or composition of subdifferentiable functions. Nevertheless, such results are true if we make stronger assumptions.

Proposition 3.8 *Let $g : \mathcal{U} \subset X \longrightarrow Y$ be a Fréchet differentiable function at x, and $f : Y \longrightarrow \mathbb{R} \cup \{\infty\}$ be a Fréchet subdifferentiable function at $g(x)$. Then $f \circ g : \mathcal{U} \longrightarrow \mathbb{R} \cup \{\infty\}$ is subdifferentiable at x, and*

$$\{p \circ dg(x) \mid p \in D^- f(g(x))\} \subseteq D^-(f \circ g)(x).$$

Proof. Since g is differentiable at x, there exists $M > 0$ such that $\|g(y) - g(x)\| \leq M\|y - x\|$ if $\|x - y\| \leq \delta_1$, for some $\delta_1 > 0$. Let $p \in D^- f(g(x))$ and $\varepsilon > 0$. There exists $\delta_2 > 0$ such that if $\|z - g(x)\| \leq \delta_2$ then

$$f(z) - f(g(x)) - \langle p, z - g(x)\rangle \geq \frac{-\varepsilon}{2M}\|z - g(x)\|.$$

On the other hand, since g is differentiable at x, there exists $\delta_3 > 0$ such that

$$\|g(y) - g(x) - dg(x)(y - x)\| \leq \frac{\varepsilon}{2(1 + \|p\|)}\|y - x\|$$

whenever $\|y - x\| \leq \delta_3$. Choose a $\delta_4 > 0$ so that $\|g(y) - g(x)\| \leq \delta_2$ if $\|y - x\| \leq \delta_4$. Then, if $\|y - x\| \leq \delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, we have that

$$\begin{aligned}
&f(g(y)) - f(g(x)) - p(dg(x)(y - x)) \\
&= f(g(y)) - f(g(x)) - p(g(y) - g(x)) + p(g(y) - g(x) - dg(x)(y - x)) \\
&\geq \frac{-\varepsilon}{2M}\|g(y) - g(x)\| - \|p\|\,\|g(y) - g(x) - dg(x)(y - x)\| \\
&\geq \frac{-\varepsilon}{2M}M\|y - x\| - \|p\|\frac{\varepsilon}{2(1 + \|p\|)}\|y - x\| \geq -\varepsilon\|y - x\|.
\end{aligned}$$

So, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \|y - x\| \leq \delta$ then

$$\frac{f(g(y)) - f(g(x)) - p(dg(x)(y-x))}{\|y-x\|} \geq -\varepsilon.$$

This means that

$$\liminf_{y \to x} \frac{f(g(y)) - f(g(x)) - p(dg(x)(y-x))}{\|y-x\|} \geq 0,$$

and therefore $p \circ dg(x) \in D^-(f \circ g)(x)$.

Proposition 3.9 *Let $f, g : \mathcal{U} \subset X \longrightarrow [0, \infty)$ be Fréchet subdifferentiable functions at x, and suppose that at least one of them is continuous at x. Then fg is Fréchet subdifferentiable at x, and $f(x)D^-g(x) + g(x)D^-f(x) \subseteq D^-(fg)(x)$.*

Proof. Note that if $A, B \subseteq \mathbb{R}$, $A \subseteq [0, \infty)$ then $\inf(A \cdot B) \geq (\inf A) \cdot (\inf B)$, which implies that, for all the functions $F, G : \mathcal{U} \longrightarrow \mathbb{R}$ with $F \geq 0$,

$$\liminf_{y \to x} F(x)G(x) \geq (\liminf_{y \to x} F(x)) \cdot (\liminf_{y \to x} G(x))$$

holds. Let $p \in D^-f(x)$, $q \in D^-g(x)$, and suppose for instance that f is continuous at x. Let us see that $f(x)q + g(x)p \in D^-(fg)(x)$. Since f is continuous at x and $q(h)/\|h\|$ is bounded for $h \neq 0$, we have that

$$\liminf_{h \to 0} \left[f(x+h) - f(x) \right] \frac{q(h)}{\|h\|} = 0.$$

On the other hand, taking into account that $f \geq 0$, we see that

$$\begin{aligned} &\liminf_{h \to 0} \left[f(x+h) \cdot \left(\frac{g(x+h) - g(x) - q(h)}{\|h\|} \right) \right] \geq \\ &\geq \liminf_{h \to 0} f(x+h) \cdot \liminf_{h \to 0} \frac{1}{\|h\|} \left(g(x+h) - g(x) - q(h) \right) \geq 0 \cdot 0 = 0. \end{aligned}$$

Then,

$$\begin{aligned} & \liminf_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x) - \langle f(x)q + g(x)p, h \rangle}{\|h\|} = \\ = \ & \liminf_{h \to 0} \frac{1}{\|h\|} \big[f(x+h)g(x+h) - f(x+h)g(x) - f(x+h)q(h) + \\ + \ & f(x+h)q(h) - f(x)q(h) + f(x+h)g(x) - f(x)g(x) - g(x)p(h) \big] = \\ = \ & \liminf_{h \to 0} \left[f(x+h) \left(\frac{g(x+h) - g(x) - q(h)}{\|h\|} \right) + \right. \\ + \ & \left. [f(x+h) - f(x)] \frac{q(h)}{\|h\|} + \left(\frac{f(x+h) - f(x) - p(h)}{\|h\|} \right) g(x) \right] \geq \\ \geq \ & \liminf_{h \to 0} \left(f(x+h) \cdot \frac{g(x+h) - g(x) - q(h)}{\|h\|} \right) + \\ + \ & \liminf_{h \to 0} [f(x+h) - f(x)] \frac{q(h)}{\|h\|} + \liminf_{h \to 0} \frac{f(x+h) - f(x) - p(h)}{\|h\|} g(x) \geq \\ \geq \ & 0 + 0 + 0 = 0, \end{aligned}$$

so that $f(x)q + g(x)p \in D^-(fg)(x)$.

Next we make a simple remark which will be useful to prove finite-dimensional subdifferentiable Rolle's and mean value theorems.

Remark 3.10 *Consider a function* $f : D \subseteq X \longrightarrow \mathbb{R} \cup \{\infty\}$.

(1) Suppose that f *attains a relative minimum at* $x_0 \in D$. *Then* f *is Fréchet subdifferentiable at* x_0, *and* $0 \in D^- f(x_0)$.

(2) Suppose that f *attains a relative maximum at* $x_0 \in D$. *Then* f *is Fréchet superdifferentiable at* x_0, *and* $0 \in D^+ f(x_0)$.

(3) Suppose that f *attains a relative extremum at* $x_0 \in D$ *and* f *is Fréchet subdifferentiable at* x_0 *(resp. superdifferentiable at* x_0*). Then* $0 \in D^- f(x_0)$ *(resp.,* $0 \in D^+ f(x_0)$*).*

Proof.

(1) If f attains a relative minimum at $x_0 \in D$, there exists $\delta > 0$ such that if $0 < \|h\| < \delta$ then $f(x_0 + h) - f(x_0) \geq 0$, and hence

$$\liminf_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{\|h\|} \geq 0,$$

which means $0 \in D^- f(x_0)$.

(2) The same proof applies to this case.

(3) Suppose that f is subdifferentiable at x_0. If f attains a relative maximum at x_0 then $0 \in D^+ f(x_0)$. Since, in addition, $D^- f(x_0) \neq \emptyset$, remark 3.2 allows us to deduce that $\{0\} = \{df(x_0)\} = D^- f(x_0) = D^+ f(x_0)$. On the other hand, if f attains a relative minimum at x_0, then, from (1) it follows that $0 \in D^- f(x_0)$. In either case, it turns out that $0 \in D^- f(x_0)$.

Proposition 3.11 (Subdifferential Rolle's theorem) *We have:*

(1) Let $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous function such that* $f(a) = f(b)$. *Then, there exists* $x_0 \in (a, b)$ *so that* $0 \in D^- f(x_0) \cup D^+ f(x_0)$.

(2) Let $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous function which is subdifferentiable on* (a, b). *Suppose that* $f(a) = f(b)$. *Then, there exists* $x_0 \in (a, b)$ *such that* $0 \in D^- f(x_0)$.

(3) Let X *be a finite-dimensional normed space, and let* B_X *and* S_X *be respectively the unit ball and unit sphere of* X. *Let* $f : B_X \longrightarrow \mathbb{R}$ *be a continuous function such that* f *is constant on* S_X. *Then, there exists* $x_0 \in \mathrm{int} B_X$ *such that* $0 \in D^- f(x_0) \cup D^+ f(x_0)$. *If, in addition,* f *is subdifferentiable on* $\mathrm{int} B_X$, *then one can assure that* $0 \in D^- f(x_0)$.

Proof. It will suffice to prove (3). Since f is continuous on the compact set B_X, f attains a maximum and a minimum in B_X. If both of them are in S_X, since f is constant on S_X, it follows that f is constant on B_X, and hence $\{0\} = \{df(x)\} = D^-f(x) = D^+f(x)$ for all $x \in \text{int}B_X$. If either the maximum or the minimum are not in S_X then f attains a relative extremum at some $x_0 \in \text{int}B_X$, and, from the preceding remark, we have either $0 \in D^+f(x_0)$ or $0 \in D^-f(x_0)$. If, in addition, f is subdifferentiable on $\text{int}B_X$, then remark 3.10(3) yields $0 \in D^-f(x_0)$.

Obviously, analogous statements hold for superdifferentiable functions. The following example shows that the conclusion $0 \in D^-f(x_0)$ in the subdifferentiable Rolle's theorem cannot be improved so as to obtain $\{0\} = D^-f(x_0)$.

Example 3.12 Let $f : [-1, 1] \longrightarrow \mathbb{R}$ be defined by $f(x) = |x|$. Then f is continuous on $[-1, 1]$ and it is subdifferentiable on $(-1, 1)$, with $f(-1) = f(1) = 1$. And yet, if $\varepsilon < 1$, there is no $x_0 \in (-1, 1)$ such that $\|p\| \leq \varepsilon$ for all $p \in D^-f(x_0)$, because for every $x \in (-1, 1)$ there exists $p \in D^-f(x)$ with $|p| = 1$. In fact,

$$D^-f(x) = \begin{cases} \{-1\} & \text{if} \quad x \in [-1, 0) \\ [-1, 1] & \text{if} \quad x = 0 \\ \{+1\} & \text{if} \quad x \in (0, 1] \,. \end{cases}$$

Now we will study the subdifferentiable mean value theorems, together with some easy consequences of them.

Theorem 3.13 (Subdifferentiable mean value theorem) *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function.*

(1) There exists $x_0 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} \in D^-f(x_0) \cup D^+f(x_0).$$

(2) Suppose that f is subdifferentiable on (a, b). Then there exists $x_0 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} \in D^-f(x_0).$$

Proof.

(1) Consider the function $g : [a, b] \longrightarrow \mathbb{R}$ defined by $g(x) = f(x) - \varphi(x)$, where $\varphi : [a, b] \longrightarrow \mathbb{R}$ is

$$\varphi(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Since g is continuous and $g(a) = 0 = g(b)$, from Rolle's theorem 3.11 we get an $x_0 \in (a, b)$ such that $0 \in D^-g(x_0) \cup D^+g(x_0)$. Suppose for instance that $0 \in D^-g(x_0)$ (the case $0 \in D^+g(x_0)$ is analogous). Then g y φ are subdifferentiable at x_0, and hence so is $f = g + \varphi$, with $D^-g(x_0) + D^-\varphi(x_0) \subset D^-f(x_0)$. As

$$D^-\varphi(x_0) = \{\frac{f(b) - f(a)}{b - a}\} = \{\varphi'(x_0)\},$$

we have that $0 + \frac{f(b)-f(a)}{b-a} \in D^- f(x_0)$, and in particular

$$\frac{f(b)-f(a)}{b-a} \in D^- f(x_0) \cup D^+ f(x_0).$$

(2) A similar proof works (using (2) of 3.11).

Proposition 3.14 *Let $f : (a,b) \longrightarrow \mathbb{R}$. We have:*

(1) If f is non-decreasing in (a,b) then $D^- f(x) \cup D^+ f(x) \subset [0,\infty)$ for all $x \in (a,b)$.

(2) If f is non-increasing in (a,b) then $D^- f(x) \cup D^+ f(x) \subset (-\infty, 0]$ for all $x \in (a,b)$.

Proof. Let us prove (1); (2) is analogous. Let $x \in (a,b)$ and $p \in D^- f(x) \cup D^+ f(x)$; let us see that $p \geq 0$. Suppose first that $p \in D^- f(x)$. Since f is non-decreasing we have $f(x+h) - f(x) \leq 0$ for all $h < 0$, and hence

$$\liminf_{h \to 0^-} \frac{f(x+h)-f(x)}{|h|} \leq 0.$$

Then,

$$\begin{aligned}
0 &\geq \liminf_{h \to 0^-} \frac{f(x+h)-f(x)}{|h|} = \liminf_{h \to 0^-} \left[\frac{f(x+h)-f(x)-ph}{|h|} + \frac{ph}{|h|} \right] \geq \\
&\geq \liminf_{h \to 0^-} \frac{f(x+h)-f(x)-ph}{|h|} + \liminf_{h \to 0^-} \frac{ph}{|h|} \\
&\geq \liminf_{h \to 0} \frac{f(x+h)-f(x)-ph}{|h|} + \liminf_{h \to 0^-} \frac{ph}{|h|} \\
&\geq 0 + \liminf_{h \to 0^-} \frac{ph}{|h|} = -p,
\end{aligned}$$

so that $p \geq 0$.

Now suppose that $p \in D^+ f(x)$. Then, as f is non-decreasing, we have that $f(x+h) - f(x) \geq 0$ for all $h > 0$, and hence

$$\limsup_{h \to 0^+} \frac{f(x+h)-f(x)}{|h|} \geq 0.$$

Therefore,

$$\begin{aligned}
0 &\leq \limsup_{h \to 0^+} \frac{f(x+h)-f(x)}{|h|} \\
&\leq \limsup_{h \to 0^+} \frac{f(x+h)-f(x)-ph}{|h|} + \limsup_{h \to 0^+} \frac{ph}{|h|} \\
&\leq \limsup_{h \to 0} \frac{f(x+h)-f(x)-ph}{|h|} + \limsup_{h \to 0^+} \frac{ph}{|h|} \\
&\leq 0 + \limsup_{h \to 0^+} \frac{ph}{|h|} = p,
\end{aligned}$$

that is, $p \geq 0$. In either case, $p \geq 0$.

Proposition 3.15 *Let $f : (a, b) \longrightarrow \mathbb{R}$ be a continuous function. Then,*

(1) If $D^- f(x) \cup D^+ f(x) \subset [0, \infty)$ for all $x \in (a, b)$ then f is non-decreasing in (a, b).

(2) If $D^- f(x) \cup D^+ f(x) \subset (-\infty, 0]$ for all $x \in (a, b)$ then f is non-increasing in (a, b).

(3) If f is subdifferentiable in (a, b) and $D^- f(x) \subset [0, \infty)$ for all $x \in (a, b)$, then f is non-decreasing in (a, b).

(4) If f is subdifferentiable in (a, b) and $D^- f(x) \subset (-\infty, 0]$ for all $x \in (a, b)$, then f is non-increasing in (a, b).

Proof. Note that it will suffice to prove (1). Indeed, (3) is a consequence of (1): if $\emptyset \neq D^- f(x) \subset [0, \infty)$ for all $x \in (a, b)$ then $D^- f(x) \cup D^+ f(x) \subset [0, \infty)$ for all x, because if $p \in D^+ f(x) \neq \emptyset$ then f is subdifferentiable and superdifferentiable at x, and hence $\{p\} = D^+ f(x) = D^- f(x) \subset [0, \infty)$; that is, if $\emptyset \neq D^- f(x) \subset [0, \infty)$ for all x, then also $D^+ f(x) \subset [0, \infty)$, and therefore $D^- f(x) \cup D^+ f(x) \subset [0, \infty)$ for all x, which implies that f is non-increasing by (1). On the other hand, (2) and (4) are immediately deduced from (1) and (3) by substituting f for $g = -f$.

So, let us prove (1). Suppose that $D^- f(x) \cup D^+ f(x) \subset [0, \infty)$ for all $x \in (a, b)$ and that f failed to be non-decreasing in (a, b). Then there would exist $x_1, x_2 \in (a, b)$, $x_1 < x_2$, so that $f(x_1) > f(x_2)$, and consequently

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0.$$

But, from the mean value theorem 3.13, there exists $x \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \in D^- f(x) \cup D^+ f(x) \subset [0, \infty),$$

and hence

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0,$$

a contradiction.

Now we will say a few words about the topology of the subdifferential sets $D^- f(x)$.

Proposition 3.16 *Let X be a Banach space, $\mathcal{U} \subset X$ an open set, and $f : \mathcal{U} \longrightarrow \mathbb{R} \cup \{\infty\}$. Then, for every $x \in \mathcal{U}$, the sets $D^- f(x)$ and $D^+ f(x)$ are closed and convex in $(X^*, \|\cdot\|)$.*

Proof. If $D^-f(x)=\emptyset$ there is nothing to say. If $D^-f(x)\neq\emptyset$, let $p,q\in D^-f(x)$, $t\in(0,1)$. Then

$$\begin{aligned}
&\liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle tp+(1-t)q,h\rangle}{\|h\|}\\
&=\liminf_{h\to 0}\left[t\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}+(1-t)\frac{f(x+h)-f(x)-\langle q,h\rangle}{\|h\|}\right]\\
&\geq t\liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}+(1-t)\liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle q,h\rangle}{\|h\|}\\
&\geq t\cdot 0+(1-t)\cdot 0=0,
\end{aligned}$$

so that $tp+(1-t)q\in D^-f(x)$. This shows that $D^-f(x)$ is convex. Let us see that $D^-f(x)$ is closed in $(X^*,\|\cdot\|)$. Let $(p_n)\subset D^-f(x)$ be such that $\|p_n-p\|\longrightarrow 0$, and check that $p\in D^-f(x)$. We have

$$\liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle p_n,h\rangle}{\|h\|}\geq 0$$

for all n, and therefore

$$\begin{aligned}
&\liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}=\\
&=\liminf_{h\to 0}\left[\frac{1}{\|h\|}\left(f(x+h)-f(x)-\langle p_n,h\rangle\right)+\frac{1}{\|h\|}\langle p_n-p,h\rangle\right]\\
&\geq\liminf_{h\to 0}\frac{1}{\|h\|}\left(f(x+h)-f(x)-\langle p_n,h\rangle\right)+\liminf_{h\to 0}\frac{1}{\|h\|}\langle p_n-p,h\rangle\\
&\geq 0+\liminf_{h\to 0}\frac{1}{\|h\|}\langle p_n-p,h\rangle=-\|p_n-p\|
\end{aligned}$$

for all n, that is,

$$\liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}\geq-\|p_n-p\|$$

for all $n\in\mathbb{N}$, and since $\|p_n-p\|\longrightarrow 0$ we deduce that

$$\liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle p,h\rangle}{\|h\|}\geq 0,$$

which means $p\in D^-f(x)$.

Remark 3.17 For Gâteaux subdifferentiability we can say something else about the topology of the subdifferential sets: $D^-_Gf(x)$ and $D^+_Gf(x)$ are ω^*-closed and convex in X^*. The proof is almost identical.

Now we proceed to study an alternative definition of subdifferentiability of functions which also generalizes the subdifferential of classical convex analysis and is equivalent to the one we have been handling for a large class of Banach spaces.

Let $\mathcal{U} \subset X$ be a non-empty open subset of a Banach space X. Let $f : \mathcal{U} \longrightarrow \mathbb{R} \cup \{\infty\}$ be a function, and let $x \in D(f)$. Let us define the sets $\mathcal{D}^- f(x) = \{\varphi'(x) \mid \varphi : \mathcal{U} \longrightarrow \mathbb{R}$ is Fréchet differentiable and $f - \varphi$ attains a local minimum at $x\}$; and $\mathcal{D}^+ f(x) = \{\varphi'(x) \mid \varphi : \mathcal{U} \longrightarrow \mathbb{R}$ is Fréchet differentiable and $f - \varphi$ attains a local maximum at $x\}$.

It is easy to see that, for a convex function f,

$$\partial f(x) = \mathcal{D}^- f(x) = D^- f(x).$$

It is also true that $\mathcal{D}^- f(x) \subseteq D^- f(x)$ for all $x \in D(f)$ and every function f; the proof is straightforward. However, the converse inclusion is not true in general, as we will see later on.

The following theorem, whose proof can be found in [33], chapter VIII, shows that there exists a wide class of Banach spaces for which $D^- f(x) = \mathcal{D}^- f(x)$ for every function f and $x \in D(f)$. Every space whose smooth structure is rich enough will belong to this class.

Theorem 3.18 *Let X be a Banach space which has a Fréchet differentiable Lipschitz bump function. Let $f : X \longrightarrow \mathbb{R}$, $x_0 \in X$, $p \in X^*$. The following are equivalent:*

(i) There exists a Fréchet differentiable function $\varphi : X \longrightarrow \mathbb{R}$ such that $f - \varphi$ attains a local minimum at x_0, $\varphi'(x_0) = p$ and φ' is $\| \cdot \| - \| \cdot \|$ continuous at x_0.

(ii) There exist $\mathcal{U}$, an open neighbourhood of x_0, and $\varphi : \mathcal{U} \longrightarrow \mathbb{R}$, a Fréchet differentiable function, so that $f - \varphi$ attains a local minimum at x_0 and $\varphi'(x_0) = p$.

(iii) $p \in D^- f(x_0)$.

If X does not have such a bump function, the result is not true. Choose for instance a separable Banach space $(X, \| \cdot \|)$ whose dual is not separable, and let $f : X \longrightarrow \mathbb{R}$ be defined by $f(x) = -\|x\|^2$. Then f is Fréchet differentiable at 0, with $f'(0) = 0$, and in particular $D^- f(0) = \{0\} \neq \emptyset$. That is, condition (iii) of theorem 3.18 is satisfied for $p = 0$. However, condition (i) is not satisfied: otherwise there exists a Fréchet differentiable function $\varphi : X \longrightarrow \mathbb{R}$ such that $f - \varphi$ attains a local minimum at 0, and $\varphi'(0) = 0$, with φ' continuous at 0. We may assume that $\varphi(0) = 0$. Then there exists $r > 0$ so that $\varphi(x) \leq -\|x\|^2$ whenever $\|x\| \leq r$. Choose $\delta > 0$ so that φ' is bounded in $B(0, \delta)$; we can assume $\delta < r/2$. Let $b : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1 smooth Lipschitz function such that $b(0) = 1$ and $b(t) = 0$ if $t \leq -\delta$. Let us define $\psi : X \longrightarrow \mathbb{R}$ by

$$\psi(x) = \begin{cases} b(\varphi(x)) & \text{if} \quad \|x\| \leq \delta \\ 0 & \text{if} \quad \|x\| > \delta. \end{cases}$$

Then ψ is Lipschitz and Fréchet differentiable, with $\psi(0) = 1$ and $\text{supp}(\psi) \subset B(0, \delta)$, which contradicts our assumption on X, because X is not an Asplund space and hence X cannot have a smooth bump function.

The following result, whose proof can be found in [33], chapter VIII, states that, in many Banach spaces, every lower semicontinuous function is Fréchet subdifferentiable on a dense subset of its domain. This fact gives us a glimpse of how large the class of subdifferentiable functions is. Recall that on the real line there are many continuous functions which are differentiable at no point.

Theorem 3.19 *Let X be a Banach having a Fréchet differentiable Lipschitz bump function. Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function, with $D(f) \neq \emptyset$. Then f is Fréchet subdifferentiable on a dense subset of $D(f)$.*

3.2 Subdifferential Mean Value Inequality Theorem

In this section we will give a subdifferential mean value inequality which holds in every Banach space and presents some advantages with respect to other subdifferential mean value theorems. In the literature there are several mean value theorems known for subdifferentiable functions. As one of the most relevant we may cite that of Robert Deville [30]:

Theorem 3.20 (Deville) *Let X be a Banach space having a Fréchet differentiable Lipschitz bump function, and let $f : X \longrightarrow \mathbb{R}$ be a lower semicontinuous function. Assume that there exists a constant $K > 0$ such that for all $x \in X$ and for all $p \in D^- f(x)$, $\|p\| \leq K$. Then*

$$|f(x) - f(y)| \leq K\|x - y\| \text{ for all } x, y \in X.$$

From this result it follows

Corollary 3.21 (Deville) *Let X be a Banach space having a Fréchet differentiable Lipschitz bump function, and let $f : X \longrightarrow \mathbb{R}$ be a continuous function. Then*

$$\sup\left\{\|p\| : p \in D^- f(x), x \in X\right\} = \sup\left\{\|p\| : p \in D^+ f(x), x \in X\right\}.$$

These quantities are finite if and only if f is Lipschitz continuous on X, and in this case they are equal to the Lipschitz constant of f.

A common feature of all the known subdifferential mean value theorems is that they demand a bound for all the lower subgradients of the considered function at each point. Here we will give a subdifferential mean value inequality for Gâteaux subdifferentiable continuous functions f which only requires a bound for one but not necessarily all of the subgradients of f at every point of its domain. That is, if for every $x \in \mathcal{U}$ there exists $p \in D_G^- f(x)$ such that $\|p\| \leq M$, then

$$|f(x) - f(y)| \leq M\|x - y\|$$

for all $x, y \in \mathcal{U}$. From this we may deduce that if a continuous function $f : \mathcal{U} \longrightarrow \mathbb{R}$ satisfies $0 \in D^- f(x)$ for all $x \in \mathcal{U}$ then f is necessarily constant. This corollary cannot be deduced from other subdifferential mean value inequalities like theorem 3.20 or that in [1].

Moreover we will show that if $f : \mathcal{U} \longrightarrow \mathbb{R}$ is a continuous function, $x, y \in \mathcal{U}$, and $M \geq 0$ is such that for every $t \in [0,1]$ there exists $p \in D_G^- f(tx + (1-t)y)$ with $\|p\| \leq M$, then $|f(x) - f(y)| \leq M\|x - y\|$.

Theorem 3.22 (Subdifferential mean value inequality) *Let X be a Banach space, $\mathcal{U}$ an open convex subset of X and $f : \mathcal{U} \longrightarrow \mathbb{R}$ a continuous function. Suppose that there exists $M \geq 0$ such that for every $x \in \mathcal{U}$ there exists $p \in D_G^- f(x)$ such that $\|p\| \leq M$. Then*

$$|f(x) - f(y)| \leq M\|x - y\|$$

for all $x, y \in \mathcal{U}$.

Proof. Let $x, y \in \mathcal{U}, x \neq y$, and fix $\varepsilon > 0$. Define $h = y - x$ and

$$A = \{\alpha \in [0,1] \mid f(x + \alpha h) - f(x) \geq -(M + \varepsilon)\alpha\|h\|\}.$$

Let us see that $A \neq \emptyset$. Taking $p \in D_G^- f(x)$ such that $\|p\| \leq M$, since

$$\liminf_{|t| \to 0} \frac{f(x + th) - f(x) - \langle p, th \rangle}{\|th\|} \geq 0,$$

there exists $\delta > 0$ such that $f(x + th) - f(x) - \langle p, th \rangle \geq -\varepsilon|t|\|h\|$ whenever $|t| \leq \delta$. Then

$$\begin{aligned} & f(x + th) - f(x) \geq \langle p, th \rangle - \varepsilon|t|\,\|h\| \geq \\ & -M|t|\,\|h\| - \varepsilon|t|\,\|h\| = \\ & -(M + \varepsilon)|t|\,\|h\|; \end{aligned}$$

and by taking $t = \delta$ we get $f(x + \delta h) - f(x) \geq -(M + \varepsilon)\delta\|h\|$, so that $\delta \in A$.

Let $\beta = \sup A \in (0,1]$. There exists $(\alpha_n) \subset [0, \beta] \cap A$ such that $\alpha_n \nearrow \beta$ and $f(x + \alpha_n h) - f(x) \geq -(M + \varepsilon)\alpha_n\|h\|$ for all $n \in \mathbb{N}$. By letting n go to infinity and using the continuity of f we get $f(x + \beta h) - f(x) = \lim_{n \to \infty} [f(x + \alpha_n h) - f(x)] \geq \lim_{n \to \infty} -(M + \varepsilon)\alpha_n\|h\| = -(M + \varepsilon)\beta\|h\|$, that is,

$$f(x + \beta h) - f(x) \geq -(M + \varepsilon)\beta\|h\|, \tag{1}$$

which means $\beta \in A$.

Let us see that $\beta = 1$. If $\beta < 1$, put $z = x + \beta h$ and choose $p \in D_G^- f(z)$ such that $\|p\| \leq M$. Since

$$\liminf_{|t| \to 0} \frac{f(z + th) - f(z) - \langle p, th \rangle}{\|th\|} \geq 0,$$

there exists $\delta > 0$ such that if $|t| \leq \delta$ then $f(z+th) - f(z) - \langle p, th\rangle \geq -\varepsilon\|th\|$, and hence

$$f(z+th) - f(z) \geq -(M+\varepsilon)|t|\,\|h\| \text{ if } |t| \leq \delta. \tag{2}$$

From (1) and (2) it follows that

$$\begin{aligned} &f(x+\beta h+th) \geq f(x+\beta h) - (M+\varepsilon)|t|\|h\| \geq \\ &f(x) - (M+\varepsilon)\beta\|h\| - (M+\varepsilon)|t|\|h\| \\ &= f(x) - (M+\varepsilon)(\beta+|t|)\|h\| \end{aligned}$$

whenever $|t| \leq \delta$. By taking $t = \delta$, we obtain

$$f(x+(\beta+\delta)h) \geq f(x) - (M+\varepsilon)(\beta+\delta)\|h\|,$$

which implies $\beta + \delta \in A$. This is a contradiction because $\beta+\delta > \beta = \sup A$. Thus, $\beta = 1$. By substituting $\beta = 1$ in (1) we get $f(x+h) - f(x) \geq -(M+\varepsilon)\|h\|$, and since $h = y - x$ this means $f(y) - f(x) \geq -(M+\varepsilon)\|y-x\|$. This reasoning proves that for all $x, y \in \mathcal{U}$ and for all $\varepsilon > 0$ we have

$$f(x) - f(y) \leq (M+\varepsilon)\|y-x\|.$$

By changing x for y we also get $f(y) - f(x) \leq (M+\varepsilon)\|y-x\|$. Therefore,

$$|f(y) - f(x)| \leq (M+\varepsilon)\|y-x\|$$

for all $x, y \in \mathcal{U}$ and for all $\varepsilon > 0$. Finally, by fixing $x, y \in \mathcal{U}$ and letting ε go to 0, we have $|f(x) - f(y)| \leq M\|x-y\|$; so it is proved that $|f(x) - f(y)| \leq M\|x-y\|$ for all $x, y \in \mathcal{U}$.

It should be noted that the preceding reasoning in fact proves the following result, which is a subdifferential mean value inequality somewhat flavoured like the classic one.

Theorem 3.23 *Let X be a Banach space and $f : \mathcal{U} \longrightarrow \mathbb{R}$ be a continuous function. If $x, y \in \mathcal{U}$ and $M \geq 0$ are so that for every $t \in [0,1]$ there exists $p \in D_G^- f(tx + (1-t)y)$ with $\|p\| \leq M$, then $|f(x) - f(y)| \leq M\|x-y\|$.*

Now the promised corollary is immediately deduced:

Corollary 3.24 *Let $\mathcal{U}$ be an open convex subset in a Banach space X, and let $f : \mathcal{U} \longrightarrow \mathbb{R}$ be a continuous function such that $0 \in D_G^- f(x)$ for all $x \in \mathcal{U}$. Then f is constant on $\mathcal{U}$.*

It is not true that if $f : X \longrightarrow \mathbb{R}$ is continuous and subdifferentiable in a dense subset $D \subseteq X$ and $0 \in D^- f(x)$ for all $x \in D$ then f is constant. Even though X is finitely dimensional and the Lebesgue measure of $X \backslash D$ is zero this is not true, as the following example proves.

Example 3.25 Let $f : [0,1] \longrightarrow \mathbb{R}$ be the Cantor-Lebesgue function (see its definition in [17], p. 55, for instance). We have that f is non-decreasing and continuous in $[0,1]$, and f is locally constant in $D = [0,1] \setminus C$, where C is Cantor's set. Hence f is differentiable in D, with $\{0\} = \{df(x)\} = D^-_G f(x)$ for all $x \in D$, and yet f is not constant.

However, if $\dim X \geq 2$, by using some cardinality reasoning one can easily deduce the following improvement of theorem 3.22.

Corollary 3.26 *Let X be a Banach space with $\dim X \geq 2$, and let $\mathcal{U} \subset X$ be an open convex subset. Let $f : \mathcal{U} \longrightarrow \mathbb{R}$ be a continuous function and let C be a countable subset of $\mathcal{U}$. Suppose that there exists $M \geq 0$ such that for every $x \in \mathcal{U} \setminus C$ there exists $p \in D^-_G f(x)$ with $\|p\| \leq M$. Then*

$$|f(x) - f(y)| \leq M\|x - y\|$$

for all $x, y \in \mathcal{U}$.

3.3 Subdifferential Approximate Rolle's Theorem

In this section we will be involved in the task of proving both Fréchet and Gâteaux subdifferential versions of the approximate Rolle's theorem given in the preceding chapter which will hold within the class of all Banach spaces having a Fréchet (respectively Gâteaux) differentiable Lipschitz bump function (the second one is quite a large class, as it includes all WCG Banach spaces). We will see that if a subdifferentiable function oscillates between $-\varepsilon$ and ε on the boundary of the unit ball then there exists a point x in the interior of the ball and there exists $p \in D^- f(x)$ (resp. $p \in D^-_G f(x)$) such that $\|p\| \leq 2\varepsilon$. In fact, for a Banach space X having a Fréchet differentiable Lipschitz bump function, it will be proved that every bounded *continuous* function $f : B_X \longrightarrow \mathbb{R}$ such that f oscillates between $-\varepsilon$ and ε on the unit sphere satisfies $\inf\{\|p\| : p \in D^- f(x) \cup D^+ f(x), \|x\| < 1\} \leq 2\varepsilon$.

In order to prove the subdifferential approximate Rolle's theorems we will need three auxiliary results. First, we will use the following equivalent statement of the formula for the subdifferential of the sum (see theorem 3.7) to prove the strongest version of the theorem in the Fréchet case.

Theorem 3.27 (Formula for the superdifferential of the sum) *Suppose that X is a Banach space with a $C^1(X)$ Lipschitz bump function. Let $f, g : X \longrightarrow \mathbb{R}$ be such that f is lower semicontinuous and g is uniformly continuous. Then, for every $x_0 \in X, p \in D^+(f+g)(x_0)$ and $\varepsilon > 0$, there exist $x_1, x_2 \in X, p_1 \in D^+ f(x_1)$ and $p_2 \in D^+ g(x_2)$ such that:*

(i) $\|x_1 - x_0\| < \varepsilon$ *and* $\|x_2 - x_0\| < \varepsilon$.

(ii) $| f(x_1) - f(x_0) | < \varepsilon$ *and* $| g(x_2) - g(x_0) | < \varepsilon$.

(iii) $\|p_1 + p_2 - p\| < \varepsilon$.

We will also need the following Variational Principle, whose proof can be found in [33], chapter I.

Theorem 3.28 (Variational Principle) *Let X be a Banach space which has a Fréchet differentiable Lipschitz bump function (respectively Gâteaux differentiable Lipschitz bump function). Let $F : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function that is bounded above, $F \not\equiv -\infty$. Then, for all $\delta > 0$ there exists a bounded Fréchet differentiable (resp. Gâteaux differentiable) Lipschitz function $\varphi : X \longrightarrow \mathbb{R}$ such that:*

1. *$F + \varphi$ attains its strong maximum in X,*
2. *$\|\varphi\|_\infty = \sup_{x \in X} \mid \varphi(x) \mid < \delta$, and $\|\varphi'\|_\infty = \sup_{x \in X} \|\varphi'(x)\| < \delta$.*

Finally, in order to prove the weaker Gâteaux version of the theorem we will also use the following version of Ekeland's Variational Principle, which is equivalent to that given in the preceding chapter (see theorem 2.6).

Theorem 3.29 (Ekeland's Variational Principle) *Let X be a Banach space, and let $f : X \longrightarrow [-\infty, \infty]$ be a proper upper semicontinuous function which is bounded above. Let $\varepsilon > 0$ and $x_0 \in X$ such that $f(x_0) > \sup\{f(x) : x \in X\} - \varepsilon$. Then for every λ with $0 < \lambda < 1$ there exists a point $z \in Dom(f)$ such that:*

(i) $\lambda\|z - x_0\| \leq f(z) - f(x_0)$

(ii) $\|z - x_0\| < \varepsilon/\lambda$

(iii) $\lambda\|x - z\| + f(z) > f(x)$ *whenever* $x \neq z$.

Now let us start with the Fréchet subdifferential approximate Rolle's theorem. Its statement is stronger and the proof is simpler than in the Gâteaux case thanks to the formula for the subdifferential of the sum. Hereafter the set $\{x \in X : \|x\| \leq R\}$ is denoted by $B(0, R)$, while $S(0, R)$ stands for $\{x \in X : \|x\| = R\}$.

Theorem 3.30 *Let X be a Banach space which has a $C^1(X)$ Lipschitz bump function, let $B = B(0, R), S = S(0, R)$ and let $f : B \longrightarrow \mathbb{R}$ be a bounded continuous function such that $f(S) \subset [-\varepsilon, \varepsilon]$. Then:*

(i) If $\sup f(B) > \sup f(S)$ then for every $\alpha > 0$ there exist $x \in int(B)$ and $p \in D^+ f(x)$ such that $\|p\| < \alpha$.

(ii) If $\inf f(B) < \inf f(S)$ then for every $\alpha > 0$ there exist $x \in int(B)$ and $p \in D^- f(x)$ such that $\|p\| < \alpha$.

(iii) If $f(B) \subseteq f(S)$ then for every $\alpha > 0$ there exist $x_1, x_2 \in int(B)$ and $p_1 \in D^+ f(x_1), p_2 \in D^- f(x_2)$ such that $\|p_1\|, \|p_2\| < 2\varepsilon/R + \alpha$.

Proof.

Case (i): let $\eta = \sup f(B) - \sup f(S) > 0$, and consider $F(x) = f(x)$ if $x \in B, F(x) = -\infty$ otherwise. Since F is upper semicontinuous and bounded above, the Variational Principle provides us with a $C^1(X)$ function g such that $\|g\|_\infty < \eta/3$, $\|g'\|_\infty < \alpha$, and $F+g$ attains its maximum at a point $x \in B$. Moreover $x \in int(B)$: otherwise, taking a such that $f(a) > \sup f(B) - \eta/3$ we would get

$$\sup f(B) - 2\eta/3 < F(a) + g(a) \leq F(x) + g(x) \leq \sup f(S) + \eta/3,$$

which is a contradiction. Therefore $x \in int(B)$ and $p = g'(x) \in D^+ f(x)$ satisfies $\|p\| < \alpha$.

Case (ii): the same proof works.

Case (iii): let us consider the function $\phi(x) = f(x) - (2\varepsilon+\alpha)\|x\|/R$. This function satisfies the conditions of case (i) and so there exist $x \in int(B)$ and $p \in D^+\phi(x)$ such that $\|p\| < \alpha$. Now, by the formula for the subdifferential of the sum, there exist $x_1, y_1 \in int(B)$ and p_1, q_1 with $p_1 \in D^+ f(x_1)$, $q_1 \in D^+(-(2\varepsilon+\alpha)/R\|y_1\|)$ such that $\|p_1 + q_1 - p\| < \alpha$, which implies

$$\|p_1\| < \alpha + \|q_1\| + \|p\| < 2\alpha + \|q_1\|.$$

Recall that $q \in D^+(-\|.\|)(v)$ if and only if $-q \in D^-(\|.\|)(v)$. Moreover, since $\|.\|$ is convex we have $\partial\|.\|(v) = D^-\|.\|(v)$, so that if $q \in D^-\|.\|(v)$ then $q(h) \leq \|v+h\| - \|v\| \leq \|h\|$ for all h, and therefore $\|q\| \leq 1$. Taking this into account we can deduce that $\|q_1\| = \|-q_1\| \leq \frac{2\varepsilon+\alpha}{R}$ and so $\|p_1\| < 2\alpha + (2\varepsilon+\alpha)/R$.

In order to find x_2 and p_2 it is enough to consider $\phi(x) = f(x) + (2\varepsilon+\alpha)\|x\|/R$ and the same proof holds using case (ii) instead of (i).

From this result it is deduced the following

Theorem 3.31 *Let $\mathcal{U}$ be a bounded connected open set in a Banach space X which has a $C^1(X)$ Lipschitz bump function, let $f : \mathcal{U} \longrightarrow \mathbb{R}$ be a bounded continuous function and let $R > 0$ and $x_0 \in \mathcal{U}$ be such that $dist(x_0, \partial\mathcal{U}) = R$. Suppose that $f(\partial\mathcal{U}) \subseteq [-\varepsilon, \varepsilon]$. Then:*

(i) If $\sup f(\overline{\mathcal{U}}) > \sup f(\partial\mathcal{U})$ then for every $\alpha > 0$ there exist $x \in \mathcal{U}$ and $p \in D^+ f(x)$ such that $\|p\| < \alpha$.

(ii) If $\inf f(\overline{\mathcal{U}}) < \inf f(\partial\mathcal{U})$ then for every $\alpha > 0$ there exist $x \in \mathcal{U}$ and $p \in D^- f(x)$ such that $\|p\| < \alpha$.

(iii) If $f(\overline{\mathcal{U}}) \subseteq [-\varepsilon, \varepsilon]$ then for every $\alpha > 0$ there exist $x_1, x_2 \in \mathcal{U}$ and $p_1 \in D^+ f(x_1), p_2 \in D^- f(x_2)$ such that $\|p_1\|, \|p_2\| < 2\varepsilon/R + \alpha$.

In each case, $\inf\{\|p\| : p \in D^- f(x) \cup D^+ f(x), x \in \mathcal{U}\} \leq 2\varepsilon/R$.

And from this we can immediately deduce

Corollary 3.32 *Let $\mathcal{U}$ be a bounded connected open subset of a Banach space X that has a $C^1(X)$ Lipschitz bump function, and let $f : \overline{\mathcal{U}} :\longrightarrow \mathbb{R}$ be continuous and bounded on $\mathcal{U}$. Suppose that f is constant on $\partial\mathcal{U}$. Then,*

$$\inf\{\|p\| : p \in D^- f(x) \cup D^+ f(x), x \in \mathcal{U}\} = 0.$$

as well as

Corollary 3.33 *Let X be a Banach space having a $C^1(X)$ Lipschitz bump function and let $f : X \longrightarrow \mathbb{R}$ be continuous and bounded on X. Then,*

$$\inf\{\|p\| : p \in D^- f(x) \cup D^+ f(x), x \in X\} = 0.$$

Finally we will study the subdifferential approximate Rolle's theorem in the Gâteaux case. Here the proof is longer and the statement weaker than in the Fréchet case. If the formula for the subdifferential of the sum were true in the Gâteaux case within the class of those Banach spaces having a Gâteaux differentiable and Lipschitz bump function, the proof of theorem 3.30 would also work in this case yielding an improvement in the statement of theorem 3.34 and its corollaries. We do not know whether such a formula is true or not within that class of Banach spaces.

Theorem 3.34 *Let X be a Banach space which has a Gâteaux differentiable Lipschitz bump function and $R, \varepsilon > 0$. Let $f : B(0,R) \longrightarrow \mathbb{R}$ be a bounded continuous function and suppose that f is Gâteaux subdifferentiable in* $\mathrm{int}B(0,R)$ *and $f(S(0,R)) \subseteq [-\varepsilon, +\varepsilon]$. Then there exist $x_\varepsilon \in intB(0,R)$ and $p \in D_G^- f(x_\varepsilon)$ so that $\|p\| \leq 2\varepsilon/R$.*

Proof. Let us suppose first that $\varepsilon < 2R$. We will consider three cases.

Case I: $f(B(0,R)) \subseteq [-\varepsilon, \varepsilon]$.

Suppose first that $f(0) > -\varepsilon$. Let $\lambda = 2\varepsilon/R$. Since $f(0) > \sup\{f(x) : x \in B(0,R)\} - 2\varepsilon$, Ekeland's Variational Principle gives us an $x_1 \in B(0,R)$ such that

(i) $\lambda\|x_1\| \leq f(x_1) - f(0)$

(ii) $\|x_1\| < 2\varepsilon/\lambda$

(iii) $\lambda\|x - x_1\| + f(x_1) > f(x)$ whenever $x \neq x_1$,

so that $x_1 \in intB(0,R)$ and, taking any $p \in D_G^- f(x_1)$, (iii) implies that $\|p\| \leq 2\varepsilon/R$. Indeed, for every h with $\|h\| = 1$ we have

$$\frac{f(x_1 + th) - f(x_1)}{|t|} < \frac{2\varepsilon}{R}$$

for every t, and also, since $p \in D_G^- f(x_1)$,

$$\liminf_{t \to 0} \frac{f(x_1 + th) - f(x_1) - tp(h)}{|t|} \geq 0$$

or equivalently

$$\limsup_{t\to 0} \frac{-f(x_1+th)+f(x_1)+tp(h)}{|t|} \leq 0,$$

and therefore

$$\begin{aligned}
|p(h)| &= \limsup_{t\to 0} \frac{p(th)}{|t|} = \limsup_{t\to 0} \frac{f(x_1+th)-f(x_1)-f(x_1+th)+f(x_1)+p(th)}{|t|} \\
&\leq \limsup_{t\to 0} \frac{f(x_1+th)-f(x_1)}{|t|} + \limsup_{t\to 0} \frac{-f(x_1+th)+f(x_1)+p(th)}{|t|} \\
&\leq \limsup_{t\to 0} \frac{f(x_1+th)-f(x_1)}{|t|} \leq \frac{2\varepsilon}{R}.
\end{aligned}$$

This proves that $\|p\| \leq \frac{2\varepsilon}{R}$.

Now suppose that $f(0) = -\varepsilon$ and pick a $p \in D_G^- f(0)$. We may suppose that $\|p\| > 2\varepsilon/R$ since we would have finished otherwise. Then there exists h with $\|h\| = 1$ such that $p(h) > 2\varepsilon/R$. As

$$\liminf_{t\to 0} \frac{f(th)-f(0)-tp(h)}{|t|} \geq 0$$

and $f(0) = -\varepsilon$ there exists $\delta > 0$ such that

$$\frac{f(\delta h)+\varepsilon-\delta p(h)}{\delta} > \frac{2\varepsilon}{R} - p(h),$$

which implies $f(\delta h)+\varepsilon > \frac{2\varepsilon\delta}{R}$. Hence $f(\delta h) > \sup f(B(0,R)) - 2\varepsilon$. Taking $\lambda = 2\varepsilon/R$ we can use again Ekeland's Variational Principle to get an $x_1 \in B(0,R)$ such that:

(i) $\lambda\|x_1 - \delta h\| \leq f(x_1) - f(\delta h)$

(ii) $\|x_1 - \delta h\| < \varepsilon/\lambda$

(iii) $\lambda\|x - x_1\| + f(x_1) > f(x)$ whenever $x \neq x_1$.

From (i) and since $f(\delta h) + \varepsilon > \frac{2\varepsilon\delta}{R}$ we get

$$\|x_1 - \delta h\| \leq \frac{f(x_1) - f(\delta h)}{2\varepsilon/R} \leq \frac{\varepsilon - f(\delta h)}{2\varepsilon/R} < \frac{2\varepsilon - \frac{2\varepsilon\delta}{R}}{2\varepsilon/R} = R - \delta,$$

which implies $\|x_1\| \leq \|x_1 - \delta h\| + \delta < R - \delta + \delta = R$ and so $\|x_1\| < R$. Now, since f is Gâteaux subdifferentiable at x_1, the same calculations as above prove that (iii) implies $\|p\| \leq 2\varepsilon/R$ for every$p \in D_G^- f(x_1)$.

Case II: $\sup f(B(0,R)) > \sup f(S(0,R))$.

Let us choose x_0 such that $\sup f(S(0,R)) < f(x_0)$, and let α, λ be such that $0 < \alpha < f(x_0) - \sup f(S(0,R))$, $\alpha \leq 2\varepsilon/R$ and $0 < \lambda < \alpha/(R+1)$. From Ekeland's Variational Principle it follows that there exists $x_1 \in \mathrm{int}B(0,R)$ such that

$$f(x) < f(x_1) + \lambda\|x - x_1\|$$

for every $x \neq x_1$, and we already know that this implies that $\|p\| \leq \lambda < \alpha$ for any $p \in D_G^- f(x_1)$.

Case III: $\inf f(B(0,R)) < \inf f(S(0,R))$.

This is the only case in which we will use the smooth variational principle. Let $\eta = \inf f(S(0,R)) - \inf f(B(0,R)) > 0$, $\alpha > 0$ such that $\alpha \leq 2\varepsilon/R$ and consider $F : X \longrightarrow \mathbb{R}\cup\{\infty\}$ defined by $F(x) = f(x)$ if $x \in B(0,R)$ and $F(x) = +\infty$ otherwise. From the smooth variational principle it follows that there exists a bounded Gâteaux differentiable Lipschitz function $\varphi : X \longrightarrow \mathbb{R}$ such that $\|\varphi\|_\infty < \eta/3$, $\|\varphi'\|_\infty < \alpha$ and $F - \varphi$ attains its minimum at a point $x_0 \in B(0,R)$. Moreover it must be $x_0 \in intB(0,R)$: otherwise, taking a such that $f(a) < \inf f(B(0,R)) + \eta/3$ we would have

$$\inf f(B(0,R)) + 2\eta/3 > F(a) - \varphi(a) \geq F(x_0) - \varphi(x_0) \geq \inf f(S(0,R)) - \eta/3,$$

which is a contradiction. Recall that the sum $g+h$ of two subdifferentiable functions g and h is subdifferentiable, and

$$D_G^- g(x) + D_G^- h(x) \subseteq D_G^-(g+h)(x).$$

We also know that if a function g attains a minimum at x then g is subdifferentiable at x and $0 \in D_G^- g(x)$. Taking this into account we can deduce

$$0 + \varphi'(x_0) \in D_G^-(F - \varphi)(x_0) + D_G^-\varphi(x_0) \subseteq D_G^- F(x_0) = D_G^- f(x_0)$$

so that $p = \varphi'(x_0)$ satisfies $p \in D_G^- f(x_0)$ and $\|p\| < \alpha \leq 2\varepsilon/R$.

Finally, consider the case in which $\varepsilon \geq 2R$. Taking into account that $p \in D_G^- f(x)$ if and only if $rp \in D_G^-(rf)(x)$ for every $r > 0$ and considering $g = \varepsilon' f/\varepsilon$, where $\varepsilon' < 2R$, we can conclude (by applying the preceding reasoning to g) that there exist an x in the interior of the ball and a subgradient $p \in D_G^- f(x)$ such that $\|p\| \leq 2\varepsilon/R$.

Remark 3.35 Note that we have only used the smooth variational principle in the proof corresponding to the case $\inf f(B(0,R)) < \inf f(S(0,R))$. Note also that in the first case we only used that $f(B(0,R)) \subseteq [-\varepsilon, \varepsilon]$. Thus, it is clear that for any Banach space X and any continuous bounded function $f : B_X(0,R) \longrightarrow \mathbb{R}$ which is Gâteaux subdifferentiable in the interior of the ball and satisfies $f \geq -\varepsilon$ and $f_{|S(0,R)} \leq \varepsilon$, there exist a point x in the interior of the ball and a subgradient $p \in D_G^- f(x)$ such that $\|p\| \leq 2\varepsilon/R$.

Remark 3.36 The conclusion of theorem 3.34 cannot be improved so as to obtain a point $x \in \text{int}B(0,R)$ such that $\|p\| \leq 2\varepsilon/R$ for all $p \in D^- f(x)$. See example 3.12.

From the preceding theorem it is deduced the more general

Theorem 3.37 *Let $\mathcal{U}$ be a bounded connected open set in a Banach space X that has a Gâteaux differentiable Lipschitz bump function. Let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be continuous and bounded, Gâteaux subdifferentiable in $\mathcal{U}$. Let $R > 0$ and $x_0 \in \mathcal{U}$ be such that $dist(x_0, \partial\mathcal{U}) = R$. Suppose that $f(\partial\mathcal{U}) \subset [-\varepsilon, \varepsilon]$. Then there exist $x_\varepsilon \in \mathcal{U}$ and $p \in D_G^- f(x_\varepsilon)$ so that $\|p\| \leq 2\varepsilon/R$.*

as well as the following two corollaries

Corollary 3.38 *Let $\mathcal{U}$ be a bounded connected open subset of a Banach space X that has a Gâteaux differentiable Lipschitz bump function, and let $f : \overline{\mathcal{U}} \longrightarrow \mathbb{R}$ be continuous, bounded, and Gâteaux subdifferentiable in $\mathcal{U}$. Suppose that f is constant on $\partial\mathcal{U}$. Then,*

$$\inf\{\|p\| : p \in D_G^- f(x), x \in \mathcal{U}\} = 0.$$

Corollary 3.39 *Let X be a Banach space having a Gâteaux differentiable Lipschitz bump function and let $f : X \longrightarrow \mathbb{R}$ be continuous, Gâteaux subdifferentiable and bounded on X. Then,*

$$\inf\{\|p\| : p \in D_G^- f(x), x \in X\} = 0.$$

Chapter 4

Smooth negligibility in Banach spaces

In this chapter we strengthen the negligibility scheme presented in chapter 1 so as to obtain diffeomorphisms removing compacta and cylinder over compacta from Banach spaces having smooth norms or seminorms. In fact we will prove that the removal of a compact set can always happen at the end of a C^1 isotopy. If a Banach space X has a real-analytic norm, we will also prove that it is possible to delete points from X by means of a real-analytic diffeomorphism. At the end of the chapter we extend some results on deleting diffeomorphisms and isotopies to the setting of differentiable manifolds modelled on infinite-dimensional Banach spaces with Fréchet differentiable norms.

4.1 Removing compact sets from a Banach space

In this section we will give a method of removing compacta smoothly from an infinite-dimensional Banach space having a (not necessarily equivalent) smooth norm. Let us state our main result.

Theorem 4.1 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a (not necessarily equivalent) C^p smooth norm ϱ. Then, for every compact set $K \subset X$, there exists a C^p diffeomorphism φ between X and $X \setminus K$. Moreover, for each open ϱ-ball B containing K, we can additionally require that φ be the identity outside B.*

In order to prove this theorem we will use the *fixed point lemma* given at the beginning of chapter 1; let us recall it.

Lemma 4.2 *Let $F : (0, \infty) \longrightarrow [0, \infty)$ be a continuous function such that, for every $\beta \geq \alpha > 0$, $F(\beta) - F(\alpha) \leq \frac{1}{2}(\beta - \alpha)$ and $\limsup_{t \to 0^+} F(t) > 0$. Then there exists a unique $\alpha > 0$ such that $F(\alpha) = \alpha$.*

We will also need sharper statements of lemmas 1.4 and 1.5. Namely,

Lemma 4.3 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space having a (not necessarily equivalent) C^p smooth norm ϱ. Then there exists a continuous functional $\omega : X \longrightarrow [0, \infty)$ which is C^p smooth on $X \setminus \{0\}$ and satisfies the following conditions:*

1. *$\omega(x + y) \leq \omega(x) + \omega(y)$ and, consequently, $\omega(x) - \omega(y) \leq \omega(x - y)$, for every $x, y \in X$;*
2. *$\omega(rx) = r\omega(x)$ for every $x \in X$ and $r \geq 0$;*
3. *$\omega(x) = 0$ if and only if $x = 0$;*
4. *$\omega(\sum_{k=1}^{\infty} z_k) \leq \sum_{k=1}^{\infty} \omega(z_k)$ for every convergent series $\sum_{k=1}^{\infty} z_k$; and*
5. *For every $\delta > 0$, there exists a sequence of linearly independent vectors (y_k) satisfying*
$$\omega(y_k) \leq \frac{\delta}{4^{k+1}}$$
for every $k \in \mathbb{N}$, and with the property that for every compact set $K \subset X$ there exists $n_0 \in \mathbb{N}$ such that
$$\inf\{\omega(z - \sum_{k=1}^{n} y_k) \mid n \geq n_0, z \in K\} > 0$$

Notice that ω need not be a norm in X; in general, $\omega(x) \neq \omega(-x)$.

and

Lemma 4.4 *Let $(X, \|\cdot\|)$ be a Banach space, and let ω be a functional satisfying conditions 1, 2, and 5 of lemma 4.3. Then, for every $\delta > 0$, there exists a C^∞ path $p = p_\delta : (0, \infty) \longrightarrow X$ such that*

1. *$\omega(p(\alpha) - p(\beta)) \leq \frac{1}{2}(\beta - \alpha)$ if $\beta \geq \alpha > 0$;*
2. *For every compact set $K \subset X$ there exists $t_0 > 0$ such that*
$$\inf\{\omega(z - p(t)) \mid 0 < t \leq t_0, z \in K\} > 0;$$
3. *$p(t) = 0$ if and only if $t \geq \delta$.*

Proof of lemmas 4.3 and 4.4:
We will recall the argument of the proof of lemmas 1.4 and 1.5, showing how the additional required properties can be obtained. We must distinguish three cases:

Case I: The norm ϱ is complete and the space X is non-reflexive.

In this case may assume that $\varrho = \|\cdot\|$, we take a continuous linear functional $T \in X^*$ with $\|T\| = 1$ such that T does not attain its norm, and we define $\omega(x) = \|x\| - T(x)$. As shown in lemma 1.4, the functional ω satisfies properties 1–4. In the proof of lemma 1.4 it was also shown that for every $\delta > 0$ there exists a sequence (y_k) such that $\|y_k\| = 1$ and

$$\omega(y_k) = \|y_k\| - T(y_k) \leq \frac{\delta}{4^{k+1}}$$

for every $k \in \mathbb{N}$, that is, ω satisfies the first part of property 5. Clearly, we may assume that the vectors (y_k) are linearly independent. We only have to check that for such a sequence (y_k) the following condition is also satisfied: for every compact set $K \subset X$ there exists $n_0 = n_0(K) \in \mathbb{N}$ such that

$$\inf\{\omega(z - \sum_{k=1}^{n} y_k) \mid n \geq n_0, z \in K\} > 0.$$

So, let K be a compact set, let $M > 0$, and take $R > 0$ such that $\|z\| \leq R$ for every $z \in K$. Since $T(y_k) \to 1$ as $k \to \infty$, we can find $n_0 \in \mathbb{N}$ such that $\sum_{k=1}^{n} T(y_k) > M + R$ for every $n \geq n_0$. Then we have

$$\begin{aligned}\omega(z - \sum_{k=1}^{n} y_k) &= \|z - \sum_{k=1}^{n} y_k\| - T(z - \sum_{k=1}^{n} y_k) \geq -T(z - \sum_{k=1}^{n} y_k) \\ &= -T(z) + T(\sum_{k=1}^{n} y_k) \geq -\|z\| + T(\sum_{k=1}^{n} y_k) = -\|z\| + \sum_{k=1}^{n} T(y_k) \\ &\geq -R + M + R = M\end{aligned}$$

whenever $n \geq n_0$, $z \in K$. This proves that

$$\inf\{\omega(z - \sum_{k=1}^{n} y_k) \mid n \geq n_0, z \in K\} \geq M > 0.$$

Case II: The norm ϱ is complete and the space X is reflexive.

As we saw in the proof of lemma 1.4, every reflexive Banach space has a C^∞ smooth non-complete norm, and hence we may go on to the case when the norm ϱ is non-complete.

Case III: The norm ϱ is non-complete.

In this case we define $\omega = \varrho$. In lemma 1.4 it was proved that ω satisfies properties 1–4. It was also seen that for every $\delta > 0$ one can find a sequence (y_k) in X such that $\omega(y_k) \leq \frac{\delta}{4^{k+1}}$ for each k, and a point $\hat{y}$ in the completion of (X, ω), denoted by $(\hat{X}, \hat{\omega})$, such that $\hat{y} \notin X$, and $\lim_n \hat{\omega}(\hat{y} - \sum_{k=1}^{n} y_k) = 0$. So the first part of property 5 is satisfied. Moreover, a revision of the proof of 1.4 shows that the sequence (y_k) can be chosen in such a way that $\{y_k \mid k = 1, 2, ...\}$ is a linearly independent set of vectors. It only remains to check that for such a sequence $(y_k) \subset X$ and for every compact set $K \subset X$ there exists $n_0 \in \mathbb{N}$ such that

$$\inf\{\omega(z - \sum_{k=1}^{n} y_k) \mid n \geq n_0, z \in K\} > 0.$$

Let K be a compact set of $(X, \|\cdot\|)$. The norm ω is continuous with respect to $\|\cdot\|$ (because it is C^p smooth). Then the linear injection of $(X, \|\cdot\|)$ into $(\hat{X}, \hat{\omega})$ is continuous, and the set K is also compact in $(\hat{X}, \hat{\omega})$. Since $\hat{y} \in \hat{X} \setminus X$ and $K \subset X$, we have that

$$\mathrm{dist}_\omega(\hat{y}, K) = \inf\{\hat{\omega}(z - \hat{y}) \mid z \in K\} > 0,$$

that is, there exists a positive number η so that $\hat{\omega}(z - \hat{y}) \geq 2\eta$ for every $z \in K$. Since $\lim_n \hat{\omega}(\hat{y} - \sum_{k=1}^{n} y_k) = 0$ we can take $n_0 \in \mathbb{N}$ such that $\hat{\omega}(\hat{y} - \sum_{k=1}^{n} y_k) \leq \eta$ whenever $n \geq n_0$. Then, for every $n \geq n_0$ and $z \in K$,

$$\begin{aligned}
&\omega(z - \sum_{k=1}^{n} y_k) = \hat{\omega}(z - \sum_{k=1}^{n} y_k) = \hat{\omega}[(z - \hat{y}) - (\sum_{k=1}^{n} y_k - \hat{y})] \\
&\geq \hat{\omega}(z - \hat{y}) - \hat{\omega}(\sum_{k=1}^{n} y_k - \hat{y}) \geq 2\eta - \hat{\omega}(\sum_{k=1}^{n} y_k - \hat{y}) \\
&\geq 2\eta - \eta = \eta > 0,
\end{aligned}$$

and hence $\inf\{\omega(z - \sum_{k=1}^{n} y_k) \mid n \geq n_0, z \in K\} > 0$. This concludes the proof of lemma 4.3.

Now let us make a sketch of the proof of lemma 4.4. For a given $\delta > 0$, choose a sequence (y_k) satisfying condition 5 of lemma 4.3, and pick a non-increasing C^∞ function $\gamma : [0, \infty) \longrightarrow [0, 1]$ such that $\gamma = 1$ in $[0, \delta/2]$, $\gamma = 0$ in $[\delta, \infty)$, and $\sup\{|\gamma'(t)| : t \in [0, \infty)\} \leq 4/\delta$. Then define a path $p : (0, \infty) \longrightarrow X$ by

$$p(t) = \sum_{k=1}^{\infty} \gamma(2^{k-1} t) y_k.$$

The proof that p satisfies condition 1 of 4.4 is the same as in lemma 1.5. Let us see that p satisfies condition 2 of 4.4. For a compact set $K \subset X$, condition 5 of lemma 4.3 provides us with numbers $\eta > 0$, $m_1 \in \mathbb{N}$ such that $\omega(z - \sum_{k=1}^{n} y_k) \geq 2\eta$ for all $n \geq m_1$ and $z \in K$. Since $\omega(y_k) \leq \delta/4^{k+1}$ for every k, we can find $m_2 \in \mathbb{N}$ so that $\sum_{k=m_2+1}^{\infty} \omega(y_k) \leq \sum_{k=m_2+1}^{\infty} \frac{\delta}{4^{k+1}} \leq \eta$. Let $n_0 = \max\{m_1, m_2\}$, and put $t_0 = \delta/2^{n_0}$. Then, taking into account that $\gamma(2^{k-1} t) = 1$ whenever $0 < t \leq t_0$ and $1 \leq k \leq n_0$, we have

$$\begin{aligned}\omega(z-p(t)) &= \omega(z-\sum_{k=1}^{\infty}\gamma(2^{k-1}t)y_k)\\ &= \omega[(z-\sum_{k=1}^{n_0}y_k)-(\sum_{k=1}^{\infty}\gamma(2^{k-1}t)y_k-\sum_{k=1}^{n_0}y_k)]\\ &\geq \omega(z-\sum_{k=1}^{n_0}y_k)-\omega(\sum_{k=1}^{\infty}\gamma(2^{k-1}t)y_k-\sum_{k=1}^{n_0}y_k)\\ &= \omega(z-\sum_{k=1}^{n_0}y_k)-\omega(\sum_{k=n_0+1}^{\infty}\gamma(2^{k-1}t)y_k)\\ &\geq \omega(z-\sum_{k=1}^{n_0}y_k)-\sum_{k=n_0+1}^{\infty}\gamma(2^{k-1}t)\omega(y_k)\\ &\geq \omega(z-\sum_{k=1}^{n_0}y_k)-\sum_{k=n_0+1}^{\infty}\omega(y_k)\geq \omega(z-\sum_{k=1}^{n_0}y_k)-\sum_{k=m_2+1}^{\infty}\omega(y_k)\\ &\geq 2\eta-\eta=\eta>0\end{aligned}$$

for every $0 < t \leq t_0$ and $z \in K$. In particular,

$$\inf\{\omega(z-p(t)) \mid 0 < t \leq t_0, z \in K\} \geq \eta > 0.$$

So condition 2 of 4.4 is satisfied as well.

Finally, it is easily seen that the fact that $\{y_k \mid k = 1, 2, ...\}$ is a linearly independent set of vectors ensures that $p(t) = 0$ if and only if $t \geq \delta$.

In order to prove 4.1 we will also need a function $f : X \longrightarrow [0, \infty)$ such that, for a given compact set K, f is C^p smooth on $X \setminus K$, it satisfies $f^{-1}(0) = K$, and $f(x) - f(y) \leq \omega(x - y)$ for every $x, y \in X$. The existence of such a function is ensured by the following lemma.

Lemma 4.5 *Let $\omega : X \longrightarrow [0, \infty)$ be a continuous functional satisfying properties 1–3 of lemma 4.3, and such that ω is C^p smooth on $X \setminus \{0\}$. Let K be a compact subset of X. For $x \in X$, write $d_K(x) = \inf\{\omega(x-y) \mid y \in K\}$. Then, for each $\varepsilon > 0$ there exists a continuous function $f = f_\varepsilon : X \longrightarrow [0, \infty)$ such that*

1. *f is C^p smooth on $X \setminus K$;*
2. *$f(x) - f(y) \leq \omega(x - y)$ for every $x, y \in X$;*
3. *$f^{-1}(0) = K$;*
4. *$\inf\{f(x) \mid d_K(x) \geq \eta\} > 0$ for every $\eta > 0$;*
5. *f is constant on the set $\{x \in X \mid d_K(x) \geq \varepsilon\}$.*

Proof of lemma 4.5

First of all let us see that the function d_K is continuous and satisfies $d_K^{-1}(0) = K$, and $d_K(x) - d_K(y) \leq \omega(x-y)$ for every $x, y \in X$. Indeed, for every $y \in X$ and for every $\varepsilon > 0$ there exists $y_\varepsilon \in K$ such that $d_K(y) + \varepsilon \geq \omega(y - y_\varepsilon)$. Then

$$\begin{aligned} & d_K(x) - d_K(y) = \inf\{\omega(x-z) \mid z \in K\} - \inf\{\omega(y-z) \mid z \in K\} \\ & \leq \omega(x - y_\varepsilon) - \omega(y - y_\varepsilon) + \varepsilon \leq \omega[(x - y_\varepsilon) - (y - y_\varepsilon)] + \varepsilon \\ & = \omega(x-y) + \varepsilon, \end{aligned}$$

so that we obtain $d_K(x) - d_K(y) \leq \omega(x-y)$ by letting ε go to zero. Since $\omega(z) \leq 2\|z\|$ for every z, this inequality implies that $d_K(x) - d_K(y) \leq 2\|x-y\|$ for every $x, y \in X$ and hence $|d_K(x) - d_K(y)| \leq 2\|x-y\|$ for every $x, y \in X$, that is, d_K is Lipschitz and therefore continuous. The same argument shows that f is Lipschitz if only it satisfies condition 2. On the other hand, if $d_K(x) = 0$ then there exists a sequence $(y_n) \subseteq K$ such that $\lim_n \omega(x - y_n) = 0$. Since K is compact we may assume that (y_n) converges to some $y \in K$. By the continuity of ω, we have that $\omega(x-y) = 0$, which implies that $x = y \in K$. This, together with the obvious fact that $d_K(x) = 0$ for every $x \in K$, implies that $d_K^{-1}(0) = K$.

Now let us define the sets $U_n = \{x \in X \mid d_K(x) < 1/n\}$ for each $n \in N$. These are open sets satisfying $U_{n+1} \subseteq U_n$ for each n, and $\bigcap_{n=1}^\infty U_n = K$. Next, for every $x \in X$ and every $r > 0$, we define the *asymmetric ω-ball* $A(x, r)$ by

$$A(x, r) = \{z \in X \mid \omega(z - x) < r\}.$$

It should be noted that the sets U_n are ω-*open*, that is, for every $x \in U_n$ there exists $r_x > 0$ such that $A(x, r_x) \subseteq U_n$. Indeed, if $x \in U_n$, choose $r = \frac{1}{n} - d_K(x) > 0$. If $\omega(z-x) < r$ then $d_K(z) - d_K(x) \leq \omega(z-x) < r = \frac{1}{n} - d_K(x)$, so that $d_K(z) < 1/n$. This means that $A(x, r)$ is contained in U_n.

So, for each $n \in N$ and each $x \in K$ choose $r_x^n > 0$ such that $r_x^n < \frac{1}{2n}$ and $A(x, r_x^n) \subseteq U_n$. Since, for each n we have $K \subset \bigcup_{x \in K} A(x, r_x^n)$, the sets $A(x, r)$ are open, and K is compact, there exist $x_j^n \in K$, $j = 1, ..., k(n)$, so that

$$K \subset \bigcup_{j=1}^{k(n)} A(x_j^n, r_j^n),$$

where r_j^n stands for $r_{x_j^n}^n$.

Next, let us see that for every ω-*ball* $A(x_0, r)$ there exists a C^p function $g : X \longrightarrow [0, 1]$ such that $A(x_0, r) = g^{-1}(0)$, $g = 1$ outside $A(x_0, 2r)$, and $g(x) - g(y) \leq M\omega(x-y)$ for some $M > 0$. Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing C^∞ function such that $h^{-1}(0) = (-\infty, r]$ and $h = 1$ on $[2r, \infty)$. Let $M = \sup\{|h'(t)| \mid t \in \mathbb{R}\}$. Define $g : X \longrightarrow [0, \infty)$ by $g(y) = h(\omega(y - x_0))$ for every $y \in X$. It is clear that $A(x_0, r) = g^{-1}(0)$ and $g = 1$ outside $A(x_0, 2r)$. If $\omega(y - x_0) - \omega(x - x_0) \geq 0$ then $g(y) = h(\omega(y - x_0)) \geq h(\omega(x - x_0)) = g(x)$ because h is non-decreasing, and then

$g(x)-g(y) \leq M\omega(x-y)$ trivially holds. If, on the contrary, $\omega(x-x_0)-\omega(y-x_0) \geq 0$ then, taking into account that $|h'(t)| \leq M$, we get

$$\begin{aligned} & g(x) - g(y) = h(\omega(x - x_0)) - h(\omega(y - x_0)) \\ & \leq M|\omega(x - x_0) - \omega(y - x_0)| = M(\omega(x - x_0) - \omega(y - x_0)) \\ & \leq M\omega(x - y). \end{aligned}$$

In either case we obtain $g(x) - g(y) \leq M\omega(x - y)$ for every $x, y \in X$.

So, for each ω-*ball* $A(x_j^n, r_j^n)$ let us pick a C^p function $g_{(n,j)} : X \longrightarrow [0,1]$ such that $A(x_j^n, r_j^n) = g_{(n,j)}^{-1}(0)$, $g_{(n,j)} = 1$ outside $A(x_j^n, 2r_j^n)$, and $g_{(n,j)}(x) - g_{(n,j)}(y) \leq M_{(n,j)}\omega(x-y)$ for every $x, y \in X$ and some $M_{(n,j)} \geq 1$. Let us note that the product of two non-negative bounded functions satisfying an inequality like $g(x) - g(y) \leq M\omega(x-y)$ also satisfies such an inequality (perhaps with a different $M > 0$). Indeed, if $g_1(x) - g_1(y) \leq M_1\omega(x - y)$ and $g_2(x) - g_2(y) \leq M_2\omega(x - y)$ then

$$\begin{aligned} & g_1(x)g_2(x) - g_1(y)g_2(y) = \\ & = g_1(x)g_2(x) - g_1(x)g_2(y) + g_1(x)g_2(y) - g_1(y)g_2(y) \\ & = g_1(x)[g_2(x) - g_2(y)] + g_2(y)[g_1(x) - g_1(y)] \\ & \leq g_1(x)M_2\omega(x - y) + g_2(y)M_1\omega(x - y) \\ & \leq \big(\|g_1\|_\infty M_2 + \|g_2\|_\infty M_1\big)\omega(x - y), \end{aligned}$$

where $\|g_i\|_\infty = \sup\{|g_i(z)| : z \in X\}$. Now, for each n, consider the product

$$\varphi_n(x) = \prod_{j=1}^{k(n)} g_{(n,j)}(x).$$

The functions $\varphi_n : X \longrightarrow [0,1]$ satisfy $\varphi_n(x) - \varphi_n(y) \leq M_n\omega(x - y)$ for every $x, y \in X$, for some $M_n \geq 1$, as well as $\varphi_n = 0$ on K, and $\varphi_n(x) = 1$ whenever $x \in X \setminus U_n$ (indeed, if $d_K(x) \geq 1/n$ then $\omega(x - x_j^n) \geq d_K(x) \geq 1/n \geq 2r_j^n$, so that $g_{(n,j)}(x) = 1$ for every $j = 1, ..., k(n)$, which yields $\varphi_n(x) = 1$).

Finally choose $m \in \mathbb{N}$ such that $1/m < \varepsilon$. For every $k \geq m$ we have $\varphi_k(x) = 1$ whenever $d_K(x) \geq \varepsilon$. Then define $f : X \longrightarrow [0,1]$ by

$$f(x) = \sum_{k=m}^{\infty} \frac{1}{2^k M_k}\varphi_k(x)$$

for every $x \in X$.

Note that for every $x \in X \setminus K$ there exist an open neighbourhood V_x of x and a positive integer $n_x \geq m$ such that $\varphi_n(y) = 1$ whenever $y \in V_x$ and $n \geq n_x$. Indeed, for each $x \in X \setminus K$ let n_x be such that $1/n_x < d_K(x)$ and put $V_x = \{y \in X \mid d_K(y) > 1/n_x\}$. It is clear that V_x is an open neighbourhood of x, and for each $y \in V_x$ we have $y \in X \setminus U_n$ for every $n \geq n_x$, so that $\varphi_n(y) = 1$ whenever $n \geq n_x$. Then all but finitely many of the functions φ_n in the series defining f are constant on a neighbourhood of each point in $X \setminus K$, which clearly implies that f is a C^p smooth function on $X \setminus K$. It is also clear that $f^{-1}(0) = K$, and $f(x) - f(y) \leq \omega(x-y)$ for every $x, y \in X$. That is, f satisfies conditions 1–3 of lemma 4.5. Let us see that f also satisfies conditions 4 and 5. For a given $\eta > 0$, take $n_0 \geq m$ such that $1/n_0 \leq \eta$. Then, for every $k \geq n_0$, we have that $\varphi_k(x) = 1$ whenever $d_K(x) \geq \eta$, and therefore

$$\inf\{f(x) \mid d_K(x) \geq \eta\} = \inf\{\sum_{k=m}^{\infty} \frac{1}{2^k M_k}\varphi_k(x) \mid d_K(x) \geq \eta\}$$
$$\geq \inf\{\sum_{k=n_0}^{\infty} \frac{1}{2^k M_k}\varphi_k(x) \mid d_K(x) \geq \eta\} = \sum_{k=n_0}^{\infty} \frac{1}{2^k M_k} > 0.$$

So condition 4 is also fulfilled. Moreover, f is constant (equal to $\sum_{k=m}^{\infty} M_k^{-1} 2^{-k}$) on the set $\{x \in X \mid d_K(x) \geq \varepsilon\}$. This concludes the proof of lemma 4.5.

With all these tools in our hands we can give a proof of theorem 4.1.

Proof of theorem 4.1

First of all let us take an *asymmetric non-complete norm* ω from lemma 4.3. Associated to this functional ω, and for a fixed $\varepsilon > 0$, let us choose a function $f = f_\varepsilon$ from lemma 4.5. Assuming $f(x) = \delta > 0$ whenever $d_K(x) \geq \varepsilon$, select a path $p = p_\delta$ from lemma 4.4. With these choices, for every $x \in X \setminus K$, define

$$\psi(x) = x + p(f(x)).$$

We will prove that $\psi : X \setminus K \longrightarrow X$ is a C^p diffeomorphism. Let y be an arbitrary vector in X, and let $F_y : (0, \infty) \longrightarrow [0, \infty)$ be defined by $F_y(\alpha) = f(y - p(\alpha))$ for $\alpha > 0$. Let us see that $F_y(\alpha)$ satisfies the conditions of 4.2. We have

$$F_y(\beta) - F_y(\alpha) = f(y - p(\beta)) - f(y - p(\alpha)) \leq \omega((y - p(\beta)) - (y - p(\alpha)))$$
$$= \omega(p(\alpha) - p(\beta)) \leq \frac{1}{2}(\beta - \alpha)$$

for every $\beta \geq \alpha > 0$. Hence, the first condition of 4.2 is fulfilled. Let us check that F_y also satisfies the second condition. Since the set $y - K = \{y - z \mid z \in K\}$ is compact, condition 2 of lemma 4.4 gives us a $t_0 = t_0(K)$ such that

$$\inf\{\omega(y - z - p(t)) \mid 0 < t \leq t_0, z \in K\} > 0;$$

that is to say, there exists a number $\eta > 0$ such that

$$\omega(y - z - p(t)) \geq 2\eta > 0$$

for every $0 < t \leq t_0$ and $z \in K$. Obviously, we may suppose that $t_0 \leq \eta$. For each $t > 0$, choose $x_t \in K$ such that $d_K(y - p(t)) \geq \omega(y - p(t) - x_t) - t$. Then, for every t with $0 < t \leq t_0$, we have

$$d_K(y - p(t)) \geq \omega(y - x_t - p(t)) - t$$
$$\geq 2\eta - t \geq 2\eta - \eta = \eta > 0,$$

that is, $d_K(y - p(t)) \geq \eta$ for $0 < t \leq t_0$. Now recall that

$$\inf\{f(x) \mid d_K(x) \geq \eta\} > 0;$$

this means that there exists some $r > 0$ such that $f(x) \geq r$ whenever $d_K(x) \geq \eta$. Then, for every $0 < t \leq t_0$ we have $f(y - p(t)) \geq r > 0$ and therefore

$$\limsup_{t \to 0^+} F_y(t) = \limsup_{t \to 0^+} f(y - p(t)) \geq r > 0,$$

so that the second condition is also satisfied.

Hence, applying 4.2, the equation $F_y(\alpha) = \alpha$ has a unique solution. This means that for any $y \in X$, a number $\alpha(y) > 0$ with the property

$$f(y - p(\alpha(y))) = \alpha(y),$$

is uniquely determined. This implies that the mapping

$$\psi(x) = x + p(f(x))$$

is one-to-one from $X \setminus K$ onto X, whose inverse satisfies

$$\psi^{-1}(y) = y - p(\alpha(y)).$$

Indeed, if $\psi(x) = \psi(z) = y$ then $f(y - p(f(x))) = f(x)$ and also $f(y - p(f(z))) = f(z)$, so that $f(x) = f(z) = \alpha(y) > 0$ by the uniqueness of $\alpha(y)$, and therefore $x = y - p(\alpha(y)) = z$. Moreover, for each $y \in X$, since $\psi(y - p(\alpha(y))) = y - p(\alpha(y)) + p(f(y - p(\alpha(y)))) = y - p(\alpha(y)) + p(\alpha(y))$, the point $x = y - p(\alpha(y))$ satisfies $\psi(x) = y$, and also $x \in X \setminus K$ (because $f(x) = \alpha(y) > 0$ and $f^{-1}(0) = K$).

As f is C^p smooth on $X \setminus K$ and p is also C^p smooth, so is ψ. Let us define $\Phi : X \times (0, \infty) \longrightarrow \mathbb{R}$ by

$$\Phi(y, \alpha) = \alpha - f(y - p(\alpha)).$$

Since for any $y \in X$ we have $y - p(\alpha(y)) \notin K$, the mapping Φ is differentiable on a neighbourhood of each point $(y_0, \alpha(y_0))$ in $X \times (0, \infty)$. On the other hand, since $F_y(\beta) - F_y(\alpha) \leq \frac{1}{2}(\beta - \alpha)$ for $\beta \geq \alpha > 0$, it is clear that $F_y'(\alpha) \leq \frac{1}{2}$ for every α on a neighbourhood of $\alpha(y)$, and

$$\frac{\partial \Phi(y, \alpha)}{\partial \alpha} = 1 - F_y'(\alpha) \geq 1 - 1/2 > 0.$$

Thus, using the implicit function theorem we obtain that the mapping $y \to \alpha(y)$ is of class C^p and therefore $\psi : X \setminus K \longrightarrow X$ is a C^p diffeomorphism. Moreover, it is obvious that $\psi(x) = x$ whenever $d_K(x) \geq \varepsilon$. So, for every $\varepsilon > 0$ we have constructed a C^p diffeomorphism $\psi_\varepsilon : X \setminus K \longrightarrow X$ such that ψ_ε is the identity outside the set $\{x \in X \mid d_K(x) \leq \varepsilon\}$. This proves, in particular, the first part of theorem 4.1.

Now let us see that if K is contained in an open ball $B = \{x \in X \mid \|x\| < r\}$ then there exists a diffeomorphism $\varphi : X \longrightarrow X \setminus K$ such that φ is the identity outside

B. Choose a C^p diffeomorphism $G : X \longrightarrow X$ transforming $\{x \in X \mid \|x\| \leq r\}$ onto $\{x \in X \mid \omega(x) \leq r\}$ (such a diffeomorphism actually exists, according to lemma 1.6). Since $G(K)$ is a compact set contained in $\{x \in X \mid \omega(x) < r\}$, it is easy to see that there exists some $\varepsilon > 0$ such that $G(K)$ is contained in $\{x \in X \mid \omega(x) \leq r - 2\varepsilon\}$. Indeed, consider the tower of open sets $A_n = \{x \in X \mid \omega(x) < r - \frac{1}{n}\}$, $n = 1, 2, \ldots$, whose union is $\{x \in X \mid \omega(x) < r\}$. By the compactness of $G(K)$, there exists n_0 such that $G(K) \subset A_{n_0}$. It is enough to choose $\varepsilon > 0$ so that $2\varepsilon < 1/n_0$. For the compact set $G(K)$, we can pick a diffeomorphism $\psi_\varepsilon : X \setminus G(K) \longrightarrow X$ such that ψ_ε is the identity outside the set $\{x \in X \mid d_{G(K)}(x) \leq \varepsilon\}$. Note that, as $G(K)$ is contained in $\{x \in X \mid \omega(x) \leq r - 2\varepsilon\}$, the set $\{x \in X \mid d_{G(K)}(x) \leq \varepsilon\}$ is contained in $\{x \in X \mid \omega(x) \leq r\}$, so that ψ_ε is the identity outside the latter. Then it is quite clear that the function $\varphi_\varepsilon : X \longrightarrow X \setminus K$ defined by $\varphi_\varepsilon = G^{-1} \circ \psi_\varepsilon^{-1} \circ G$ is a C^p diffeomorphism between X and $X \setminus K$ satisfying $\varphi_\varepsilon(x) = x$ whenever $\|x\| \geq r$.

4.2 Smooth negligibility of subspaces and cylinders over compacta

The main results of this section show how to construct diffeomorphisms between an infinite-dimensional Banach space and the space minus one of its infinite codimensional subspaces, provided that this space has a smooth seminorm whose set of zeros is the subspace we want to delete. Recall that, for a real linear space X, a function $\varrho : X \longrightarrow [0, \infty)$ is said to be a seminorm in X provided ϱ satisfies the following properties:

(i) $\varrho(x + y) \leq \varrho(x) + \varrho(y)$ for every $x, y \in X$; and

(ii) $\varrho(\lambda x) = |\lambda|\varrho(x)$ for every $x \in X$ and every real number λ; in particular, $\varrho(0) = 0$.

Note that the set of zeros of such a function is always a linear space. Let ϱ be a seminorm in a linear space X, let $F = \varrho^{-1}(0)$ be its set of zeros, and consider the canonical projection $\pi : X \longrightarrow X/F$. It is clear that ϱ induces a norm, denoted by $\bar{\varrho}$, on the quotient space X/F such that $\varrho = \pi \circ \bar{\varrho}$. The seminorm ϱ is said to be non-complete provided that the normed space $(X/F, \bar{\varrho})$ is not complete. For a Banach space $(X, \|\cdot\|)$, a continuous seminorm $\varrho : X \longrightarrow [0, \infty)$ is said to be C^p smooth (resp. real-analytic) if it is so away from its set of zeros $F = \varrho^{-1}(0)$, which is a closed linear subspace of X in this case. By a ϱ-cylindrical set in X we will mean a subset $A \subseteq X$ such that $A = \pi^{-1}(\pi(A))$. The set A is said to be a ϱ-cylinder over a compact subset K of the quotient space X/F provided A is a ϱ-cylindrical set such that $\pi(A) = K$.

It is worth noting that the norm $\bar{\varrho}$ induced by a C^p smooth (resp. real-analytic) seminorm ϱ in the quotient space $X/\varrho^{-1}(0)$ is C^p smooth (resp. real-analytic) too. In fact this is true if only ϱ satisfies conditions (1) and (2) of lemma 4.3 and $\varrho^{-1}(0)$ is a linear subspace of X. We will use this fact in order to deduce a generalization of lemma 4.5 which we will need in the proof of the main result of this section. Let us note that a functional $\omega : X \longrightarrow [0,\infty)$ satisfying (1) and (2) of lemma 4.3 and such that $F = \omega^{-1}(0)$ is a linear subspace of X has also the following property: $\omega(x+z) = \omega(x)$ for every $z \in F$, $x \in X$, and hence ω induces a quotient functional $\bar{\omega} : X/F \longrightarrow [0,\infty)$ satisfying (1) and (2) of lemma 4.3, as well as $\bar{\omega}(\bar{x}) = 0$ if and only if $\bar{x} = \bar{0}$, and such that $\omega = \bar{\omega} \circ \pi$ (where $\pi : X \longrightarrow X/F$ is the canonical projection).

Lemma 4.6 *Let $\omega : X \longrightarrow [0,\infty)$ be a functional satisfying (1) and (2) of lemma 4.3 and such that $F = \omega^{-1}(0)$ is a linear subspace of X. Suppose that ω is C^n smooth (resp. real-analytic) on $X \setminus F$. Then, the induced quotient functional $\bar{\omega} : X/F \longrightarrow [0,\infty)$ is also C^n smooth (resp. real-analytic) on $(X/F) \setminus \{\bar{0}\}$.*

Proof. The points of X/F will be denoted by $\bar{x} = \pi(x)$, for some $x \in X$. Choose a point $x \in X \setminus F$. Define $P_k^x(y) = \frac{1}{k!} d^k\omega(x)(y)^k$ for every $y \in X$, $1 \leq k \leq n$. Taking into account that $\omega(x+z) = \omega(x)$ for every $z \in F$, we have

$$\lim_{\|h\|\to 0} \frac{\omega(x+z+h) - \omega(x+z) - \sum_{k=1}^n P_k^x(h)}{\|h\|^n}$$
$$= \lim_{\|h\|\to 0} \frac{\omega(x+h) - \omega(x) - \sum_{k=1}^n P_k^x(h)}{\|h\|^n} = 0$$

for every $z \in F$. By the uniqueness of Taylor's polynomial this implies that $k!P_k^x = d^k\omega(x) = d^k\omega(x+z) = k!P_k^{x+z}$ for every $z \in F$ and $1 \leq k \leq n$. Moreover,

$$d\omega(x)(h) = \lim_{t\to 0} \frac{\omega(x+th) - \omega(x)}{t} = \lim_{t\to 0} \frac{\omega(x+t(h+z)) - \omega(x)}{t} = d\omega(x)(h+z)$$

for every $z \in F$ and $h \in X$. Using induction in this fashion, one can easily see that $d^k\omega(x)(h+z)^k = d^k\omega(x)(h)^k$ for every $z \in F$ and $h \in X$, $k = 1, ..., n$. Then it is clear that each derivative $d^k\omega(x) = k!P_k^x$ induces a k-homogeneous polynomial $\bar{P}_k^x$ on X/F such that $P_k^x = \bar{P}_k^x \circ \pi$.

We will prove that $\bar{\omega}$ is C^n smooth, and $d^k\bar{\omega}(\bar{x})(\bar{y})^k = d^k\omega(x)(y)^k$ for every $x \in X \setminus F$, $y \in X$, $k = 1, ..., n$. Arguing by induction, let us suppose that for $0 \leq k \leq n-1$ we know that $\bar{\omega}$ is C^k smooth on $(X/F) \setminus \{\bar{0}\}$, with $d^k\bar{\omega}(\bar{x})(\bar{h})^k = d^k\omega(x)(h)^k$, and see that $\bar{\omega}$ is also C^{k+1} smooth, with $d^{k+1}\bar{\omega}(\bar{x})(\bar{y})^{k+1} = d^{k+1}\omega(x)(y)^{k+1}$. As it is customary, $d^0\omega(x)$ denotes $\omega(x)$. Let $\varepsilon > 0$. Since

$$\lim_{\|y\|\to 0} \frac{\|d^k\omega(x+y) - d^k\omega(x) - d^{k+1}\omega(x)(y)\|}{\|y\|} = 0,$$

there exists $\delta > 0$ such that if $\|y\| \leq 2\delta$ then $\|d^k\omega(x+y) - d^k\omega(x) - d^{k+1}\omega(x)(y)\| \leq \varepsilon\|y\|$. Suppose that $\|\bar{y}\| \leq \delta$. For each $m \in \mathbb{N}$ we have $\delta \geq \|\bar{y}\| = \inf\{\|y+z\| : z \in F\} = \inf\{\|y+z\| : z \in F, \|y+z\| \leq \|\bar{y}\| + 1/m\}$, so that we can choose $z_m \in F$

satisfying $\|y + z_m\| \leq \min\{2\delta, \|\bar{y}\| + 1/m\}$. Then,

$$\begin{aligned} &\|d^k\bar{\omega}(\bar{x}+\bar{y}) - d^k\bar{\omega}(\bar{x}) - d^{k+1}\omega(x)(y)\| \\ &= \|d^k\omega(x+z_m+y) - d^k\omega(x) - d^{k+1}\omega(x)(z_m+y)\| \\ &\leq \varepsilon\|z_m+y\| \leq \varepsilon(\|\bar{y}\| + \frac{1}{m}) \end{aligned}$$

for every m, and hence, by letting m go to ∞ we get

$$\|d^k\bar{\omega}(\bar{x}+\bar{y}) - d^k\bar{\omega}(\bar{x}) - d^{k+1}\omega(x)(y)\| \leq \varepsilon\|\bar{y}\|.$$

So, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|d^k\bar{\omega}(\bar{x}+\bar{y}) - d^k\bar{\omega}(\bar{x}) - d^{k+1}\omega(x)(y)\| \leq \varepsilon\|\bar{y}\|$$

whenever $\|\bar{y}\| \leq \delta$, which means that $d^k\bar{\omega}$ is differentiable at $\bar{x}$, with $d^{k+1}\bar{\omega}(\bar{x})(\bar{y}) = d^{k+1}\omega(x)(y)$. Moreover, using the continuity of $d^{k+1}\omega$ one can easily see that $\bar{x} \mapsto d^{k+1}\bar{\omega}(\bar{x})$ is also continuous. Indeed, fix a point $x \in X \setminus F$ and let $\varepsilon > 0$; since $d^{k+1}\omega$ is continuous at x, there exists $\delta > 0$ such that

$$\sup\{|d^{k+1}\omega(y)(h)^{k+1} - d^{k+1}\omega(x)(h)^{k+1}| : \|h\| \leq 2\} \leq \varepsilon$$

whenever $\|y - x\| \leq 2\delta$. Then, if $\|\bar{x} - \bar{y}\| \leq \delta$, we have, for some $z \in F$ such that $\|y + z - x\| \leq 2\delta$,

$$\begin{aligned} &\sup\{|d^{k+1}\bar{\omega}(\bar{y})(\bar{h})^{k+1} - d^{k+1}\bar{\omega}(\bar{x})(\bar{h})^{k+1}| : \|\bar{h}\| \leq 1\} \\ &\leq \sup\{|d^{k+1}\omega(y)(h+u)^{k+1} - d^{k+1}\omega(x)(h+u)^{k+1}| : \|h+u\| \leq 2, u \in F\} \\ &= \sup\{|d^{k+1}\omega(y+z)(h+u)^{k+1} - d^{k+1}\omega(x)(h+u)^{k+1}| : \|h+u\| \leq 2, u \in F\} \\ &\leq \varepsilon. \end{aligned}$$

Therefore $d^{k+1}\bar{\omega}$ is continuous on $(X/F) \setminus \{0\}$. So it is shown that $\bar{\omega}$ is of class C^n.

Finally, let us see that if ω is real-analytic then so is $\bar{\omega}$. Choose a point $x \in X \setminus F$, and put $P_n = P_n^x$ for every n. Since ω is real-analytic on $X \setminus F$, there exists $R > 0$ such that the series $\sum_{n=0}^{\infty} P_n(y)$ converges to $\omega(x+y)$ uniformly on $\{y \in X \mid \|y\| \leq 2R\}$. Now, it is easy to see that the series $\sum_{n=0}^{\infty} \bar{P}_n(y)$ converges to $\bar{\omega}(\bar{x}+\bar{y})$ uniformly on $\{\bar{y} \in X/F \mid \|\bar{y}\| \leq R\}$. Indeed, let $\varepsilon > 0$ and choose n_0 such that

$$|\sum_{k=0}^{n} P_k(y) - \omega(x+y)| \leq \varepsilon$$

whenever $n \geq n_0$ and $\|y\| \leq 2R$. Then, if $\|\bar{y}\| \leq R$ and $n \geq n_0$, since $R \geq \|\bar{y}\| = \inf\{\|y+z\| \mid z \in F\} = \inf\{\|y+z\| \mid z \in F, \|y+z\| \leq 2R\}$, we have, for some $z \in F$ so that $\|y + z\| \leq 2R$,

$$|\sum_{k=0}^{n} \bar{P}_k(\bar{y}) - \bar{\omega}(\bar{x}+\bar{y})| = |\sum_{k=0}^{n} P_k(y+z) - \omega(x+y+z)| \leq \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there exists n_0 such that $|\sum_{k=0}^{n} \bar{P}_k(\bar{y}) - \bar{\omega}(\bar{x}+\bar{y})| \leq \varepsilon$ whenever $n \geq n_0$ and $\|\bar{y}\| \leq R$. That is, $\sum_{n=0}^{\infty} \bar{P}_n(\bar{y}) = \bar{\omega}(\bar{x}+\bar{y})$ uniformly for every $\bar{y}$ in a neighbourhood of $\bar{0}$. Since this reasoning holds for every $x \in X \setminus F$, we have proved that the functional $\bar{\omega}$ is real-analytic on X/F.

As said before, we will need a generalization (for cylinders over compacta) of lemma 4.5.

Lemma 4.7 *Let ϱ be a continuous seminorm on X, and F be its set of zeros. Let ω be a functional satisfying 1 and 2 of lemma 4.3, and such that $F = \omega^{-1}(0)$. Suppose that ω is C^p smooth on $X \setminus F$. Let A be a ϱ-cylinder over a compact set $K \subset X/F$. For each $x \in X$, write $d_A(x) = \inf\{\omega(x-y) \mid y \in A\}$. Then, for each $\varepsilon > 0$ there exists a continuous function $f = f_\varepsilon : X \longrightarrow [0,\infty)$ such that*

1. *f is C^p smooth on $X \setminus A$;*
2. *$f^{-1}(0) = A$;*
3. *$f(x) - f(y) \leq \omega(x-y)$ for every $x, y \in X$;*
4. *$\inf\{f(x) \mid d_A(x) \geq \eta\} > 0$ for every $\eta > 0$; and*
5. *f is constant on the set $\{x \in X \mid d_A(x) \geq \varepsilon\}$.*

Proof: This result is an immediate consequence of lemmas 4.5 and 4.6 above. Indeed, the function f required in 4.7 is nothing else but $f(x) = \bar{f}(\pi(x))$, where $\bar{f}$ is a corresponding function given by lemma 4.5 for the space X/F and the functional $\bar{\omega}$, which is C^p smooth thanks to lemma 4.6.

Now we can state our main result of this section.

Theorem 4.8 *Let $(X, \|\cdot\|)$ be a Banach space with a C^p smooth seminorm ϱ whose set of zeros is a subspace F such that the quotient space X/F is infinite-dimensional. Let $A \subset X$ be a cylinder over a compact set $K \subset X/F$. Then there exists a C^p diffeomorphism φ between X and $X \setminus A$. Moreover, assuming that A is contained in an open cylinder $C = \{x \in X \mid \varrho(x) < r\}$, we can additionally require that φ be the identity outside C.*

Proof. First of all let us fix an $\varepsilon > 0$. Without loss of generality we can assume that A is contained in the *unit cylinder* $\{x \in X \mid \varrho(x) \leq 1\}$ and $0 \in A$. Consider the canonical projection $\pi : X \longrightarrow X/F$. Since ϱ is a C^p smooth seminorm in X with $\varrho^{-1}(0) = F$, it follows from lemma 4.6 that the induced norm $\bar{\varrho} : X/F \longrightarrow \mathbb{R}$ is C^p smooth on $(X/F) \setminus \{\bar{0}\}$. Then there exists a C^p smooth functional $\bar{\omega} : X/F \longrightarrow \mathbb{R}$ which satisfies properties 1–5 of lemma 4.3. Let us define $\omega : X \longrightarrow \mathbb{R}$ by $\omega = \bar{\omega} \circ \pi$. It is easy to check that the so-defined functional ω is C^p smooth on $X \setminus F$ and satisfies the following conditions:

1. $\omega(x+y) \leq \omega(x) + \omega(y)$, and $\omega(x) - \omega(y) \leq \omega(x-y)$, for every $x, y \in X$;

2. $\omega(rx) = r\omega(x)$ whenever $r \geq 0$;

3. $\omega^{-1}(0) = F$;

4. For every $\delta > 0$ there exists a sequence (y_k) of vectors of X satisfying $\omega(y_k) \leq \delta/4^{k+1}$ for every $k \in \mathbb{N}$, and with the property that for every compact set $K \subset X/F$ there exists $n_0 \in \mathbb{N}$ such that

$$\inf\{\omega(z - \sum_{k=1}^{n} y_k) \mid n \geq n_0, z \in \pi^{-1}(K)\} > 0.$$

For this functional ω and our fixed $\varepsilon > 0$, choose a function $f = f_\varepsilon$ from lemma 4.7. Assume $f(x) = \delta > 0$ whenever $d_A(x) \geq \varepsilon$. Now, by imitating the proof of lemma 4.4, it is easy to construct a path $p = p_\delta : (0, \infty) \longrightarrow X$ that satisfies the following properties (with respect to the above ω):

1. $\omega(p(\alpha) - p(\beta)) \leq \frac{1}{2}(\beta - \alpha)$ if $\beta \geq \alpha > 0$;

2. For every compact set $K' \subset X/F$ there exists $t_0 > 0$ such that

$$\inf\{\omega(z - p(t)) \mid 0 < t \leq t_0, z \in \pi^{-1}(K') = A\} > 0;$$

3. $p(t) = 0$ if and only if $t \geq \delta$.

Next, define $H(x) = x + p(f(x))$ for every $x \in X \setminus A$, and repeat a suitable argument from the proof of 4.1 to show that H is a required diffeomorphism. Namely, for each $y \in X$, consider the function $F_y : (0, \infty) \longrightarrow [0, \infty)$, $F_y(t) = f(y - p(t))$. Using the properties of ω and p, check that F_y satisfies the conditions of 4.2. Hence, there exists a unique $\alpha = \alpha(y) > 0$ such that $F_y(\alpha) = \alpha$. From this, by appealing to the implicit function theorem, deduce that H is a C^p diffeomorphism from $X \setminus A$ onto X, such that $H(x) = x$ whenever $d_A(x) \geq \varepsilon$. This proves, in particular, the first part of theorem 4.8.

Finally, using lemma 1.6 as at the end of the proof of theorem 4.1, one can show that if A is contained in an open cylinder $C = \{x \in X \mid \varrho(x) < r\}$ then there exists a C^p diffeomorphism φ between X and $X \setminus A$ such that φ is the identity outside C. This concludes the proof of 4.8.

In the case when $K = \{\bar{0}\}$, where $\bar{0}$ denotes the origin of X/F, theorem 4.8 reads that the linear space F is C^p negligible in X.

Corollary 4.9 *Let $(X, \|\cdot\|)$ be a Banach space with a C^p smooth seminorm ϱ whose set of zeros is a subspace F such that the quotient space X/F is infinite-dimensional. Then, for every $\varepsilon > 0$ there exists a C^p diffeomorphism φ between X and $X \setminus F$ satisfying $\varphi(x) = x$ whenever $\varrho(x) \geq \varepsilon$.*

4.3 Real-analytic negligibility of points and subspaces

Here we give a method of constructing real-analytic diffeomorphisms between an infinite-dimensional Banach space X and the space minus an infinite-codimensional subspace F, provided the space X has a real-analytic seminorm whose set of zeros is F. As a result, singletons are real-analytic negligible in every infinite-dimensional Banach space having a real-analytic norm. It should be noted that the class of Banach spaces having (not necessarily equivalent) real-analytic norms is large. For instance, it is easy to show that every Banach space which is linearly injectable into some $\ell_p(\Gamma)$ $(1 < p < \infty)$ has a (not necessarily equivalent) real-analytic norm. Taking into account that every superreflexive Banach space is linearly injectable into some $\ell_p(\Gamma)$ with $1 < p < \infty$ (see [53], proof of Lemma 2, p. 133) and the same is true for all separable spaces, it follows that all superreflexive spaces and all separable spaces have such norms.

In order to prove the main result of this section will need the following lemma.

Lemma 4.10 *Let $(X, \|\cdot\|)$ be a Banach space, and let (y_k) be a sequence of vectors such that $\|y_k\| \leq 1$ for all k. Consider the function $G : \mathbb{R} \longrightarrow \mathbb{R}$, $G(s) = \frac{1}{1+s^2}$, and define the path $p : (0, \infty) \longrightarrow X$ by*

$$p(t) = \sum_{k=1}^{\infty} G(2^{k-1}t) y_k.$$

Then p is a real-analytic function from $(0, \infty)$ to X.

Proof. Let Y be the complexification of the space X, and let $\Omega = \{z \in \mathbb{C} : \mathrm{Re} z > 2|\mathrm{Im} z|\}$. It is clear that Ω is an open subset of the complex plane containing the interval $(0, \infty)$ of the real line. Since the zeros of the complex function $z \longrightarrow \frac{1}{1+4^{k-1}z^2}$ are outside Ω for each $k \in \mathbb{N}$, every function

$$g_k(z) = \frac{y_k}{1 + 4^{k-1}z^2} = G(2^{k-1}z)$$

is holomorphic in Ω. Let us see that the series $\sum_{k=1}^{\infty} g_k$ converges uniformly on the compact subsets of Ω. Let $K \subset \Omega$ be a compact set. Then there exists $t_0 \in \mathbb{R}^+$ such that $K \subseteq \{z \in \Omega : \mathrm{Re} z \geq t_0\}$. For each $z \in K$ let us write $z = a + bi$. Then, taking into account that $a > 2|b|$ implies $a^2 - b^2 \geq \frac{3}{4}a^2$, we have

$$\left|\frac{1}{1 + 4^{k-1}z^2}\right| = \left|\frac{1}{1 + 4^{k-1}(a^2 + 2abi - b^2)}\right|$$
$$= \left|\frac{1}{\left([1 + 4^{k-1}(a^2 - b^2)]^2 + [4^{k-1}2ab]^2\right)^{1/2}}\right|$$
$$\leq \frac{1}{1 + 4^{k-1}(a^2 - b^2)} \leq \frac{1}{1 + 4^{k-1}\frac{3}{4}a^2} \leq \frac{1}{1 + 4^{k-1}\frac{3}{4}t_0^2},$$

and hence, for every $z \in K$,

$$\sum_{k=1}^{\infty} \|g_k(z)\| \leq \sum_{k=1}^{\infty} \left| \frac{1}{1+4^{k-1}z^2} \right|$$
$$\leq \sum_{k=1}^{\infty} \frac{1}{1+4^{k-1}\frac{3}{4}t_0^2} \leq \sum_{k=1}^{\infty} \frac{4}{4^{k-1}3t_0^2} = \frac{16}{9t_0^2}.$$

This means that $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on K, for every compact set $K \subset \Omega$. In particular the function

$$z \longrightarrow p(z) = \sum_{k=1}^{\infty} g_k(z)$$

is continuous in Ω. Moreover, for every $y^* \in Y^*$, the preceding inequalities show that the series

$$\sum_{k=1}^{\infty} \frac{y^*(y_k)}{1+4^{k-1}z^2}$$

converges uniformly on the compact subsets of Ω, and, since the functions

$$z \longrightarrow \frac{y^*(y_k)}{1+4^{k-1}z^2}$$

are all holomorphic, this implies that the functions

$$z \longrightarrow y^*(p(z)) = \sum_{k=1}^{\infty} \frac{y^*(y_k)}{1+4^{k-1}z^2}$$

are holomorphic in Ω. That is, $p : \Omega \longrightarrow Y$ is continuous and $y^* \circ p : \Omega \longrightarrow \mathbb{C}$ is holomorphic for every $y^* \in Y^*$. This implies that $p : \Omega \longrightarrow Y$ is also holomorphic. In particular, its restriction to the interval $(0, \infty)$ of the real line is real-analytic.

Proposition 4.11 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a real-analytic seminorm ϱ whose set of zeros is a subspace F such that the normed space $(X/F, \bar{\varrho})$ is non-reflexive. Then there exists a real-analytic diffeomorphism between X and $X \setminus F$.*

Proof. Let π, $\bar{\varrho}$ and F be as in the definitions preceding the statement of lemma 4.6. Since the normed space $(X/F, \bar{\varrho})$ is non-reflexive, according to James's theorem [52], there exists a linear functional $S : X/F \longrightarrow \mathbb{R}$ which is continuous from $(X/F, \bar{\varrho})$ onto $\mathbb{R}$ and such that S does not attain the supremum

$$\sup\{S(\bar{z}) \mid \bar{z} \in X/F, \bar{\varrho}(\bar{z}) = 1\} = 1.$$

It should be noted that the norm $\bar{\varrho}$ is continuous with respect to the usual quotient norm in X/F (recall that $\bar{\varrho}$ is real-analytic by virtue of lemma 4.6, and hence continuous in X/F). Therefore the linear functional S is also continuous from X/F (with its usual quotient norm) onto $\mathbb{R}$.

Next, put $T = S \circ \pi \in X^*$, and define the functionals $\bar{\omega} : X/F \longrightarrow [0, \infty)$ and $\omega : X \longrightarrow [0, \infty)$ by

$$\bar{\omega}(x) = \bar{\varrho}(\bar{x}) - S(\bar{x}), \quad \text{and} \quad \omega(x) = \varrho(x) - T(x) = \bar{\omega}(\pi(x)).$$

It is quite clear that the functionals $\bar{\omega}$ and ω are real-analytic on the sets $(X/F)\setminus\{\bar{0}\}$ and $X \setminus F$ respectively. We can select vectors (y_k) of X such that $\varrho(y_k) = 1$ and $\omega(y_k) \leq 1/4^k$ for every k. For $t > 0$, write $G(t) = 1/(1 + t^2)$ and consider the path

$$p(t) = \sum_{k=1}^{\infty} G(2^{k-1}t)y_k.$$

According to lemma 4.10, the path $p : (0, \infty) \longrightarrow X$ is real-analytic. Now, let us define $H : X \setminus F \longrightarrow X$ by

$$H(x) = x + p(w(x)).$$

We will check that for every $y \in X$ the function $F_y : (0, \infty) \longrightarrow [0, \infty)$ defined by $F_y(\alpha) = \omega(y - p(\alpha))$ satisfies the conditions of 4.2 and, therefore, has a unique fixed point. Note that G is a decreasing function satisfying $\sup\{|G'(t)| : t \in (0, \infty)\} \leq 2$. Then we have

$$\begin{aligned}
&F_y(\beta) - F_y(\alpha) = \omega(y - p(\beta)) - \omega(y - p(\alpha)) \leq \omega((y - p(\beta)) - (y - p(\alpha))) \\
&= \omega(p(\alpha) - p(\beta)) = \omega(\sum_{k=1}^{\infty}(G(2^{k-1}\alpha) - G(2^{k-1}\beta))y_k) \\
&\leq \sum_{k=1}^{\infty}(G(2^{k-1}\alpha) - G(2^{k-1}\beta))\omega(y_k) \leq \sum_{k=1}^{\infty} 2|2^{k-1}\alpha - 2^{k-1}\beta|\omega(y_k) \\
&= \sum_{k=1}^{\infty} 2^k\omega(y_k)|\beta - \alpha| \leq \sum_{k=1}^{\infty} 2^{k+1}\frac{1}{4^{k+1}}|\beta - \alpha| = \frac{1}{2}(\beta - \alpha)
\end{aligned}$$

for every $\beta \geq \alpha > 0$. So, the first condition of 4.2 is satisfied. Let us check that F_y also satisfies the second condition. Let $M > 0$ and choose $k_0 \in \mathbb{N}$ such that $\sum_{j=1}^{k_0} T(y_j) > 2(M + T(y))$ (this is clearly possible, as $T(y_k) \to 1$ when $k \to \infty$). Then, if $0 < \alpha < 1/2^{k_0}$, $G(2^{j-1}\alpha) \geq 1/2$ for $j = 1, 2, ..., k_0$, which implies

$$\begin{aligned}
&F_y(\alpha) = \omega(y - p(\alpha)) = \varrho(y - p(\alpha)) - T(y) + T(p(\alpha)) \\
&\geq -T(y) + T(p(\alpha)) = -T(y) + \sum_{k=1}^{\infty} G(2^{k-1}\alpha)T(y_k) \\
&\geq -T(y) + \sum_{j=1}^{k_0} G(2^{j-1}\alpha)T(y_j) \geq -T(y) + \sum_{j=1}^{k_0} \frac{1}{2}T(y_j) \\
&> -T(y) + M + T(y) = M
\end{aligned}$$

for every $\alpha > 0$ such that $\alpha < 1/2^{k_0}$. This proves that

$$\lim_{t\to 0^+} F_y(t) = +\infty,$$

and the second condition of 4.2 is satisfied as well. Therefore, for every $y \in X$, there exists a unique $\alpha = \alpha(y) > 0$ such that $F_y(\alpha) = \alpha$, and this means that the mapping

$$H(x) = x + p(\omega(x))$$

is one-to-one from $X \setminus F$ onto X, with

$$H^{-1}(y) = y - p(\alpha(y)).$$

Since the functions ω and p are real-analytic, so is H. Now, by appealing to the real-analytic version of the implicit function theorem, one can deduce, as in the proof of 4.1, that H is a real-analytic diffeomorphism from $X \setminus F$ onto X.

Proposition 4.12 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a real-analytic seminorm ϱ whose set of zeros is a subspace F such that the normed space $(X/F, \bar{\varrho})$ is infinite-dimensional and reflexive. Then there exists a real-analytic diffeomorphism between X and $X \setminus F$.*

Proof. Let us denote $Z = X/F$, and let $\pi : X \longrightarrow Z$ be the canonical projection. By lemma 3.1, the norm $\bar{\varrho} : Z \longrightarrow \mathbb{R}$ induced by a real-analytic seminorm ϱ satisfying $\varrho^{-1}(0) = F$ is also real-analytic on Z. In particular, the norm $\bar{\varrho}$ is continuous with respect to the usual quotient norm of $Z = X/F$. On the other hand, the norm $\bar{\varrho}$ is complete (indeed, the normed space $(X/F, \bar{\varrho})$ is reflexive and, hence, complete). Then, $\bar{\varrho}$ is an equivalent real-analytic norm on Z. Consequently, the space Z is C^∞ smooth and, since it is reflexive, theorem 4.1 in chapter V of [33] gives us a $2k$-homogeneous polynomial h on Z and constants $K, L > 0$ such that

$$K\bar{\varrho}(z)^{2k} \le h(z) \le L\bar{\varrho}(z)^{2k}$$

for every $z \in Z$; in particular, for such real-analytic $h : Z \to [0, \infty)$, we have $h^{-1}(0) = 0$. According to [34], Theorem 3, p. 149, every reflexive (in general, every WCG) Banach space has a separable infinite-dimensional complemented subspace. Then we can write $Z = W \times V$, where W is a separable infinite-dimensional subspace of Z. Since W is separable, W admits a non-complete norm g such that g^2 is real-analytic on the whole W (see [35], Proposition 4.1). For every $z = (u, v) \in Z = W \times V$, let us define

$$Q(z) = \sqrt{g(u)^2 + h(v)}.$$

It is clear that the function $Q : Z \to [0, \infty)$ is real-analytic on $Z \setminus \{0\}$ and satisfies $Q|_W = g$ and $Q^{-1}(0) = 0$. Since the norm g is non-complete we can find a $\bar{\varrho}$-bounded sequence (u_k) in W such that $g(u_k) \le \frac{1}{4^{k+1}}$ for each k, and a point u_0 in the completion of (W, g), denoted by $(\hat{W}, \hat{g})$, such that $u_0 \notin W$, and

$$\lim_n g(u_0 - \sum_{k=1}^n u_k) = 0.$$

Let us choose a bounded sequence (x_k) in X such that $\pi(x_k) = (u_k, 0)$ for every k, put $G(t) = 1/(1+t^2)$, and define a path $q : (0, \infty) \longrightarrow X$ by

$$q(t) = \sum_{k=1}^{\infty} G(2^{k-1}t)x_k$$

for $t > 0$. By lemma 4.1, the path q is real-analytic. Now define $H : X \setminus F \longrightarrow X$ by

$$H(x) = x + q(Q(\pi(x)))$$

for each $x \in X \setminus F$. Let us see that H is a bijection between $X \setminus F$ and X. For each $y \in X$ consider the function $F_y : (0, \infty) \longrightarrow [0, \infty)$, $F_y(t) = Q(\pi(y - q(t)))$, and check that F_y satisfies the conditions of 4.2. Writing $\pi(y) = (u, v) \in W \times V$ and noting that $\pi(q(t)) \in W \times \{0\} \simeq W$ for every $t > 0$, we have that

$$\begin{aligned}
&F_y(t) - F_y(s) = Q(\pi(y - q(t))) - Q(\pi(y - q(s))) \\
&= \sqrt{g(u - \pi(q(t)))^2 + h(v)} - \sqrt{g(u - \pi(q(s)))^2 + h(v)} \\
&\leq |g(u - \pi(q(t))) - g(u - \pi(q(s)))| \leq g(\pi(q(s) - q(t))) \\
&= g(\sum_{k=1}^{\infty}(G(2^{k-1}s) - G(2^{k-1}t))u_k) \leq \sum_{k=1}^{\infty} g((G(2^{k-1}s) - G(2^{k-1}t))u_k) \\
&= \sum_{k=1}^{\infty}(G(2^{k-1}s) - G(2^{k-1}t))g(u_k) \leq \sum_{k=1}^{\infty} 2|2^{k-1}s - 2^{k-1}t|g(u_k) \\
&= \sum_{k=1}^{\infty} 2^k g(u_k)|t - s| \leq \sum_{k=1}^{\infty} 2^{k+1}\frac{1}{4^{k+1}}|t - s| = \frac{1}{2}(t - s)
\end{aligned}$$

for every $t \geq s > 0$. On the other hand,

$$\begin{aligned}
&\limsup_{t \to 0^+} F_y(t) = \limsup_{t \to 0^+} Q(\pi(y - q(t))) \\
&= \limsup_{t \to 0^+} \sqrt{g(\pi(y - q(t)))^2 + h(v)} \\
&\geq \limsup_{t \to 0^+} g(\pi(y - q(t))) = \hat{g}(\pi(y) - u_0) > 0
\end{aligned}$$

because $u_0 \in \hat{W} \setminus W$ and $\pi(y) \in W$. So, by 4.2, for every $y \in X$ there exists a unique $\alpha = \alpha(y) > 0$ such that $F_y(\alpha) = \alpha$, and from this it follows that $H : X \setminus F \longrightarrow X$ is a bijection, with

$$H^{-1}(y) = y - q(\alpha(y)).$$

Indeed, if $H(x) = H(z) = y$ then $Q(\pi(y - q(Q(\pi(x))))) = Q(\pi(x))$ and also $Q(\pi(y - q(Q(\pi(z))))) = Q(\pi(z))$, so that $Q(\pi(x)) = Q(\pi(z)) = \alpha(y) > 0$ by the uniqueness of $\alpha(y)$, and therefore $x = y - q(\alpha(y)) = z$. Moreover, for each $y \in X$, since $Q(\pi(y - q(\alpha(y)))) = \alpha(y)$, the point $x = y - q(\alpha(y))$ satisfies $H(x) = y$, and also $x \in X \setminus F$ because $Q(\pi(x)) = \alpha(y) > 0$, $Q^{-1}(0) = 0$, and $\pi^{-1}(0) = F$.

Clearly, the function H is real-analytic on $X \setminus F$. Furthermore, by using the real-analytic version of the implicit function theorem, one can deduce, as in the proof of 4.1, that H is a real-analytic diffeomorphism from $X \setminus F$ onto X.

By combining the preceding results, we obtain the following

Theorem 4.13 *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a real-analytic seminorm ϱ whose set of zeros is a subspace F such that the quotient space X/F is infinite-dimensional. Then there exists a real-analytic diffeomorphism between X and $X \setminus F$.*

as well as

Corollary 4.14 *Let X be an infinite-dimensional Banach space with a (not necessarily equivalent) real-analytic norm. Then there exists a real-analytic diffeomorphism between X and $X \setminus \{0\}$.*

4.4 A characterization of convex negligibility of points

Let X be an infinite-dimensional Banach space with a (not necessarily equivalent) C^p smooth norm ϱ. Let us briefly give the gist of the negligibility technique that we used throughout this chapter. If X is non-reflexive, find a continuous linear functional $T \in X^*$ with $\sup_{\varrho(x)=1} T(x) = 1$ such that T does not attain this supremum, and put $\omega(x) = \varrho(x) - T(x)$ for every $x \in X$. If X is reflexive, there exists a C^p smooth non-complete norm ω in X. In either case we construct a deleting path $p : (0, \infty) \longrightarrow X$ with the properties $p(t) = 0$ if and only if $t \geq 1$, $\omega(p(\alpha) - p(\beta)) \leq \frac{1}{2}(\beta - \alpha)$ if $\beta \geq \alpha > 0$, and $\limsup_{t \to 0^+} \omega(y - p(t)) > 0$ for every $y \in X$. We define $H : X \setminus \{0\} \longrightarrow X$ by

$$H(x) = x + p(\omega(x)).$$

The mapping $\psi = H^{-1}$ is a C^p diffeomorphism between X and $X \setminus \{0\}$ whose support is the C^p smooth convex body $U = \{x \in X \mid \omega(x) \leq 1\}$, which satisfies $ccU = \{0\}$. Recall that the support of a mapping $\psi : X \longrightarrow X$ is defined as $\text{supp}\psi = \overline{X \setminus \text{Fix}\psi}$, where $\text{Fix}\psi = \{x \in X \mid \psi(x) = x\}$.

On the other hand, it is clear that the existence in X of a C^p smooth convex body U satisfying $ccU = \{0\}$ implies the existence of a (not necessarily equivalent) C^p smooth norm ϱ in X (it is enough to take the Minkowski functional q_U of U, and define $\varrho(x) = q_U(x) + q_U(-x)$). Therefore, for a Banach space X, the following statements are equivalent.

(a) X has a (not necessarily equivalent) C^p smooth norm; and

(b) there exists a C^p diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ whose support is a C^p smooth convex body containing no rays.

Furthermore, these statements remain equivalent if one changes the words "containing no rays..." and "not necessarily equivalent" for "bounded" and "equivalent", respectively. That is, for a Banach space X, the following are equivalent.

(i) X has an equivalent C^p smooth norm; and

(ii) there exists a C^p diffeomorphism $\varphi : X \longrightarrow X\setminus\{0\}$ whose support is a bounded C^p smooth convex body.

In the latter case, we will add another equivalent condition which is related to the failure of Rolle's theorem in infinite-dimensional Banach spaces. Recall that, as we showed in chapter 2, Rolle's theorem fails for those smooth infinite-dimensional Banach spaces which can be injected in a Banach space having a differentiable norm, although an approximate Rolle's theorem remains true for all Banach spaces. Next we show that, in fact, this failure can be viewed as a third equivalent condition to (i) and (ii) above.

Proposition 4.15 *For an infinite-dimensional Banach space X, the following statements are equivalent:*

(i) X has an equivalent C^p smooth norm;

(ii) there exists a C^p diffeomorphism $\varphi : X \longrightarrow X\setminus\{0\}$ whose support is a bounded C^p smooth convex body; and

(iii) there exists a C^p smooth function $f : X \longrightarrow [0,\infty)$ such that the set $U = f^{-1}([0,1])$ is convex and bounded, whose boundary is $f^{-1}(1)$, and yet $f'(x) \neq 0$ for every $x \in X$. In particular, Rolle's theorem fails for the space X.

Proof. We already know that (i) and (ii) are equivalent, and it is clear that (i) follows from (iii) (indeed, consider the Minkowski functional $q_U(x) = \inf\{\lambda > 0 : x \in \lambda U\}$ of U, and define $\|x\| = q_U(x) + q_U(-x)$; a standard use of the implicit function theorem shows that the conditions on U and f imply that the functional q_U is C^p smooth, and therefore so is the equivalent norm $\|\cdot\|$). Let us see that (i) implies (iii). Let $\|\cdot\|$ be an equivalent C^p smooth norm on X. From theorem 4.1 we can choose a diffeomorphism $\varphi : X \longrightarrow X\setminus\{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1/2$. Then define $f : X \longrightarrow [0,\infty)$ by $f(x) = \|\varphi(x)\|$ for every $x \in X$. It is easy to see that $f'(x) \neq 0$ for every $x \in X$, and the set $f^{-1}([0,1]) = \{x \in X \mid \|x\| \leq 1\}$ is obviously bounded and convex.

4.5 Deleting isotopies in Banach manifolds

In the remaining part of this chapter we will be involved in the task of extending some results on smooth deleting isotopies obtained for Hilbert spaces by D. Burghelea and N. H. Kuiper in [16] to a larger class of spaces, namely, that of all Banach spaces having equivalent Fréchet differentiable norms.

Before stating those results we will recall some topological concepts. A *homotopy* of a topological space $\mathcal{X}$ into a topological space $\mathcal{Y}$ is a continuous map $G : \mathcal{X}\times I \longrightarrow \mathcal{Y}$, where $I = [0,1]$. The homotopy G determines a parametric family of maps $(g_t)_{t\in I}$ defined by

$$g_t(x) = G(x,t)$$

for $x \in \mathcal{X}$, $t \in I$. We say that the homotopy G connects the map g_0 with the map g_1. We will often identify the homotopy G with the family $(g_t)_{t \in I}$. Two maps $f, g : \mathcal{X} \longrightarrow \mathcal{Y}$ are said to be *homotopic* if there exists a homotopy of $\mathcal{X}$ into $\mathcal{Y}$ which connects f with g. By an (invertible) *isotopy* on a topological space $\mathcal{X}$ we mean a homotopy $G : \mathcal{X} \times I \longrightarrow \mathcal{X}$ such that the map $\overline{G} : \mathcal{X} \times I \longrightarrow \mathcal{X} \times I$ defined by

$$\overline{G}(x, t) = (G(x, t), t)$$

is a self-homeomorphism of $\mathcal{X} \times I$.

If $\mathcal{M}$ is a manifold of class C^1, modelled on a Banach space X, and K is a subset of $\mathcal{M}$, then, by a *smooth isotopy deleting K from $\mathcal{M}$* we mean a mapping $G : \mathcal{M} \times I \longrightarrow \mathcal{M}$ such that G satisfies the following conditions:

(i) The map $g_0(x) = G(x, 0)$ is the identity in $\mathcal{M}$.

(ii) For every $t \in (0, 1)$, the map $g_t(x) = G(x, t)$ is a self-diffeomorphism of $\mathcal{M}$.

(iii) For $t = 1$, the map $g_1(x) = G(x, 1)$ is a diffeomorphism from $\mathcal{M}$ onto $\mathcal{M} \setminus K$.

(iv) The map H defined by $H(x, t) = (G(x, t), t)$ is a C^1 diffeomorphism from $\mathcal{M} \times I$ onto $(\mathcal{M} \times I) \setminus (K \times \{1\})$.

If U is an open neighbourhood of the set K, we say that the isotopy G has support in U provided that $G(x, t) = x$ whenever $x \in \mathcal{M} \setminus U$, $t \in I$.

Let us state our main results.

Theorem 4.16 *Let X be an infinite-dimensional Banach space having a Fréchet differentiable equivalent norm $\|\cdot\|$. For every $x_0 \in U \subset X$, where U is an open neighbourhood of x_0, there exists a C^1 smooth isotopy deleting x_0 from X with support in U.*

Theorem 4.17 *Let X be an infinite-dimensional Banach space having a Fréchet differentiable equivalent norm $\|\cdot\|$.*

(1) If $\mathcal{M}$ is a Banach manifold of class C^1 modelled on the space X, and U is an open neighbourhood of a point $x_0 \in \mathcal{M}$, then there exists a smooth isotopy deleting x_0 from $\mathcal{M}$ with support in U.

(2) If $\mathcal{M}$ is a Banach manifold of class C^1 with boundary $\mathcal{N} = \partial\mathcal{M}$, modelled on the space X, and V is an open neighbourhood of a point x_0 in $\partial\mathcal{M}$, then there exists a diffeomorphism from the pair $(\mathcal{M}, \partial\mathcal{M})$ onto $(\mathcal{M} \setminus \{x_0\}, \partial\mathcal{M} \setminus \{x_0\})$, with support in V.

Proof of theorem 4.16:
We may assume that $x_0 = 0$. From lemmas 1.4 and 1.5, pick a C^1 smooth *non-complete asymmetric norm* ω and a deleting path p associated to this ω and satisfying $p(t) = 0$ whenever $t \geq 1/\sqrt{2}$. Let us define $\Psi : (X \times I) \setminus \{(0, 1)\} \longrightarrow X$ by

$$\Psi(x, t) = x + p(f_t(x)),$$

where $f_t(x) = \big(t^2\omega(x)^2 + (1-t)^2\big)^{1/2}$, and put $\varphi_t(x) = \Psi(x,t)$. First of all let us see that, for $0 \leq t < 1$, φ_t is a diffeomorphism from X onto X while, for $t = 1$, φ_1 is a diffeomorphism from $X \setminus \{0\}$ onto X.

Let (y,t) be an arbitrary point of $X \times I$, and let $F = F_{(y,t)} : (0,\infty) \longrightarrow [0,\infty)$ be defined by $F(\alpha) = f_t(y - p(\alpha))$ for $\alpha > 0$. Let us see that $F_{(y,t)}(\alpha)$ satisfies the conditions of lemma 1.3. In order to check that F satisfies the first condition, since $F(\beta) - F(\alpha) \leq \frac{1}{2}(\beta - \alpha)$ trivially holds for $\beta \geq \alpha > 0$ when $F(\beta) - F(\alpha) < 0$, we may assume that $F(\beta) - F(\alpha) \geq 0$. This implies that $\omega(y - p(\beta)) \geq \omega(y - p(\alpha))$. Then, taking into account the properties of ω and p listed in lemmas 1.4 and 1.5, we may deduce

$$\begin{aligned}
&F(\beta) - F(\alpha) = f_t(y - p(\beta)) - f_t(y - p(\alpha)) \\
&= \big(t^2\omega(y - p(\beta))^2 + (1-t)^2\big)^{1/2} - \big(t^2\omega(y - p(\alpha))^2 + (1-t)^2\big)^{1/2} \\
&\leq |t\omega(y - p(\beta)) - t\omega(y - p(\alpha))| = t\big(\omega(y - p(\beta)) - \omega(y - p(\alpha))\big) \\
&\leq t\omega(p(\alpha) - p(\beta)) \leq \omega(p(\alpha) - p(\beta)) \leq \frac{1}{2}(\beta - \alpha)
\end{aligned}$$

for all $\beta \geq \alpha > 0$. Hence the first condition of lemma 1.3 is satisfied. On the other hand, for $t > 0$ we have

$$\begin{aligned}
&\limsup_{\alpha \to 0^+} F_{(y,t)}(\alpha) = \limsup_{\alpha \to 0^+} \big(t^2\omega(y - p(\alpha))^2 + (1-t)^2\big)^{1/2} \\
&\geq t \limsup_{\alpha \to 0^+} \omega(y - p(\alpha)) > 0,
\end{aligned}$$

while for $t = 0$ we have $F_{(y,0)}(\alpha) \equiv 1$. In either case, F satisfies the second condition of lemma 1.3.

Now, by lemma 1.3, the equation $F_{(y,t)}(\alpha) = \alpha$ has a unique solution. This means that for each $(y,t) \in X \times I$, a number $\alpha(y,t) > 0$ with the property

$$f_t(y - p(\alpha(y,t))) = \alpha(y,t),$$

is uniquely determined. This implies that, for $0 \leq t < 1$, the mapping $\varphi_t(x) = x + p(f_t(x))$ is a bijection from X onto X, while, for $t = 1$, $\varphi_1(x) = x + p(\omega(x))$ defines a bijection from $X \setminus \{0\}$ onto X. In either case, the inverse of φ_t satisfies

$$\varphi_t^{-1}(y) = y - p(\alpha(y,t)).$$

Indeed, if $\varphi_t(x) = \varphi_t(z) = y$ then $f_t(y - p(f_t(x))) = f_t(x)$ and also $f_t(y - p(f_t(z))) = f_t(z)$, so that $f_t(x) = f_t(z) = \alpha(y,t) > 0$ by the uniqueness of $\alpha(y,t)$, and therefore $x = y - p(\alpha(y,t)) = z$. Moreover, for each $y \in X$, since $\varphi_t(y - p(\alpha(y,t))) = y - p(\alpha(y,t)) + p(f_t(y - p(\alpha(y,t)))) = y - p(\alpha(y,t)) + p(\alpha(y,t))$, the point $x = y - p(\alpha(y,t))$ satisfies $\varphi_t(x) = y$. Note that this point x is in X when $t < 1$, while for $t = 1$ we have $x \in X \setminus \{0\}$ (because $f_1(x) = \alpha(y,1) > 0$ and $f_1^{-1}(0) = \omega^{-1}(0) = \{0\}$).

It is plain that Ψ and φ_t are C^1 smooth, for each t. Now we see that the mapping $(y,t) \to \alpha(y,t)$ is of class C^1. Let us define $\Phi : X \times I \times (0,\infty) \longrightarrow \mathbb{R}$ by

$$\Phi(y,t,\alpha) = \alpha - \big[t^2\omega(y - p(\alpha))^2 + (1-t)^2\big]^{1/2} = \alpha - F_{(y,t)}(\alpha).$$

It is clear that $t^2\omega(y-p(\alpha))^2+(1-t)^2>0$ on a neighbourhood of any point $(z,s,\alpha(z,s))$ in $X\times I\times(0,\infty)$. Then the mapping Φ is differentiable on a neighbourhood of any point $(y,t,\alpha(y,t))$ in $X\times I\times(0,\infty)$. On the other hand, since $F_{(y,t)}(\beta)-F_{(y,t)}(\alpha)\leq\frac{1}{2}(\beta-\alpha)$ for $\beta\geq\alpha>0$, it is clear that $F'_{(y,t)}(\alpha)\leq\frac{1}{2}$ for every α in a neighbourhood of $\alpha(y,t)$, and

$$\frac{\partial\Phi(y,t,\alpha)}{\partial\alpha}=1-F'_{(y,t)}(\alpha)\geq 1-1/2>0.$$

Thus, using the implicit function theorem we obtain that the function $(y,t)\mapsto\alpha(y,t)$ is of class C^1. In particular φ_t^{-1} is also C^1 for all $t\in I$.

So far we have proved that the mappings φ_t are C^1 self-diffeomorphisms of X for $0\leq t<1$, and φ_1 is a C^1 diffeomorphism from $X\setminus\{0\}$ onto X. It should be noted that, for every $t\in I$, $\varphi_t(x)=x$ whenever $\omega(x)\geq 1$. Moreover, φ_0 is the identity on X. Now, fix an $\varepsilon>0$ such that $\{x\in X\mid \|x\|\leq\varepsilon\}\subseteq U$, and, from lemma 1.6, choose a C^1 self-diffeomorphism $h: X\longrightarrow X$ transforming $\{x\in X\mid\|x\|\leq\varepsilon\}$ onto $\{x\in X\mid\omega(x)\leq 1\}$ and preserving the rays emanating from the origin. Define $g_t=h^{-1}\circ\varphi_t^{-1}\circ h$. It is clear that g_t is a self-diffeomorphism of X for $0\leq t<1$, and g_1 is a diffeomorphism from X onto $X\setminus\{0\}$. Moreover, $g_t(x)=x$ whenever $x\in X\setminus U$, for all $t\in I$; and $g_0(x)=x$ for all $x\in X$.

Finally, define $G: X\times I\longrightarrow X$ by $G(y,t)=g_t(y)$. Taking into account that the mappings p, h, $(y,t)\longrightarrow\alpha(y,t)$, and $(x,t)\longrightarrow f_t(x)$ are of class C^1, it is quite clear that the map H defined by $H(x,t)=(G(x,t),t)$ is a C^1 diffeomorphism from $X\times I$ onto $(X\times I)\setminus\{(0,1)\}$. Therefore G is the desired C^1 isotopy deleting 0 from X.

Proof of theorem 4.17:

(1) We can take a chart mapping x_0 into $0\in X$ and such that the unit ball in X is covered by the image of U. Then we can transport an isotopy from theorem 4.16, which deletes 0 from X and has its support in the unit ball, to our manifold $\mathcal{M}$, and we can extend it by the constant identity map in all the remaining points of $\mathcal{M}\times I$. In this way we get an isotopy deleting x_0 from $\mathcal{M}$, with support in U.

(2) We may consider (and identify) a collar

$$\mathcal{N}\times I\subset\mathcal{M}$$

of the boundary $\mathcal{N}\times\{1\}=\partial\mathcal{M}$, so thin near $x_0\in\partial\mathcal{M}\subset\mathcal{M}$ that there exists an open set $U\subset\mathcal{N}$ with

$$x_0\in U\times\{1\}\subset U\times I\subset V\subset\mathcal{M}.$$

Then we apply an isotopy from theorem 4.16, but we consider it as a diffeomorphism

$$\mathcal{N}\times I\longrightarrow(\mathcal{N}\times I)\setminus\{(x_0,1)\},$$

and we extend it by the identity outside the collar. The diffeomorphism $\mathcal{M}\longrightarrow\mathcal{M}\setminus\{x_0\}$ obtained in this way fulfills the requirements of the statement.

It should be noted that theorem 4.16 can be generalized so as to delete isotopically compact sets in infinite-dimensional Banach spaces with Fréchet differentiable norms. Indeed, let K be a compact subset of X. From lemma 1.4 pick a C^1 smooth *non-complete asymmetric norm* ω and, associated to this ω, get a function f from lemma 4.5 satisfying

1. f is C^p smooth on $X \setminus K$, and Lipschitz continuous on X;
2. $f(x) - f(y) \leq \omega(x - y)$ for every $x, y \in X$;
3. $f^{-1}(0) = K$;
4. $\inf\{f(x) \mid d_K(x) \geq \eta\} > 0$ for every $\eta > 0$;
5. f is constant on the set $\{x \in X \mid d_K(x) \geq \varepsilon\}$.

Finally, assuming that $f(x) = \delta > 0$ whenever $d_K(x) \geq \varepsilon$, select a path $p = p_\delta$ from lemma 4.4. With these choices define $\Psi : (X \times I) \setminus \{(0, 1)\} \longrightarrow X$ by

$$\Psi(x, t) = x + p(f_t(x)),$$

where $f_t(x) = \left(t^2 f(x)^2 + (1-t)^2\right)^{1/2}$, and put $\varphi_t(x) = \Psi(x, t)$.

It is easy to check that for every Lipschitz function f which is 0 on a compact set K and is of class C^1 outside K, the function f^2 is of class C^1 on the whole of X. That is why the map $(x, t) \mapsto f_t(x)$ is of class C^1, and hence so is Ψ. By combining the ideas of the proofs of 4.1 and 4.16 one can easily show that, for $0 \leq t < 1$, φ_t is a diffeomorphism from X onto X, while, for $t = 1$, φ_1 is a diffeomorphism from $X \setminus K$ onto X. By imitating the final part of the proof of 4.16 one can then conclude the following.

Theorem 4.18 *Let X be an infinite-dimensional Banach space having a Fréchet differentiable equivalent norm $\|\cdot\|$, and let K be a compact subset of X. Then, for every ball U containing K, there exists a C^1 smooth isotopy deleting K from X, with support in U.*

We will finish this chapter with a remark on the possibility of proving theorems 4.17 and 4.16 for higher orders of smoothness. Assume that our space X is C^m smooth, and suppose that we have a *non-complete (perhaps asymmetric) norm* $\omega : X \longrightarrow [0, \infty)$ such that $\omega(\cdot)^p$ is C^m smooth on the whole of X for some $p > 1$. Then, by changing f_t for

$$f_t(x) = \left(t^p \omega(x)^p + (1-t)^p\right)^{1/p}$$

in the proof of 4.16, we obtain a C^m smooth isotopy deleting x_0 from X. It is easy to see that, if X is either a separable Banach space or a superreflexive Banach space, and X is infinite-dimensional, then X has a non-complete norm ω such that $\omega(\cdot)^p$ is of the same class of smoothness as X *on the whole space* X, for some $p > 0$. Indeed, for any superreflexive space X, according to [53] (proof of Lemma 2, p. 133) there exists a linear injection of X into some $\ell_q(\Gamma)$ which, in its turn, is a dense subspace of some $\ell_{2n}(\Gamma)$. Hence, there exists a linear injection $T : X \longrightarrow \ell_{2n}(\Gamma)$ so that $T(X)$ is a dense non-closed subspace of $\ell_{2n}(\Gamma)$. On the other hand, it is clear that every separable infinite-dimensional Banach space admits a dense non-closed linear injection into ℓ_2. In either case, by taking the standard C^∞ equivalent norm $\|\cdot\|$ of $\ell_{2n}(\Gamma)$ or ℓ_2 and defining $\omega(x) = \|T(x)\|$ for all $x \in X$, we obtain a C^∞ non-complete norm ω in X. Thus, we may state the following result, its proof being almost the same as those of 4.16 and 4.17.

Theorem 4.19 *Let X be either an infinite-dimensional separable Banach space or an infinite-dimensional superreflexive Banach space. Assume that the space X has a C^m smooth bump function.*

(1) If $x_0 \in U \subset X$, where U is an open neighbourhood of x_0, then there exists a C^m smooth isotopy deleting x_0 from X with support in U.

(2) If $\mathcal{M}$ is a Banach manifold of class C^m modelled on the space X, and U is an open neighbourhood of a point $x_0 \in \mathcal{M}$, then there exists a C^m smooth isotopy deleting x_0 from $\mathcal{M}$ with support in U.

(3) If $\mathcal{M}$ is a Banach manifold of class C^m with boundary $\mathcal{N} = \partial\mathcal{M}$, modelled on the space X, and V is an open neighbourhood of a point x_0 in $\partial\mathcal{M}$, then there exists a C^m diffeomorphism from the pair $(\mathcal{M}, \partial\mathcal{M})$ onto $(\mathcal{M}\setminus\{x_0\}, \partial\mathcal{M}\setminus\{x_0\})$, with support in V.

Chapter 5

Classification of convex bodies and starlike bodies in Banach spaces

In this chapter we give a complete classification of the smooth convex bodies of every Banach space. In particular we see that every smooth convex body whose characteristic cone is a subspace of infinite codimension is relatively diffeomorphic to a half-space. As a consequence, such smooth convex bodies are smoothly negligible. We also give a partial classification of the smooth starlike bodies of every WCG Banach space.

5.1 Classification of smooth convex bodies

Making use of his pioneering results on negligibility, V. L. Klee [56] gave a topological classification of the convex bodies of a Hilbert space. This result was generalized to every Banach space with the help of Bessaga's non-complete norm technique (see the book by Bessaga and Pelczynski [12]). To get a better insight in the history of the topological classification of convex bodies the reader should also look at the papers by Stocker [65], Corson and Klee [18], and Bessaga and Klee [10, 11]. In [37], T. Dobrowolski gave a C^p smooth version of that result which held within the class of WCG Banach spaces. The results of chapter 4 enable us to eliminate this restriction, yielding a general result on the smooth classification of the smooth convex bodies of every Banach space.

Recall that a convex body U in a Banach space X is said to be a C^p body if U is a C^p submanifold with one-codimensional boundary ∂U. If U_1, U_2 are C^p convex bodies in a Banach space X, we say that U_1 and U_2 are C^p relatively diffeomorphic provided there exists a C^p diffeomorphism $\varphi : X \longrightarrow X$ such that $\varphi(U_1) = U_2$. Given a C^p convex body U in X we can always assume without loss of generality that $0 \in \text{int}U$, and define the characteristic cone of U by $ccU = \{x \in X \mid rx \in U \text{ for all } r > 0\}$. For the definition of the Minkowski functional of a convex body and further elementary properties of convex bodies, we refer to the text preceding lemma 1.6 on page 23.

Theorem 5.1 *Let U be a C^p convex body in a Banach space X.*

(a) If ccU is a linear subspace of finite codimension (say $X = ccU \oplus Z$, with Z finite-dimensional), then U is C^p relatively diffeomorphic to $ccU + \{z \in Z : |z| \leq 1\}$, where $|\cdot|$ is an Euclidean norm in Z.

(b) If ccU is not a linear subspace or ccU is a linear subspace such that the quotient space X/ccU is infinite-dimensional, then U is C^p relatively diffeomorphic to a closed half-space (that is, $\{x \in X \mid x^*(x) \geq 0\}$, for some $x^* \in X^*$).

Proof: We will follow the argument in [37], making the appropriate changes.
(a) It is enough to apply lemma 1.6 for $U_1 = U$ and $U_2 = ccU + \{z \in Z : |z| \leq 1\}$.

(b) We will consider two cases.

Case I: ccU is not a linear space.

Recall that $0 \in \text{int}U$. We may proceed as follows. Pick $y \in \text{int}U$ such that $-y \in \partial U$ and $\{ry \mid r \in \mathbb{R}^+\} \subset U$. Let f be a supporting functional for U at $-y$, say $f(-y) = -1$, and let $X_1 = \text{Ker}f$, so that $-y + X_1$ is the supporting hyperplane of U at $-y$. We have that $U \subset [-1, \infty) \times X_1$, $(-1, 0) \in U$ and $(-1, \infty) \times \{0\} \subset \text{int}U$. Now take a function $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ of class C^∞ with the following properties:

(i) $\gamma(t) = 0$ if t belongs to some neighbourhood of 0;

(ii) $\lim_{t\to\infty} \gamma(t) = \infty$; and

(iii) $0 \leq \gamma'(t) \leq \frac{\gamma(t)+1}{t}$ for $t > 0$.

We will also need a non-decreasing real function λ of class C^∞, defined for $t > 0$, such that $\lambda(t) = 0$ for $t \leq 1/2$ and $\lambda(t) = 1$ for $t \geq 1$. Let us define $q_1 : U \times X \longrightarrow [0, \infty)$ by

$$q_1(u, x) = \inf\{\lambda > 0 \mid u + \frac{1}{\lambda}(x - u) \in U\},$$

and define also $q_2 : U \times X \longrightarrow [0, \infty)$ by

$$q_2(u, x) = \inf\{\lambda > 0 \mid u + \frac{1}{\lambda}(x - u) \in [-1, \infty) \times X_1\}.$$

Note that $q_1(u, \cdot)$ and $q_2(u, \cdot)$ are the Minkowski functionals of the sets U and $[-1, \infty) \times X_1$ with respect to the point $u \in U$. Then put

$$u(x) = (\gamma(q_1(0, x)), 0).$$

Making use of the above listed properties of γ and the implicit function theorem, it is easy to check that the map

$$\Phi(s, z) = u(z) + s\big((-1, z) - u(z)\big)$$

defines a C^p diffeomorphism between $(0,\infty)\times X_1$ and $V = \big(\mathbb{R}\times X_1\big)\setminus\big([0,\infty)\times\{0\}\big)$. Let $\pi_1 : X \longrightarrow X_1$ be the projection $\pi_1(s,y) = y$ and, for each $x \in V$, put $z = \pi_1\circ\Phi^{-1}(x)$. Next define

$$H(x) = u(z) + \big[\lambda\circ q_1(u(z),x)\frac{q_1(u(z),x)}{q_2(u(z),x)} + 1 - \lambda\circ q_1(u(z),x)\big]\big(x - u(z)\big)$$

for $x \in V$, and $H(x) = x$ for $x \in X\setminus V$. It is not difficult to see that H is a C^p self-diffeomorphism of X such that $H(U) = [-1,\infty)\times X_1$.

Case II: ccU is a linear subspace such that the quotient space X/ccU is infinite dimensional.

Let us put $F = ccU$. Again we may write $X = \mathbb{R}\times X_1$, where X_1 is one-codimensional, and $F = ccU \subset \{0\}\times X_1$. Let $\pi_1 : X \longrightarrow X_1$ be the projection $\pi_1(s,z) = z$. Let $q = q_{\pi_1(U)} : X_1 \longrightarrow [0,\infty)$ be the Minkowski functional of the set $\pi_1(U)$, and put

$$\varrho(x) = q_{\pi_1(U)}(x) + q_{\pi_1(U)}(-x)$$

for every $x \in X_1$. Then ϱ is a seminorm on X_1 such that $\varrho^{-1}(0) = F$ and ϱ is C^p smooth on $X_1\setminus F$; we identify F with its projection into X_1. Let us define the C^p convex bodies

$$V = \{(t,x)\in\mathbb{R}\times X_1 \mid t^2+\varrho^2(x)\le 1\},\quad\text{and}\quad V_1 = \{x\in X_1 \mid \varrho(x)\le 1\}.$$

We will construct the following C^p diffeomorphisms:

(1) $H_1 : \mathbb{R}\times X_1\setminus F \longrightarrow \mathbb{R}\times X_1$, with $H_1(V\setminus F) = V$;

(2) $H_2 : \mathbb{R}\times X_1\setminus F \longrightarrow \mathbb{R}\times\partial V$, with $H_2(V\setminus F) = [0,\infty)\times\partial V$; and

(3) $H_3 : \partial V \longrightarrow X_1$.

Suppose that H_1, H_2, and H_1 are constructed. Then it is clear that the composition

$$H = (\mathrm{id}\times H_3)\circ H_2\circ H_1^{-1} : \mathbb{R}\times X_1 \longrightarrow \mathbb{R}\times X_1$$

is a self-diffeomorphism of X which satisfies $H(V) = [0,\infty)\times X_1$. Hence, applying lemma 1.6 for $U_1 = U$ and $U_2 = V$, the theorem is proved.

Finally, let us explain the way one can obtain the diffeomorphisms H_i, $i = 1,2,3$. Since X_1 has a C^p smooth seminorm ϱ such that $\varrho^{-1}(0) = F$, the existence of H_1 follows from corollary 4.9. On the other hand, H_2 can be defined by

$$H_2(z) = \big(-\log(q_V(z)),\frac{1}{q_V(z)}z\big)$$

for $z \in \mathbb{R}\times X_1\setminus F$. Let us describe H_3. According to corollary 4.9, there exists a C^p diffeomorphism $H_0 : X_1\setminus F \longrightarrow X_1$ with $H_0(x) = x$ for $\varrho(x)\ge\frac{1}{4}$. Then the map

$$G_0(t,x) = \begin{cases} \big(\sqrt{1-\varrho^2(H_0(x))},\ H_0(x)\big) & \text{for } t>0\\ (t,x) & \text{for } t\le 0\end{cases}$$

establishes a C^p diffeomorphism between $\partial V \setminus \{1\} \times F$ and ∂V. Next take a C^∞ convex body U_0 of the plane $\mathbb{R}^2$ such that

$$\{(t,s) \in \mathbb{R}^2 \mid t^2 + s^2 = 1, t \geq 0\} \cup \{(-1,s) \in \mathbb{R}^2 \mid |s| \leq 1/2\} \subset \partial U_0,$$

and put $p(t,x) = q_{U_0}(t, \varrho(x))$ for $(t,x) \in \mathbb{R} \times X_1$, where q_{U_0} is the Minkowski functional of the set U_0. According to lemma 1.6 there exists a C^p diffeomorphism $G_1 : X \longrightarrow X$ with $G_1(\partial V) = D = \{(t,x) \in \mathbb{R} \times X_1 \mid p(t,x) = 1\}$. The stereographic projection G_2 from the point $(1,0) \in D$, followed by a suitable translation, defines a C^p diffeomorphism from $D \setminus \{1\} \times F$ onto X_1. Then, if we put

$$H_3 = G_2 \circ G_1 \circ G_0^{-1},$$

we obtain the desired C^p diffeomorphism from ∂V onto X_1.

5.2 Removing convex bodies from a Banach space

Once we know how to delete points or subspaces in spaces having smooth norms or seminorms, it is not difficult to delete smooth convex bodies. One can give a straightforward proof of this fact, but it will be more convenient for us to deduce it from theorem 5.1.

Theorem 5.2 *Let X be a Banach space, and let U be a C^p smooth convex body such that its characteristic cone, ccU, is either a linear subspace of infinite codimension in X or it is not a linear subspace of X. Then there exists a C^p diffeomorphism from X onto $X \setminus U$.*

Proof. According to theorem 5.1, there exists a C^p self-diffeomorphism of X mapping U onto a closed half-space. Therefore $X \setminus U$ is C^p diffeomorphic to an open half-space. Since an open half-space is obviously C^∞ diffeomorphic to the whole space, we may conclude that $X \setminus U$ and X are C^p diffeomorphic.

In the case when our convex body U is bounded we can find a diffeomorphism between X and $X \setminus U$ which restricts to the identity outside a (large enough) ball.

Theorem 5.3 *Let X be an infinite-dimensional Banach space, and let U be a bounded C^p smooth convex body. Then, there exists a C^p diffeomorphism from X onto $X \setminus U$ which has bounded support.*

Proof. On the one hand it is clear that the existence in X of a bounded C^p smooth convex body U implies the existence of a equivalent C^p smooth norm $\|\cdot\|$ in X (it is enough to take the Minkowski functional q_U of U, and define $\|x\| = q_U(x) + q_U(-x)$). Then, according to 1.2, there exists a C^p diffeomorphism φ from X onto $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$. We may assume that U is contained in the unit ball of X. Then $\|x\| \leq q_U(x)$ for all $x \in X$, where q_U is the Minkowski functional of the set U.

On the other hand, we can easily construct a C^p diffeomorphism g from $X \setminus \{0\}$ onto $X \setminus U$ satisfying $g(x) = x$ whenever $\|x\| \geq 2$ (indeed, take a C^∞ non-decreasing function $\lambda : (0, \infty) \longrightarrow (1, \infty)$ such that $\lambda(t) = t$ for $t \geq 2$, and put

$$g(x) = \frac{\lambda(q_U(x))}{q_U(x)} x$$

for $x \in X \setminus \{0\}$). Then, the mapping $H = g \circ \varphi$ establishes a C^p diffeomorphism from X onto $X \setminus U$ such that $H(x) = x$ whenever $\|x\| \geq 2$.

5.3 Classification of smooth starlike bodies

In this section we will give a partial classification of the smooth starlike bodies of a WCG Banach space. As we will see, such a result can be easily deduced from theorem 5.1 if we note that all WCG Banach spaces have (not necessarily equivalent) C^∞ norms. However, our result does not classify all the smooth starlike bodies of a WCG Banach space. At first glance one might fancy that theorem 5.1 should be readily extended to the category of smooth starlike bodies, but a moment's reflection shows us that part (b) of 5.1 is not true for starlike bodies whose characteristic cones are not linear subspaces. The convexity is an essential condition here. There is a wide variety of smooth starlike bodies none of which is diffeomorphic to a half space or to any other fixed body. This is due to the fact that the characteristic cone of a starlike body may be a rather complicated set which in general will bear no resemblance to the characteristic cone of a convex set.

In order to be more clear we need some definitions. A non-empty closed subset A of a Banach space X is said to be a starlike body provided A has a non-empty interior and there exists a point $x_0 \in X$ such that each ray emanating from x_0 meets the boundary of A at most once. In this case we will say that A is *starlike with respect to* x_0. When dealing with starlike bodies, we can always assume that they are starlike with respect to the origin (up to a suitable translation), and we will do so unless otherwise stated. For a starlike body A, we define the characteristic cone of A as

$$ccA = \{x \in X \mid rx \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A as

$$q_A(x) = \inf\{\lambda > 0 \mid \frac{1}{\lambda} x \in A\}$$

for all $x \in X$. It is easily seen that for every starlike body A its Minkowski functional q_A is a continuous function which satisfies $q_U(rx) = rq_U(x)$ for every $r \geq 0$ and $q_A^{-1}(0) = ccA$.

We will say that A is a C^p smooth starlike body provided its Minkowski functional q_A is C^p smooth on the set $X \setminus ccA = X \setminus q_A^{-1}(0)$. If A_1, A_2 are C^p starlike bodies in a Banach space X, we will say that A_1 and A_2 are C^p relatively diffeomorphic provided there exists a C^p diffeomorphism $\varphi : X \longrightarrow X$ such that $\varphi(A_1) = A_2$.

The following example shows that, as we said above, one cannot hope to extend part (b) of theorem 5.1 to the category of smooth starlike bodies, and it also gives us a hint of how complex the characteristic cone of a starlike body can be.

Example 5.4 Let $A = \{(x, y) \in \mathbb{R}^2 \mid |xy| \leq 1\}$. It is plain that A is a starlike body in the plane $\mathbb{R}^2$, and its characteristic set is the pair of lines defined by the equation $xy = 0$. Then A cannot be relatively diffeomorphic (not even relatively homeomorphic) to a half-plane of $\mathbb{R}^2$. Indeed, $X \setminus A$ is not connected, while the complement of a closed half-space (that is to say, an open half-space) is always connected. Similar examples show that for every $n \in \mathbb{N}$ there exists a starlike body A_n in the plane $\mathbb{R}^2$ such that $X \setminus A_n$ has exactly n connected components. Hence A_n is not relatively homeomorphic to A_m whenever $n \neq m$.

Next we give an elementary result concerning smooth starlike bodies which will be very useful in order to prove the result on classification of smooth starlike bodies. Moreover, this proposition, stating that all smooth starlike bodies with the same characteristic cones are pairwise diffeomorphic, somewhat unravels the tangle of starlike bodies. We omit the proof of this result, since it is an easy adaptation of that of lemma 1.6.

Proposition 5.5 *Let X be a Banach space, and let A_1, A_2 be C^p smooth starlike bodies such that $ccA_1 = ccA_2$. Then there exist a C^p diffeomorphism $g : X \longrightarrow X$ such that $g(A_1) = A_2$, $g(0) = 0$, and $g(\partial A_1) = \partial A_2$, where ∂A_j stands for the boundary of A_j. Moreover, $g(x) = \mu(x)x$, where $\mu : X \longrightarrow [0, \infty)$, and hence g preserves the rays emanating from the origin.*

Now we may deduce from 5.1 the following result on a partial classification of the smooth starlike bodies of a WCG Banach space.

Theorem 5.6 *Let A be a C^p starlike body in a WCG Banach space X.*

(a) If ccA is a linear subspace of finite codimension (say $X = ccA \oplus Z$, with Z finite-dimensional), then A is C^p relatively diffeomorphic to $ccA + \{z \in Z : |z| \leq 1\}$, where $|\cdot|$ is an Euclidean norm in Z.

(b) If ccA is a linear subspace such that the quotient space X/ccA is infinite dimensional, then A is C^p relatively diffeomorphic to a closed half-space.

Proof.
(a) It is enough to apply proposition 5.5 for $A_1 = A$ and $A_2 = ccA + \{z \in Z : |z| \leq 1\}$.

(b) Let $F = ccA$, and consider the quotient space $Z = X/F$. It is easy to see that the quotient of a WCG Banach space over one of its subspaces is WCG too. Since Z is WCG, it is linearly injectable into some $c_0(\Gamma)$ and, hence, it has a (not necessarily equivalent) C^∞ norm. By composing this norm with the canonical projection $\pi : X \longrightarrow Z$ we get a C^{∞} seminorm $\varrho : X \longrightarrow [0, \infty)$ whose set of zeros is $F = ccA$. Now consider the C^p convex body $B = \{x \in X \mid \varrho(x) \leq 1\}$. Since $ccB = ccA = F$, by proposition 5.5 the starlike bodies A and B are C^p relatively diffeomorphic. On the other hand, according to theorem 5.1, B is C^p relatively diffeomorphic to a half-space. Thus, A is C^p relatively diffeomorphic to a half-space too.

Chapter 6

Other applications of smooth negligibility

Apart from the classification of smooth convex bodies and starlike bodies in Banach spaces and the failure of Rolle's theorem in infinite dimensions, negligibility theory has found a lot of applications in several branches of mathematics, such as Ordinary Differential Equations in Banach spaces, free group actions on Banach spaces, and Infinite-Dimensional Differential Topology. In this final chapter we will give a sample of other applications of negligibility, pointing out how the results of chapter 4 enlarge the class of spaces within which those applications hold. Most of the theorems stated below were known for the separable Hilbert space, and some of them also for WCG Banach spaces having smooth or real-analytic norms. Our results imply that they also hold for every Banach space having a (not necessarily equivalent) smooth norm.

6.1 Garay's phenomena for ODE's in Banach spaces

Perhaps one of the most unexpected applications of negligibility theory is that found by Barnabas M. Garay [45, 46] concerning some strange topological properties of cross-sections of solution funnels for ordinary differential equations in infinite-dimensional Banach spaces. Garay made use of negligibility theory to study the geometry of the failure of Knesser and Peano's theorems in infinite-dimensional Banach spaces. He showed that, for several classes of infinite-dimensional Banach spaces, including the separable Hilbert space, every compact set can be represented as a cross-section of a solution funnel to some ordinary differential equation. He also found other applications of deleting homeomorphisms to topological dynamics (parallelizable dynamical systems with uniformly bounded trajectories in c_0, and infinite-dimensional aspects of Coleman's conjecture; see [47]). The results of chapter 4 enable us to extend Garay's theorems to the class of all Banach spaces having C^p smooth norms, with $p \in \mathbb{N} \cup \{\infty\}$.

Before stating those results formally we will recall some concepts and introduce some notation. Let X be a Banach space. For a continuous function $F : \mathbb{R} \times X \longrightarrow$

X, consider the ordinary differential equation (ODE)

$$D_t x = F(t, x). \tag{1}$$

For $(t_0, x_0) \in \mathbb{R} \times X$, a function $x \in C^1(I_x, X)$ is said to be a solution of (1) through (t_0, x_0) provided I_x is an open interval in $\mathbb{R}$ containing t_0, $x(t_0) = x_0$, and $D_t x(s) = F(s, x(s))$ for all $s \in I_x$. The solutions whose domain is $\mathbb{R}$ are called *global*. Hereafter $\mathcal{F}(X)$ will stand for the class of all the continuous functions $F : \mathbb{R} \times X \longrightarrow X$ that satisfy the following conditions:

(2) for each $(t_0, x_0) \in \mathbb{R} \times X$, the ODE (1) has at least one solution through (t_0, x_0);

(3) all the solutions of (1) extend to global solutions.

As it is well-known, Peano's theorem ensures that all continuous functions $F : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfy (2). Unfortunately, Peano's theorem does not remain true in infinite dimensions. For a counterexample we refer to [29]. Given an infinite-dimensional Banach space X, (2) is satisfied only for those continuous functions $F : \mathbb{R} \times X \longrightarrow X$ which fulfil some additional requirements (usually compactness assumptions or hypothesis of dissipative type; see [29]).

For given $F \in \mathcal{F}(X)$, $(t_0, x_0) \in \mathbb{R} \times X$, the *cross-section of the solution funnel* at the time t is the set

$$S_t\big(F, (t_0, x_0)\big) = \{x(t) \in X \mid x \text{ is a solution of (1) through } (t_0, x_0)\},$$

while the *solution funnel* (or *integral funnel*) is defined by

$$S\big(F, (t_0, x_0)\big) = \{(t, x(t)) \in \mathbb{R} \times X \mid x(t) \in S_t(F, (t_0, x_0))\}.$$

It was Kneser [57] that began to study the topological properties of cross-sections of solution funnels. He proved the following result

Theorem 6.1 (Kneser) *Let $F \in \mathcal{F}(\mathbb{R}^n)$. Then the cross-section of the solution funnel $S_t\big(F, (t_0, x_0)\big)$ is a nonempty, compact and connected subset of $\mathbb{R}^n$, for every $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $t \in \mathbb{R}$.*

There is an enormous amount of literature on generalizations of Kneser's theorem for certain classes of integral equations and functional differential equations on Banach spaces, manifolds, etc. However, the problem of characterizing cross-sections of solution funnels is far from being settled, in spite of several necessary or sufficient conditions (formulated in terms of Algebraic or Differential Topology) given by Pugh [60] and Rogers [61]. Perhaps the best result is due to Pugh, who considered the problem of classifying the cross sections of solution funnels in terms of cobordism theory in Algebraic Topology. A point $x_0 \in \mathbb{R}^n$ and a compact set $A \subset \mathbb{R}^n$ are said to be *funnel cobordant* in $\mathbb{R}^n$ provided there exists an $F \in \mathcal{F}(\mathbb{R}^n)$ such that $S_1(F, (0, x_0)) = A$ and $S_0(F, (1, a)) = \{x_0\}$ for every $a \in A$.

Theorem 6.2 (Pugh)

(a) If $\{x_0\}$ and A are funnel cobordant, then there exists a C^∞ diffeomorphism from $\mathbb{R}^n \setminus A$ onto $\mathbb{R}^n \setminus \{x_0\}$.

(b) If $A \subset \mathbb{R}^n$ is compact, $x_0 \in \mathbb{R}^n$, and there exists a C^∞ diffeomorphism from $\mathbb{R}^n \setminus A$ onto $\mathbb{R}^n \setminus \{x_0\}$, then $\{x_0\}$ and A are funnel cobordant by means of some $F \in \mathcal{F}(\mathbb{R}^n)$.

The construction of this F in [60] is based on the existence of a C^∞ diffeomorphism from $\mathbb{R}^n \setminus A$ onto $\mathbb{R}^n \setminus \{x_0\}$ which fixes all the points outside a ball containing A. It is worth mentioning that the function F constructed by Pugh has the nice additional property that, when A consists of at least two points, $(0, x_0)$ is the only point of non-uniqueness for the equation (1).

In this setting, it is natural to ask whether Kneser's theorem remains true in infinite dimensions. The answer is negative. It was observed by Binding [14] that if X is an infinite-dimensional Banach space then there exists $F \in \mathcal{F}(X)$ such that $S_1(F, (0,0)) = \{x \in X : \|x\| \leq 1\}$. Hence, cross-sections of solution funnels need not be compact. On the other hand, Binding [14] also constructed a continuous function $F : \mathbb{R} \times X \longrightarrow X$ such that $S_1(F, (0,0))$ is not connected. Unfortunately, this function F does not belong to $\mathcal{F}(X)$. In fact, the disconnectedness of $S_1(F, (0,0))$ in [14] is caused by a strong violation of the local existence condition (2).

B. M. Garay [45] investigated to what extent Kneser's theorem fails in infinite-dimensional Banach spaces and, by generalizing part (b) of Pugh's theorem 6.2, he provided a wide class of examples of disconnected cross-sections of solution funnels in those infinite-dimensional Banach spaces X which admit deleting diffeomorphisms. He showed that the disconnectedness of cross-sections of solution funnels may be caused by a very complicated global behaviour of the trajectories as well and not only by the failure of local existence (2). For several types of infinite-dimensional Banach spaces X he showed the existence of differential equations $D_t x = F(t, x)$, $F \in \mathcal{F}(X)$, with disconnected cross-sections of a solution funnel. His construction depended upon the existence of deleting diffeomorphisms and therefore was effective only for those smooth infinite-dimensional Banach spaces which are linearly injectable into some $c_0(\Gamma)$ (that is, the class of Banach spaces within which Dobrowolski's results [35] on deleting diffeomorphisms held true). Now, thanks to our results of chapters 4 and 5 on smooth negligibility of compacta and smooth convex bodies, Garay's results can be extended to the class of all infinite-dimensional Banach spaces having Fréchet differentiable equivalent norms.

The most general–but less definite–form of Garay's theorem [45] is the following

Theorem 6.3 (Garay) *Let $(X, \|\cdot\|)$ be a Banach space and let A be a nonempty bounded closed subset of X. Let $\alpha = \sup\{\|a\| : a \in A\}$. Assume that there exists a C^1 diffeomorphism h mapping $X \setminus A$ onto $X \setminus \{0\}$ such that $h(x) = x$ whenever $\|x\| \geq \alpha + 1$. Then there exists an $F \in \mathcal{F}(X)$ such that $S_1(F, (0,0)) = A$. Moreover, if A consists of at least two points, then $F \in \mathcal{F}(X)$ can be chosen is such a way that $(0,0) \in \mathbb{R} \times X$ is the only point of non-uniqueness for (1).*

By combining this result with theorems 4.1 and 5.3, we obtain

Theorem 6.4 (Garay) *Let X be an infinite-dimensional Banach space having a Fréchet differentiable equivalent norm. Let A be either a compact subset of X or a bounded smooth convex body in X. Then there exists a function $F \in \mathcal{F}(X)$ such that $S_1(F,(0,0)) = A$.*

Moreover, if A consists of at least two points, then $F \in \mathcal{F}(X)$ can be chosen so that $(0,0) \in \mathbb{R} \times X$ is the only point of non-uniqueness for (1).

Proof. From theorems 4.1 and 5.3 we know that there are diffeomorphism with bounded support from X onto $X \setminus A$, on one hand, and from X onto $X \setminus \{0\}$, on the other hand. Therefore there exists a diffeomorphism from $X \setminus \{0\}$ onto $X \setminus A$ which restricts to the identity outside a ball. Thus, the result follows from theorem 6.3.

At this point we cannot refrain from giving an outline of the proof of 6.3. We will devote the remaining part of this section to explaining how, by using a deleting diffeomorphism, one can construct an ordinary differential equation $D_t x = F(t,x)$ which has a unique global solution through each point other than the origin, while the solutions through $(0,0)$ are not unique and reach all the points of a predetermined compact set or convex body. We will follow Garay's argument in [45]; see also [9].

Theorem 6.3 is a relatively easy consequence of the following result, which is interesting in itself.

Theorem 6.5 (Garay) *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with an equivalent Fréchet differentiable norm $\|\cdot\|$. Let A be either a compact set or a bounded C^1 smooth convex body in X. We may assume that A is contained in the unit ball of X. Then, there exists a continuous function $f : X \longrightarrow X$ such that $f^{-1}(0) = A$, $f(x) = x$ whenever $\|x\| \geq 2$, and such that, for every $(t_0, x_0) \in \mathbb{R} \times (X \setminus A)$, the differential equation*

$$x' = f(x) \tag{4}$$

has a unique solution passing through (t_0, x_0), and the solution is global and unbounded.

Outline of the proof.

Let h be a C^1 diffeomorphism from $X \setminus \{0\}$ onto $X \setminus A$ which satisfies $h(x) = x$ if $\|x\| \geq 2$ (h does exist thanks to theorems 4.1 and 5.3). Let us consider the family of curves

$$x(t) = h^{-1}(h(x_0)e^t), \quad t \in \mathbb{R}, \quad x_0 \in X \setminus \{0\}, \tag{5}$$

which are pairwise disjoint and cover the set $X \setminus A$. They provide the solutions to the differential equation

$$x' = g(x), \tag{6}$$

where

$$g(x) = [(Dh^{-1})(h(x))]h(x).$$

Let $f_1 : X \longrightarrow X$ be the extension of g defined by letting $f_1 = 0$ on the set A. Then the differential equation

$$x' = f_1(x) \tag{7}$$

almost satisfy the statement above, except that f_1 might be discontinuous at the points of A. One can correct this flaw by putting

$$f(x) = \phi(x)f_1(x),$$

where $\phi : X \longrightarrow [0,1]$ is a continuous function such that $\phi^{-1}(0) = A$ and $\phi(x) = 1$ whenever $\|x\| \geq 2$. Then the equation

$$x' = f(x) \tag{8}$$

has a unique solution passing through each point $(t_0, x_0) \in \mathbb{R} \times (X \setminus A)$, and the solution is global and unbounded, while, if x is a bounded global solution of (2) then there exists a point $a \in A$ such that $x(t) = a$ for every $t \in \mathbb{R}$.

Remark 6.6 If, in 6.5, the space X has an equivalent C^p smooth norm and, moreover, for a compactum A there exists a C^p smooth real-valued function ϕ with $\phi^{-1}(0) = A$, then f may be chosen to be C^{p-1} smooth. As shown in [35], for every compactum K of a separable Banach space X there exists a C^∞ smooth function $\phi : X \to [0,1]$ with $\phi^{-1}(0) = K$. However, for a nonseparable Banach space X, even if X admits a C^∞ norm, such functions need not exist, as a recent result of Hajek's [51] shows. It would be quite an interesting problem to identify the Banach spaces X which admit such functions.

As said above, from theorem 6.5 it is possible to deduce Garay's theorem. We refer to [45] for the details.

Theorem 6.7 (Garay) *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space having a Fréchet differentiable norm $\|\cdot\|$, and let $A \subset X$ be either a compact set with at least two points or a bounded C^1 smooth convex body. Then there exists a continuous map $F : \mathbb{R} \times X \longrightarrow X$ such that the Cauchy problem*

$$x'(t) = F(t, x), \qquad x(t_0) = x_0$$

admits a unique (global) solution through each point $(t_0, x_0) \neq (0, 0)$, while the solutions through $(0, 0)$ are not unique and given by

$$x(t) = \frac{1}{2}(t^2 + t|t|)a, \qquad a \in A,$$

which means that at the time $t = 1$ the solutions through $(0, 0)$ reach all the points of A.

Remark 6.8 As in the remark above, if the space X has an equivalent C^p smooth norm and for a compactum A there exists a C^p smooth real-valued function ϕ with $\phi^{-1}(0) = A$, then the map F may be chosen to be C^{p-1} smooth.

6.2 Periodic diffeomorphisms without fixed points, and free group actions on Banach spaces

V. L. Klee also used his results on negligibility [56] to prove that if X is either a non-reflexive Banach space or an infinite dimensional L^p space, there exists a two-periodic homeomorphism $f : X \longrightarrow X$ without fixed points. This was somewhat surprising because, for a finite-dimensional space X, P. A. Smith [64] had proved that every prime-periodic homeomorphism of X must have a fixed point. Klee even showed that for the Hilbert space H and for each integer $n \geq 2$ there exists a periodic homeomorphism $f : H \longrightarrow H$ of pure period n that has no fixed points. By using the results of chapter 4, in many Banach spaces these results can be sharpened so as to obtain periodic real-analytic diffeomorphisms of arbitrary period n having no fixed points. This holds for every Banach space having a complemented separable subspace which is isomorphic to its cartesian square. For $n = 2$ the result is more general while, for $n \geq 3$, smooth and real-analytic versions of Klee's theorems are obtained as corollaries of new results on free actions of the n-torus on Banach spaces.

Theorem 6.9 *Let X be an infinite-dimensional Banach space having a (not necessarily equivalent) C^p smooth norm ϱ. Then there exists a two-periodic C^p diffeomorphism $f : X \longrightarrow X$ such that f has no fixed points and f transforms the ball $\{x \in X \mid \varrho(x) \leq 1\}$ onto itself. If we assume that X has a (not necessarily equivalent) real-analytic norm, we obtain a two-periodic real-analytic diffeomorphism $f : X \longrightarrow X$ without fixed points.*

Proof. From theorem 1.2 we get a C^p diffeomorphism $\varphi : X \longrightarrow X \setminus \{0\}$ such that φ is the identity outside the ball $B = \{x \in X \mid \varrho(x) \leq 1\}$. Put $A(x) = -x$ for every $x \in X$ (note that A is a two-periodic linear isomorphism whose only fixed point is the origin, and A takes the ball B onto itself). Define $f : X \longrightarrow X$ by $f(x) = \varphi^{-1}(A(\varphi(x)))$ for every $x \in X$. Then it is clear that f is the desired diffeomorphism.

Let us recall that a Lie group G is said to act on a space X if there exists a continuous map $\Phi : G \times X \to X$ such that $\Phi(e, x) = x$ and $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ for all $g, h \in G$ and all $x \in X$. Here e denotes the neutral element of the group G. If X is a smooth (or real-analytic) manifold and Φ is C^p smooth (resp. real-analytic) then we say that G acts on X in a C^p smooth (resp. real-analytic) way. In such a case, for every $g \in G$, $x \mapsto \Phi(g, x)$ is a C^p (resp. real-analytic) self-diffeomorphism of X (and G can be identified with a subgroup of the group of diffeomorphisms of X). If for every $g \neq e$ and $x \in X$ we have $\Phi(g, x) \neq x$, then the action is called free.

Hereafter T denotes the unit circle $\{s \in \mathbb{C} : |s| = 1\}$, and T^n stands for the n-torus $\{(s_1, ..., s_n) \in \mathbb{C}^n : |s_j| = 1, j = 1, ..., n\}$; T^n will be considered with its natural group structure. For $n \geq 2$ we can obtain a sharper version of Klee's result as a corollary of the following

Theorem 6.10 *Let X be a Banach space of the form $X = Y \times Z$, where Z is an infinite-dimensional space which admits a complex structure and is C^p smooth (resp. real-analytic) diffeomorphic to $Z \setminus \{0\}$. Then, there exists a C^p smooth (resp. real-analytic) free action Φ of T on the space X.*

Proof. As $X = Y \times Z$, and Z and $Z \setminus \{0\}$ are C^p (resp. real-analytic) diffeomorphic, it is obvious that so are X and $X \setminus Y$. Let us choose a diffeomorphism $h : X \longrightarrow X \setminus Y$. Since Z has a complex structure, the standard action $\varphi : T \times (Y \times Z) \longrightarrow Y \times Z$ given by

$$\varphi(s, (y, z)) = (y, sz)$$

is well-defined. It is clear that for every $s \in T$, $s \neq 1$, the set of fixed points of $x \mapsto \varphi(s, x)$ is precisely Y. Now, it suffices to define $\Phi(s, x) = h^{-1}(\varphi(s, h(x)))$ for every $s \in T$ and $x \in X$.

Corollary 6.11 *Let X be a Banach space of the form $X = Y \times Z$, where Z is a separable infinite-dimensional space which is isomorphic to its cartesian square. Then, for each integer $n \geq 2$ there exists a real-analytic diffeomorphism $f : X \longrightarrow X$ of pure period n such that f has no fixed points.*

Proof. Since Z is separable, Z has a real-analytic non-complete norm (see [35], proposition 4.1) and it is real-analytic diffeomorphic to $Z \setminus \{0\}$. On the other hand, Z admits a complex structure, because $Z \simeq Z \times Z$. Hence, from theorem 6.10, $x \mapsto \Phi(e^{2\pi i/n}, x)$ is a real-analytic self-diffeomorphism of pure period n and without fixed points.

We can improve the last result in the following way.

Theorem 6.12 *Let X be a space from 6.11. Then, for each $n \in \mathbb{N}$, there exists a real-analytic free action Φ of the n-torus T^n on X.*

Proof. For the sake of simplicity we will write the proof only for the case $n = 2$. The reader will immediately see that the same argument, with obvious modifications, holds in the general case. Since the space Z is separable we can take a separating sequence of continuous functionals $(z_n^*) \subset Z^*$ such that $\|z_n^*\| = 1$ for every n. Define $\omega : Z \times Z \longrightarrow [0, \infty)$ by

$$\omega(u, v) = \big(\sum_{n=1}^{\infty} \frac{1}{2^n} (|z_n^*(u)|^2 + |z_n^*(v)|^2) \big)^{1/2},$$

where $(u, v) \in Z \times Z$. It is clear that ω is a prehilbertian norm in $Z \times Z$ which is compatible with the natural complex structure that the isomorphy of Z and $Z \times Z$ induces on Z (as the formula $\omega(u + iv) = \omega(u, v)$ defines a norm on Z when considered as a complex space). Choose a linear isomorphism $L : Z \longrightarrow Z \times Z$, define a prehilbertian norm $\varrho : Z \longrightarrow [0, \infty)$ by $\varrho(z) = \omega(L(z))$, and consider the ϱ-sphere of X, $S = \{z \in Z \mid \varrho(z) = 1\}$. In this setting, according to [36], there exists a real-analytic diffeomorphism from Z onto $S \times \mathbb{R}$. Using once again the fact that

Z is isomorphic to $Z \times Z$, it is clear that there exists a real-analytic diffeomorphism between Z and $\mathbb{R}^2 \times S \times S$. By means of the isomorphism $L : Z \longrightarrow Z \times Z$ we may identify the sphere S with the sphere $\hat{S} = \{u+iv \mid \omega(u,v) = 1\}$ of the complex space Z. As noticed above, the norm ω is complex-symmetric, that is, for every complex s with $|s| = 1$, and for every $z = u + iv \in \hat{S}$, the product sz belongs to $\hat{S}$. We have the following natural real-analytic free action of T^2 on $Y \times \mathbb{R}^2 \times \hat{S} \times \hat{S}$:

$$(g, (y, w, (z_1, z_2)) \mapsto (y, w, (s_1 z_1, s_2 z_2)),$$

where $g = (s_1, s_2) \in T^2$, $y \in Y$, $w \in \mathbb{R}^2$, and $(z_1, z_2) \in \hat{S} \times \hat{S}$. It follows from our discussion above that $Y \times \mathbb{R}^2 \times \hat{S} \times \hat{S}$ is real-analytic diffeomorphic to $Y \times Z = X$. Hence the proof is complete.

We will finish the chapter with a remark on the Borsuk-Ulam coincidence-type theorem in infinite dimensions. It is easy to see that the following infinite-dimensional version of the Borsuk-Ulam theorem follows from the classical, finite-dimensional one. For every $n \in \mathbb{N}$, and every mapping $f : S \to \mathbb{R}^n$ of a unit sphere S in an infinite-dimensional normed space X, there exists $x \in S$ so that $f(-x) = f(x)$. Ulam [62], Problem 167, asked whether this can be extended so as to obtain $f(Tx) = f(x)$ for some $x \in S$, where T is a self-mapping of S (and X is a Hilbert space). In a commentary following the statement of Problem 167 in [62], Klee answered this question in the negative by exhibiting a self-diffeomorphism T of S and a smooth mapping $f : S \to \mathbb{R}$ so that $f(Tx) = f(x)$ for no $x \in S$. Below we show that the mappings T and f may actually be taken real-analytic.

Remark 6.13 *Let X be an infinite-dimensional separable Banach space and let ω be a real-analytic norm on X. There exist a real-analytic self-diffeomorphism of $S = \{x \in X \mid \omega(x) = 1\}$ and a real-analytic mapping $f : S \to \mathbb{R}$ so that $f(Tx) = f(x)$ for no $x \in S$.*

Proof. By [36], S is real-analytic diffeomorphic to a (codimension one) linear subspace E of X. Now it suffices to exhibit the required T and f on E. This is trivial. Take a continuous linear functional x^* on E and a vector $x_0 \in E$ so that $x^*(x_0) \neq 0$, and let T be the shift $Tx = x + x_0$. Clearly, $x^*(Tx) = x^*(x)$ for no x.

Bibliography

[1] D. Aussel, J.-N. Corvellec, M. Lassonde, *Mean value property and subdifferential criteria for lower semicontinuous functions*, Transactions of the Amer. Math. Soc., 347 (1995), 4147-4161.

[2] D. Azagra, *Diffeomorphisms between spheres and hyperplanes in infinite-dimensional Banach spaces*, Studia Math. 125 (2) (1997), p. 179-186.

[3] D. Azagra, T. Dobrowolski, *Smooth and real-analytic negligibility of compact sets and subspaces in infinite-dimensional Banach spaces, and applications*, Preprint.

[4] D. Azagra, R. Deville, *Subdifferential Rolle's and Mean Value Inequality Theorems*, Bull. Austral. Math. Soc. 56 (1997), p. 319-329.

[5] D. Azagra, J. Gómez, J. A. Jaramillo, *Rolle's Theorem and Negligibility of Points in Infinite Dimensional Banach Spaces*, To appear in Journal of Mathematical Analysis and Applications.

[6] D. Azagra, J. Gómez, J. A. Jaramillo, *Rolle's theorem and Negligibility of Points in Infinite-Dimensional Banach Spaces, Research Announcement*, Extracta Mathematicae, Vol. 11, Núm. 3, 480-484 (1996).

[7] C. Bessaga, *Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. et Phys.14 (1966), pp. 27-31.

[8] C. Bessaga, *Negligible sets in linear topological spaces*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. et Phys. 16 (1968), pp. 117-119.

[9] C. Bessaga, *Interplay Between Infinite-Dimensional Topology and Functional Analysis. Mappings Defined by Explicit Formulas and Their Applications*, Topology Proceedings, 19 (1994).

[10] C. Bessaga and V. L. Klee, *Two topological properties of topological linear spaces*, Israel J. Math. 2 (1964), pp. 211-220.

[11] C. Bessaga and V. L. Klee, *Every non-normable Fréchet space is homeomorphic with all of its closed convex bodies*, Math. Ann. 163 (1966), pp. 161-166.

[12] C. Bessaga and A. Pelczynski, *Selected topics in infinite-dimensional topology*, Monografie Matematyczne, Warszawa 1975.

[13] J. Bés and J. Ferrera, *private communication.*

[14] P. Binding, *On infinite-dimensional differential equations*, J. Differential Equations 24 (1977), 349-354.

[15] W. A. Blankinship, *Generalization of a construction by Antoine*, Ann. of Math. 53 (1951), pp. 276-297.

[16] D. Burghelea and N. H. Kuiper, *Hilbert manifolds*, Annals of Math. 90 (1969), pp. 379-417.

[17] D. L. Cohn, *Measure Theory*, Birkhäuser, Boston, 1980.

[18] H. H. Corson and V. L. Klee, *Topological classification of convex sets*, Proc. Symp. Pure Math. 7, Convexity, Amer. Math. Soc., Providence, R. I. (1963), pp. 37-51.

[19] M. G. Crandall, L. C. Evans and P.L. Lions, *Some properties of viscosity solutions to Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 282 (1984), 487-502.

[20] M. G. Crandall, H. Ishii and P.L. Lions, *User's guide to viscosity solutions of second order to fully nonlinear partial differential equations*, Bull. Amer. Math. Soc., 27 (1992), 1-67.

[21] M. G. Crandall, H. Ishii and P.L. Lions, *Uniqueness of viscosity solutions revisited*, J. Math. Soc. Japan, 39 (1987), 581-596.

[22] M. G. Crandall and P.L. Lions, *Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre*, C. R. Acad. Sci. Paris, 292 (1984), 183-186.

[23] M. G. Crandall and P.L. Lions, *Viscosity solutions of hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), 1-42.

[24] M. G. Crandall and P.L. Lions, *Solutions de viscosité non bornées des équations de Hamilton-Jacobi du premier ordre*, C. R. Acad. Sci. Paris, 198 (1981), 217-220.

[25] M. G. Crandall and P.L. Lions, *On existence and uniqueness of solutions of Hamilton-Jacobi equations*, Non. Anal. Theor. Math. Appl., 10 (1986), 353-370.

[26] M. G. Crandall and P.L. Lions, *Unbounded viscosity solutions of Hamilton-Jacobi equations*, Illinois J. Math., 31 (1987), 665-688.

[27] M. G. Crandall and P.L. Lions, *Solutions de viscosité pour les équations de Hamilton-Jacobi en dimension infinie intervenant dans le contrôle optimal de problèmes d'évolution*, C. R. Acad. Sci. Paris, 305 (1987), 233-236.

[28] M. G. Crandall and P.L. Lions, *Hamilton-Jacobi equations in infinite dimensions*, J. Funct. Anal., *Part I: Uniqueness of viscosity solutions*, 62 (1985), 379-396; *Part II: existence of viscosity solutions*, 65 (1986), 368-405; *Part III* 68 (1986), 214-247; *Part IV: Unbounded linear terms*, 90 (1990), 237-283; *Part V:B-continuous solutions*, 97 (1991), 417-465.

[29] K. Deimling, *Ordinary Differential Equations in Banach spaces*, Lecture Notes in Mathematics, 596, Springer Verlag, Berlin 1977.

[30] R. Deville, *A mean value theorem for the non differentiable mappings*, Serdica Math. J. 21 (1995), 59-66.

[31] R. Deville, *Stability of Subdifferentials of Nonconvex Functions in Banach Spaces*, Set-Valued Analysis 2 (1994), 141-157.

[32] R. Deville and El Mahjoub El Haddad, *The subdifferential of the sum of two functions in Banach spaces II. Second order case*, Bull. Austral. Math. Soc., 51 (1995), 235-248.

[33] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, vol. 64, Pitman Monographies and Surveys in Pure and Applied Mathematics, 1993.

[34] J. Diestel, *Geometry of Banach Spaces –Selected Topics*, Lecture Notes in Mathematics, 485, Springer-Verlag, New York 1975.

[35] T. Dobrowolski, *Smooth and R-analytic negligibility of subsets and extension of homeomorphism in Banach spaces*, Studia Math. 65 (1979), 115-139.

[36] T. Dobrowolski, *Every Infinite-Dimensional Hilbert Space is Real-Analytically Isomorphic with Its Unit Sphere*, Journal of Functional Analysis, 134 (1995), 350-362.

[37] T. Dobrowolski, *Relative Classification of Smooth Convex Bodies*, Bull. Acad. Polon. Sci., Sér. Sci. Math. 25 (1977), 309-312.

[38] J. Dugundji, *An extension of Tietze's theorem*, Pacific Journal of Mathematics, 1 (1951), pp. 353-367.

[39] J. Eells and K. D. Elworthy, *Open embeddings of certain Banach manifolds*, Ann. of Math. 91 (1970), 465-485.

[40] S. Eilenberg, *On the problems of topology*, Ann. of Math. 50 (1949), pp. 247-260.

[41] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (New series), vol. 1, no. 3 (1979), 443-474.

[42] E. M. El Haddad, *Calcul sous-differentiel et solutions de viscosité des équations de Hamilton-Jacobi*, Thèse doctorale présentée à l'Université Bordeaux I en 1994.

[43] M. Fabian, P. Hájek, and J. Vanderwerff, *On smooth variational principles in Banach spaces*, Journal of Mathematical Analysis and Applications 197, 153-172 (1996).

[44] J. Ferrer, *Rolle's theorem fails in* ℓ_2, Am. Math. Monthly, vol. 103, n. 2 (1996), 161-165.

[45] B. M. Garay, *Cross sections of solution funnels in Banach spaces*, Studia Math. 97 (1990), 13-26.

[46] B. M. Garay, *Deleting Homeomorphisms and the Failure of Peano's Existence Theorem in Infinite-Dimensional Banach Spaces*, Funkcialaj Ekvacioj, 34 (1991), 85-93.

[47] B. M. Garay, *Parallelizability in Banach spaces I-II-III*, Acta Math. Hungar., Proc Royal Soc. Edinburgh, Proc. Banach Center Sem.

[48] K. Goebel and J. Wośko, *Making a hole in the space*, Proc. Amer. Math. Soc. 114 (1992), 475-476.

[49] W. T. Gowers, *A solution to Banach's hyperplane problem*, Bulletin London Math. Soc. 26 (1994), 523-530.

[50] W. T. Gowers, B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851-874.

[51] P. Hájek, *Smooth functions on* c_0, To appear in Israel J. Math.

[52] R. C. James, *Weakly compact sets*, Trans. Amer. Math. Soc. 113 (1964), 129-140.

[53] K. John, H. Toruńczyck, and V. Zizler, *Uniformly smooth partitions of unity on superreflexive Banach spaces*, Studia Math. 70 (1981), 129-137.

[54] S. Kakutani, *Topological properties of the unit sphere of a Hilbert space*, Proc. Imp. Acad. Tokyo, 19 (1943), pp. 269-271.

[55] O. H. Keller, *Die Homoiomorphie der kompakten konvexen Mengen im Hilbertschen Raum*, Math. Ann. 105 (1931), pp. 748-758.

[56] V. L. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. 74 (1953), 10-43.

[57] H. Kneser, *Über die Lösungen eines Systems gewöhnlichen Differentialgleichungen das der Lipschitzen Bedingung nicht genügt*, S.-B. Preuss. Akad. Wiss. Math. Kl. (1923), 171-174.

[58] N. Moulis, *Sur les variétés hibertiennes et les fonctions non-dégénérées*, Indagationes Math. 30 (1968), 497-511.

[59] R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Mathematics, Vol. 1364, Springer-Verlag, Berlin/New York, 1993.

[60] C. C. Pugh, *Funnel sections*, J. Differential Equations 19 (1975), 270-295.

[61] J. T. Rogers, *The shape of a cross section of the solution funnel of an ordinary differential equation*, Illinois J. Math. 21 (1977), 420-426.

[62] R. D. Mauldin (Editor), *The Scottish Book*, Birkhäuser, Boston 1981.

[63] S. A. Shkarin, *On Rolle's theorem in infinite-dimensional Banach spaces*, translation from Matematicheskie Zametki, vol. 51, no.3, pp. 128-136, March, 1992.

[64] P. A. Smith, *Fixed-point theorems for periodic transformations*, Amer. J. Math., 63 (1941), 1-8.

[65] J. J. Stoker, *Unbounded convex point sets*, Amer. J. Math. 62 (1940), pp. 165-179.

[66] S. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. 37 (1971), 173-180.

[67] A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann. 111 (1935), pp. 767-776.