UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS FÍSICAS

DISCRETE QUANTUM SYSTEMS: COMPLEMENTARITY, PHASES AND ALL THAT

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Introduction

In the emmerging field of quantum information the concept of interference plays a prominent role. There are also other reasons that justify the importance of this notion: quantum interference is used to improve the precision of the measuring, to increase the speed of data transmission in communications, to get a secure distribution of cryptographic keys, or to the improvement of the resolution in lithography. Even though some of these applications can become part of our daily reality in a near future (like the case of cryptography), the principles in which they are based have been tested in a number of crucial experiments.

From a classical point of view, the natural way of understanding the phenomena of interference is in terms of phase, which, can be measured without any type of ambiguity, in experiments that are more than one hundred and fifty years old.

According to the basic principles of quantum mechanics, each observable is represented by Hermitian operator that acts on the Hilbert space of the system. The search for an operator represent us the phase of an harmonic oscillator is an old problem. Although a complete quantum description does not need any operator asociated to that variable (since the same information is given by a probability distribution function of this variable), this do not implies that the problem of finding such an operator does not have importance.

In last years, a lot of works have been dedicated to the analisis of this variable for the electromagnetic field, however, there are other many fields where the phase is an essential variable.

On the other hand, in a classical domain, one needs to work with beams of light that have a well defined phase between modes. (we are refering to the polarization properties of light).

The main objetive of this Thesis is to present a simple solution to the definition of a phase operator in some models commonly used in quantum optics and accordingly, to introduce a consistent description of the polarization properties of light.

The outline of this Thesis can be summarized as follows: in Chapter 1 we describe the formal framework for discrete systems (two- level and three-level systems) and the definition of the phase for these systems, which will be used in the next chapters. In chapter 2 we analize the interaction between two- and three-level atoms and the

electromagnetic field, and introduce a definition of the relative phase operator for these models. In chapter 3 we use the existence of deformed algebras to show a method that allow us to obtain effective hamiltonians in a systematic way. In chapter 4 we analize the phase between two-mode of the fields from a quantum point of view, and we introduce the concept of visibility, which allows us to study the polarization properties of ligth including the polarization degree. The last Chapter summarizes the main obtained results.

1 Resumen

1.1 Sistemas cuánticos discretos

1.1.1 Sistemas de dos niveles

El estado puro más general correspondiente a un sistema de dos niveles o qubit (espín 1/2, átomo con dos niveles de energía, polarización de un fotón, etc...) se representa como la combinación lineal

$$|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle \,, \tag{1.1}$$

donde $|1\rangle$ y $|2\rangle$ son los vectores de una base (llamada, a menudo, *computacional*) en el espacio de Hilbert. A menudo, este vector se describe mediante el espinor

$$|\psi\rangle = \begin{pmatrix} c_2 \\ c_1 \end{pmatrix}. \tag{1.2}$$

Puesto que el estado está normalizado a la unidad, y se define salvo fase global, los coeficientes se pueden representar como

$$c_1 = \sin(\vartheta/2), \qquad c_2 = \cos(\vartheta/2)e^{i\varphi}. \tag{1.3}$$

Cualquier observable en el espacio de Hilbert bidimensional se reduce a una matriz 2×2 compleja que se puede expresar en términos de las matrices de Pauli (y la identidad)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 2| + |2\rangle\langle 1|, \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i(|1\rangle\langle 2| - |2\rangle\langle 1|), \tag{1.4}$$

$$\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |2\rangle\langle 2| - |1\rangle\langle 1|, \qquad \hat{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| + |2\rangle\langle 2|. \tag{1.5}$$

A menudo resulta cómodo trabajar con los operadores de subida y bajada

$$\hat{\sigma}_+ = \frac{1}{2}(\hat{\sigma}_1 + i\hat{\sigma}_2) = |2\rangle\langle 1|, \qquad \hat{\sigma}_- = \frac{1}{2}(\hat{\sigma}_1 - i\hat{\sigma}_2) = |1\rangle\langle 2|, \tag{1.6}$$

que cumplen las relaciones de conmutación

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_3, \qquad [\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm, \tag{1.7}$$

distintivas del álgebra su(2).

El paso siguiente es un conjunto de N qubits idénticos. El espacio de Hilbert de qubits es ahora $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2 \cong \mathbb{C}^{2N}$, y su base natural está formada por los vectores $\{|1\rangle_1 \otimes ... \otimes |1\rangle_N$, ..., $|1\rangle_1 \otimes ... \otimes |2\rangle_j$, ..., $|2\rangle_1 \otimes ... \otimes |2\rangle_N\}$. donde el subínice j denota el sistema j-ésimo.

Puesto que los qubits son indistiguibles, el estado debería venir descrito por una combinación lineal de la forma

$$|S, m\rangle = \sqrt{\frac{m!(N-m)!}{N!}} \sum |2\rangle_1 ... |2\rangle_m |1\rangle_{m+1} ... |1\rangle_N, \tag{1.8}$$

donde la suma se efectúa sobre todas las posibles permutaciones. Una vez definidos los estados colectivos, los operadores colectivos de este sistema de qubits toman la forma

$$\hat{\mathbf{S}} = \frac{1}{2}\sum_{j=1}^{N} \hat{\boldsymbol{\sigma}}^j \quad \hat{S}_\pm = \sum_{j=1}^{N} \hat{\sigma}_\pm^j = \hat{S}_1 \pm i\hat{S}_2, \tag{1.9}$$

donde $\hat{\boldsymbol{\sigma}}^j$ son las matrices de Pauli correspondientes al j-ésimo qubit. Los operadores $\hat{\mathbf{S}}$ cumplen también las relaciones de conmutación del álgebra su(2).

En muchas aplicaciones resulta esencial considerar la fase de un qubit. La imagen de "clásica" de la esfera de Bloch parece identificar φ en (1.4) como dicha fase. Sin embargo, esta aparece como un parámetro asociado al estado y no como una variable cuántica asociada a un operador autoadjunto. Con esta finalidad, obsérvese que

$$\langle \hat{\sigma}_- \rangle = \sin(\vartheta/2)\cos(\vartheta/2)e^{i\varphi}, \tag{1.10}$$

por lo que, $\langle \hat{\sigma}_- \rangle$ define la exponencial $e^{i\varphi}$ (excepto por un factor constante) sobre la esfera de Bloch. Parece apropiado trabajar la exponencial compleja de la fase $\hat{E}$ del qubit como la solución de la descomposición polar

$$\hat{\sigma}_- = \sqrt{\hat{\sigma}_-\hat{\sigma}_+}\ \hat{E}. \tag{1.11}$$

Una vez resuelta esta ecuación para $\hat{E}$ se puede obtener un operador de fase $\hat{\phi}$ definido por $\hat{E} = e^{i\hat{\phi}}$. En la base stándar, las soluciones unitarias de la ecuación anterior son de la forma

$$\hat{E} = |1\rangle\langle 2| - |2\rangle\langle 1|. \tag{1.12}$$

1.1.2 Sistemas de tres niveles

En esta sección, generalizamos los resultados anteriores al caso de un sistema de tres niveles (*qutrit*). En la base computacional el estado puro del qutrit se representa como la combinación

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle, \tag{1.13}$$

que puede ser descrita como

$$|\psi\rangle = \begin{pmatrix} c_3 \\ c_2 \\ c_1 \end{pmatrix}. \tag{1.14}$$

La normalización de este estado junto con una elección para la fase global nos permite representar estos coeficentes como

$$c_1 = \sin(\xi/2)\cos(\vartheta/2), \quad c_2 = \sin(\xi/2)\sin(\vartheta/2)e^{i\varphi_{12}}, \quad c_3 = \cos(\xi/2)e^{i\varphi_{13}}, \tag{1.15}$$

y estas cuatro coordenadas se mueven en los rangos $0 \leq \vartheta, \xi \leq \pi$, y $0 \leq \varphi_{12}, \varphi_{13} \leq 2\pi$. Para trabajar con operadores que actúan sobre el espacio de Hilbert del qutrit, utilizamos las matrices de Gellmann:

$$\hat{\lambda}_1 = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix} 0&-i&0 \\ i&0&0 \\ 0&0&0 \end{pmatrix}, \quad \hat{\lambda}_3 = \begin{pmatrix} 1&0&0 \\ 0&-1&0 \\ 0&0&0 \end{pmatrix}, \quad \hat{\lambda}_4 = \begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{pmatrix},$$
$$\hat{\lambda}_5 = \begin{pmatrix} 0&0&-i \\ 0&0&0 \\ i&0&0 \end{pmatrix}, \quad \hat{\lambda}_6 = \begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}, \quad \hat{\lambda}_7 = \begin{pmatrix} 0&0&0 \\ 0&0&-i \\ 0&i&0 \end{pmatrix}, \quad \hat{\lambda}_8 = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&-2 \end{pmatrix}. \tag{1.16}$$

que constituyen los operadores de traza cero de su(3).

A veces resulta útil trabajar con los operadores

$$\hat{S}^{ij} = |j\rangle\langle i|, \tag{1.17}$$

los cuales pueden ser expresados en términos de las matrices de Gellmann como

$$\begin{aligned} S^{21} &= |1\rangle\langle 2| = \frac{1}{2}(\hat{\lambda}_1 + i\hat{\lambda}_2), & S^{12} &= |2\rangle\langle 1| = \frac{1}{2}(\hat{\lambda}_1 - i\hat{\lambda}_2), \\ S^{31} &= |1\rangle\langle 3| = \frac{1}{2}(\hat{\lambda}_4 + i\hat{\lambda}_5), & S^{13} &= |3\rangle\langle 1| = \frac{1}{2}(\hat{\lambda}_4 - i\hat{\lambda}_5), \\ S^{32} &= |2\rangle\langle 3| = \frac{1}{2}(\hat{\lambda}_6 + i\hat{\lambda}_7), & S^{23} &= |3\rangle\langle 2| = \frac{1}{2}(\hat{\lambda}_6 - i\hat{\lambda}_7). \end{aligned} \tag{1.18}$$

y sus relaciones de conmutación son

$$[\hat{S}^{ij}, \hat{S}^{kl}] = \delta_{il}\hat{S}^{kj} - \delta_{kj}\hat{S}^{il}. \tag{1.19}$$

Obviamente, los tres operadores "diagonales" $\hat{S}^{ii}$ miden *poblaciones*, mientras que los operadores escalera "no-diagonales" $\hat{S}^{ij}$ generan *transiciones* del nivel i al j. Para enfatizar esta idea, definimos los operadores de subida y bajada como

$$\begin{cases} \hat{S}_{+}^{ij} = \hat{S}^{ij} & \text{if } \ j > i, \\ \\ \hat{S}_{-}^{ji} = \hat{S}^{ij} & \text{if } \ j < i, \end{cases} \tag{1.20}$$

en completa analogía con los operadores $\hat{\sigma}_{\pm}$ para el qutrit.

De los tres operadores diagonales sólo dos son independientes debido a $\hat{S}^{11} + \hat{S}^{22} + \hat{S}^{33} = \hat{\mathbb{1}}$, por ello trabajaremos con dos operadores independientes de traza cero construidos como [los cuales constituyen un subálgebra de Cartan maximal]

$$\hat{S}_{z}^{12} = \frac{1}{2}(\hat{S}^{22} - \hat{S}^{11}), \qquad \hat{S}_{z}^{23} = \frac{1}{2}(\hat{S}^{33} - \hat{S}^{22}), \tag{1.21}$$

que miden las inversiones de población entre niveles correspondientes.

Es de esperar que los operadores de inversión sean conjugados a los operadores de fase del qutrit. De hecho, notemos que $(\hat{S}_{\pm}^{12}, \hat{S}_{z}^{12})$ y $(\hat{S}_{\pm}^{23}, \hat{S}_{z}^{23})$ corresponden a los qubits $1 \leftrightarrow 2$ y $2 \leftrightarrow 3$, respectivamente. No obstante, estos qubits no son independientes ya que la ecuación (1.19) impone un acoplamiento altamente no trivial entre ellos. Por analogía con la descomposición polar (del qubit), parece apropiado definir

$$\hat{S}_{-}^{12} = \sqrt{\hat{S}_{-}^{12}\hat{S}_{+}^{12}}\ \hat{E}^{12}. \tag{1.22}$$

Aquí $\hat{E}^{12} = \exp(i\hat{\phi}^{12})$, $\hat{\phi}^{12}$ es el operador hermítico que representa a la fase del qubit $1 \leftrightarrow 2$. La solución unitaria de la ecuación (1.22) viene dada, salvo fase global, por

$$\hat{E}^{12} = |1\rangle\langle 2| - |2\rangle\langle 1| + |3\rangle\langle 3|. \tag{1.23}$$

1.2 Fase relativa en interacciones átomo-campo

1.2.1 Átomos de dos niveles interactuando con campos cuánticos

El modelo de Dicke describe la interacción de una coleción de A átomos idénticos de dos niveles con un campo cuántico en una cavidad sin pérdidas. Las dimensiones espaciales del sistema atómico son más pequeñas que la longitud de onda del campo, por lo que podemos suponer que todos los átomos sienten el mismo campo. El modelo desprecia la interación dipolo-dipolo entre átomos (o lo que es igual, sus funciones de onda no se solapan). El hamiltoniano para este modelo (en unidades $\hbar = 1$) es

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}, \tag{1.24}$$

con

$$\begin{aligned} \hat{H}_0 &= \omega\ \hat{N}, \\ \hat{H}_{\rm int} &= \Delta\ \hat{S}_3 + g\left(\hat{a}^\dagger \hat{S}_- + \hat{a}\hat{S}_+\right). \end{aligned} \tag{1.25}$$

Aquí $\hat{N} = \hat{a}^\dagger\hat{a} + \hat{S}_3$ es el operador número de excitaciones, g es la constante de acoplamiento (la cual en esta aproximación es la misma para todos los átomos y puede ser elegida real), y $\Delta = \omega - \omega_0$ es la desintonía entre la frecuencia atómica y la del campo.

Por simplicidad restringiremos nuestra atención al caso de resonancia entre la frecuencia atómica y la del campo $\omega_0 = \omega \equiv \omega$. Ya que el campo se describe con el habitual espacio de Fock $|n\rangle_{\rm f}$, la base natural para el sistema total es $|n, m\rangle \equiv |n\rangle_{\rm f} \otimes |m\rangle_{\rm a}$. No obstante, es directo probar que

$$[\hat{H}_0, \hat{H}_{\rm int}] = 0, \tag{1.26}$$

lo que implica que ambas son constantes de movimiento.

El hamiltoniano $\hat{H}_0$ (o, equivalentemente, el número de excitaciones $\hat{N}$) determina la energía total proporcionada, por el campo de radiación y el sistema atómico, la cual es conservada por la interacción. Esto quiere decir que la aparición de m átomos excitados requiere la aniquilación de m fotones. Por tanto, podemos reescribir la base total como

$$|N, m\rangle \equiv |N - m\rangle_{\rm f} \otimes |m\rangle_{\rm a}. \tag{1.27}$$

Con esta terminología $|N, m-1\rangle$ significa $|N-(m-1)\rangle_{\rm f} \otimes |m-1\rangle_{\rm a}$.

Siguiendo el espíritu de la sección anterior, describiremos la fase relativa átomo-campo en términos de una descomposición polar de las amplitudes complejas. Con este fin, introduzcamos los operadores

$$\hat{X}_+ = \hat{a}\hat{S}_+, \qquad \hat{X}_- = \hat{a}^\dagger\hat{S}_-, \qquad \hat{X}_3 = \hat{S}_3. \tag{1.28}$$

Estos operadores mantienen la primera relación de conmutación de su(2) en (1.7), $[\hat{X}_3, \hat{X}_\pm] = \pm\hat{X}_\pm$, no obstante la segunda queda modificada de la siguiente manera:

$$[\hat{X}_-, \hat{X}_+] = P(\hat{X}_3), \tag{1.29}$$

donde $P(\hat{X}_3)$ representa una función polinómica de segundo grado en el operador $\hat{X}_3$. Esto es un ejemplo típico de las llamadas deformaciones polinómicas del álgebra su(2). En particular, el estado $|N, 0\rangle$ desempeña el papel de *vacío cuántico*, ya que

$$\hat{X}_-|N,0\rangle = 0. \tag{1.30}$$

Por lo tanto, podemos construir subespacios invariantes, como en la teoría estándar del momento angular en la forma

$$|N,m\rangle = \frac{1}{\mathcal{N}}\ \hat{X}_+^m|N,0\rangle, \tag{1.31}$$

donde $\mathcal{N}$ es una constante de normalización. Se puede probar que

$$\hat{X}_+^{\mathcal{D}+1}|N,0\rangle = 0, \tag{1.32}$$

confirmando que el número de estados accesibles es $\mathcal{D}+1$, siendo $\mathcal{D}$ el valor mínimo entre el número de total de excitaciones y el número total de átomos.

Como consecuencia, el espacio completo del sistema puede ser dividido en suma directa $\mathcal{H} = \oplus_{N=0}^{\infty}\mathcal{H}_N$ de subespacios invariantes bajo la acción de los operadores $(\hat{X}_+, \hat{X}_-, \hat{X}_3)$, además cada uno de ellos tiene un número fijo de excitaciones.

En cada uno de estos subespacios invariantes el operador $\hat{X}_3$ es diagonal, mientras que $\hat{X}_+$ y $\hat{X}_-$ son operadores escalera representados por matrices de dimensión finita. Esto sugiere introducir una descomposición polar de la forma

$$\hat{X}_- = \sqrt{\hat{X}_+\hat{X}_-}\ \hat{E} \tag{1.33}$$

Ahora podemos garantizar que el operador $\hat{E} = e^{i\hat{\phi}}$, representa la exponencial de la fase relativa, es unitario y conmuta con el número de excitaciones

$$\begin{gathered}\hat{E}\hat{E}^\dagger = \hat{E}^\dagger\hat{E} = \hat{\mathbb{1}},\\ [\hat{E},\hat{N}] = 0.\end{gathered} \tag{1.34}$$

Por lo tanto, podemos estudiar mejor su restricción a cada subespacio invariante $\mathcal{H}_N$, la cual denotaremos por $\hat{E}^{(N)}$. Con estas condiciones, la acción del operador $\hat{E}^{(N)}$ en cada subespacio viene dada por

$$\hat{E}^{(N)} = \sum_{m=0}^{\mathcal{D}} |N,m\rangle\langle N,m+1| + e^{i(\mathcal{D}+1)\phi^{(N)}}|N,\mathcal{D}\rangle\langle N,0|, \tag{1.35}$$

donde $\phi^{(N)}$ es una fase arbitraria.

Para el estado general inicial

$$|\Psi(0)\rangle = \sum_{N,M} Q_{N-M}A_M|N-M,M\rangle. \tag{1.36}$$

tenemos

$$P(N,\phi,t)=\frac{1}{2\pi}\left|\sum_{m',m=0}^{\mathcal{D}} Q_{N-m}A_m\ C^N_{m'm}(t)e^{im'\phi}\right|^2, \tag{1.37}$$

y entonces llegamos a la distribución de probabilidad:

$$P(\phi,t)=\frac{1}{2\pi}\sum_{N=0}^{\infty}\left|\sum_{m',m=0}^{\mathcal{D}} Q_{N-m}A_m\ C^N_{m'm}(t)e^{im'\phi}\right|^2. \tag{1.38}$$

Este es un resultado básico y compacto que utilizaremos para analizar la evolución de las propiedades de la fase del modelo de Dicke.

En la figura 1.1 hemos evaluado numéricamente esta distribución $P(\phi,t)$ como una función de ϕ y el tiempo adimensional gt, para el caso en el que todos los átomos están inicialmente desexcitados y el campo se encuentra en un estado coherente con varios valores del número medio de fotones $\bar{n}$. En todos los casos, cuando $\tau=0$ tenemos que $C^N_{m'm}(0)=\delta_{m'm}$ y

$$P(\phi,t=0)=\frac{1}{2\pi}\sum_{N=0}^{\infty}\left|\sum_{m=0}^{\mathcal{D}} Q_{N-m}A_m\ e^{im\phi}\right|^2. \tag{1.39}$$

Dos comportamientos diferentes son evidentes a partir de estas gráficas. El primero ocurre en la región de campo débil cuando el número de excitaciones en el sistema es mucho más pequeño que el número de átomos, $N\ll A$. Si, por simplicidad, asumimos que todos los átomos están desexcitados y que en el estado inicialmente coherente del campo el número medio de fotones es pequeño, digamos $\bar{n}\sim 1$, entonces podemos quedarnos solamente con los términos dominantes en la ecuación (1.38), obteniendo

$$P(\phi,t)\simeq\frac{1}{2\pi}\{1+\bar{n}[|C^1_{00}(t)|^2+|C^1_{01}(t)|^2+2\mathrm{Re}(C^1_{00}(t)C^1_{01}(t)^*e^{i\phi})]\}e^{-\bar{n}}. \tag{1.40}$$

Vemos que, debido a la dependencia periódica temporal de los términos $C^N_{m'm}(t)$, esta distribución es oscilante para todos los tiempos, lo que es corroborado numéricamente en la figura 1.1.a.

El segundo caso (y quizá más interesante) corresponde a la región de campo fuerte, el número de átomos $A\ll N$. Se puede demostrar que los coeficientes $C^N_{m'm}(t)$ pueden ser aproximados, hasta orden $A/\sqrt{\bar{n}}$, por

$$C^N_{m'm}(t)\simeq d^A_{m'm}(-\Omega_N t), \tag{1.41}$$

donde

$$\Omega_N=2g\sqrt{N-A/2+1/2}, \tag{1.42}$$

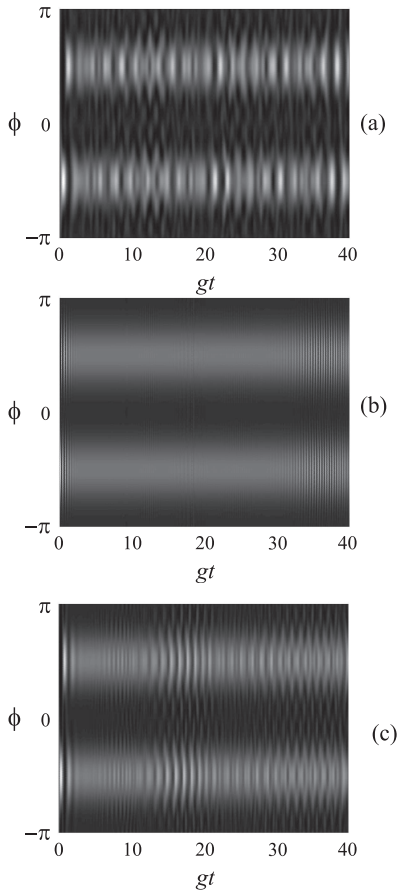

Fig. 1.1. Representación de la distribución de probabilidad $P(\phi, t)$ como función de ϕ y el tiempo adimensional gt utilizando contornos grises para el caso de $A = 5$ átomos inicialmente desexcitados y el campo en un estado coherente con los valores siguientes del número medio de fotones: a) $\bar{n} = 1$ (campo débil), b) $\bar{n} = 50$ (campo fuerte), y $\bar{n} = 5$ (campo intermedio).

y $d^A_{m'm}$ son funciones d de Wigner, definidas como los elementos de matriz para las rotaciones finitas, a partir de los operadores de las representaciones el grupo SU(2)

$$d^A_{m'm}(\vartheta) = d^A_{mm'}(\vartheta) = \langle m'|e^{i\vartheta \hat{S}_x}|m\rangle, \tag{1.43}$$

donde $m, m' = 0, 1, \ldots, A$. El punto ahora es que, en esencia, únicamente un subespacio de dimensión $A + 1$ domina la dinámica. Además, poniendo la forma explícita de estas funciones d, un cálculo simple da

$$P(\phi,t)=\frac{1}{2\pi}\sum_{N=0}^{\infty}Q_N^2\left|\sum_{m=0}^{A}\sqrt{\frac{A!}{(A-m)!m!}}[\tan(\Omega_N t/2)]^m e^{im(\phi-\pi/2)}\right|^2[\cos(\Omega_N t/2)]^{2A}. \tag{1.44}$$

Cuando $A\gg 1$ y cuando las oscilaciones quedan bien resueltas, se puede realizar

$$P(\phi,t)=\sqrt{\frac{A}{2\pi}}\sum_N \phi_N(t)\ \exp\left[-\frac{A}{2}(\phi-\pi/2+\delta_N)^2\right], \tag{1.45}$$

donde $\phi_N(t)$ es una fución del tiempo de estructura complicada que lleva información sobre los colapsos y resurgimientos pero que tiene poco interés para lo objetivos que nos ocupan , y $\delta_N=\arg[\tan(\Omega_N t/2)]$. Ahora, esta claro que, ya que δ_N toma únicamente los valores 0 y π, las distribuciones gaussianas anteriores tienden a tener dos picos en $\phi=\pm\pi/2$, en acuerdo con las expectativas clásicas. La presencia de colapsos y resurgimientos es evidente en la figura 1.1.b, lo cual confirma la anterior evidencia numérica y analítica. El conocido comportamiento de la cuasi-independencia del tiempo en la ventana temporal entre colapso y resurgimiento es también clara. Como podemos ver, la distribución tiende a ser más aleatoria a medida que evoluciona, aunque los dos picos en $\pm\pi/2$ se siguen conservando.

En la región intermedia, cuando $N\sim A$, el comportamiento es más complejo, como se muestra en la figura 1.1.c, y no hay aproximaciones analíticas disponibles.

1.2.2 Átomos de tres niveles en interacción con campos cuánticos

Queremos explorar en detalle las propiedades de la fase de un sistema de tres niveles. Para ser más concretos, consideraremos una configuración Λ configuración, con niveles de energía $\omega_1<\omega_2<\omega_3$ y transiciones dipolares permitidas $1\leftrightarrow 3$ y $2\leftrightarrow 3$, pero no $1\leftrightarrow 2$.

El hamiltoniano para este sistema es

$$\hat{H}=\hat{H}_{\rm a}+\hat{H}_{\rm f}+\hat{V}, \tag{1.46}$$

donde

$$\begin{aligned}\hat{H}_{\rm a}&=\sum_i \omega_i\hat{S}^{ii},\\ \hat{H}_{\rm f}&=\omega_a\hat{a}^\dagger\hat{a}+\omega_b\hat{b}^\dagger\hat{b},\\ \hat{V}&=g_a(\hat{a}\hat{S}_+^{13}+\hat{a}^\dagger\hat{S}_-^{13})+g_b(\hat{b}\hat{S}_+^{23}+\hat{b}^\dagger\hat{S}_-^{23}).\end{aligned} \tag{1.47}$$

Aquí $\hat{H}_{\rm a}$ describe la dinámica del átomo libre y $\hat{H}_{\rm f}$ representa los modos de frecuencia ω_a y ω_b de la cavidad, con operadores de aniquilación $\hat{a}$ y $\hat{b}$, respectivamente. Finalmente, en el término de interacción $\hat{V}$, escrito en las aproximaciones dipolar y de onda

rotante, asumimos que la transición $1 \leftrightarrow 3$ se acopla de manera casi resonante con el modo a y la transición $2 \leftrightarrow 3$ se acopla con el modo b, con constantes de acoplamiento g_a y g_b que serán considerados números reales.

La base para el sistema total es $|i\rangle_{\mathrm{a}} \otimes |n_a, n_b\rangle_{\mathrm{f}}$, donde $|n_a, n_b\rangle_{\mathrm{f}}$, es la base de Fock bimodal habitual. No obstante, se puede probar que los dos operadores número de excitaciones

$$\hat{N}_a = \hat{a}^\dagger \hat{a} - \hat{S}^{11} + \hat{\mathbb{1}}, \qquad \hat{N}_b = \hat{b}^\dagger \hat{b} - \hat{S}^{22} + \hat{\mathbb{1}}, \tag{1.48}$$

son cantidades conservadas.

Las desintonías se definen como

$$\Delta_a = \omega_{31} - \omega_a, \qquad \Delta_b = \omega_{32} - \omega_b, \tag{1.49}$$

con $\omega_{ij} = \omega_i - \omega_j$. Por lo tanto, el problema puede ser reducido a estudiar la restricción de $\hat{H}_{\mathrm{int}}$ a cada subespacio $\mathcal{H}^{(N_a,N_b)}$ con valores fijos de los pares de número de excitaciones (N_a, N_b). En cada uno de estos subespacios $\mathcal{H}^{(N_a,N_b)}$ hay tres vectores de la base que pueden ser escritos como

$$|i; n_a = N_a - \mu_i, n_b = N_b - \nu_i\rangle, \tag{1.50}$$

donde los valores de μ_i y ν_i son definidos como

$$(\mu_1, \mu_2, \mu_3) = (0, 1, 1), \qquad (\nu_1, \nu_2, \nu_3) = (1, 0, 1). \tag{1.51}$$

Notemos que cuando $N_a = 1$ y $N_b = 0$ o $N_a = 0$ y $N_b = 1$ algunos estados pueden tener un número de ocupación de fotones negativo y deben ser eliminados.

Definamos los operadores

$$\begin{aligned} \hat{X}_+^{13} &= \hat{a}\ \hat{S}_+^{13}, \qquad \hat{X}_z^{13} = \hat{S}_z^{13}; \\ \hat{X}_+^{23} &= \hat{b}\ \hat{S}_+^{23}, \qquad \hat{X}_z^{23} = \hat{S}_z^{23}. \end{aligned} \tag{1.52}$$

Estos operadores satisfacen la mayoría de las relaciones de conmutación de su(3) con tal de que una de ella se reescriba como

$$[\hat{X}_+^{13}, \hat{X}_-^{23}] = -\hat{Y}_+^{12}, \tag{1.53}$$

donde

$$\hat{Y}_+^{12} = -\hat{a}\hat{b}^\dagger \hat{S}_+^{12}. \tag{1.54}$$

No obstante, algunas de ellas deben ser modificadas de la forma siguiente

$$\begin{aligned} [\hat{X}_+^{13}, \hat{X}_-^{13}] &= \hat{N}_a(\hat{\mathbb{1}} - 2\hat{S}^{11} - \hat{S}^{22}), \\ [\hat{X}_+^{23}, \hat{X}_-^{23}] &= \hat{N}_b(\hat{\mathbb{1}} - 2\hat{S}^{22} - \hat{S}^{11}), \\ [\hat{Y}_+^{12}, \hat{Y}_-^{12}] &= \hat{N}_a\hat{N}_b(\hat{S}^{11} - \hat{S}^{22}), \end{aligned} \tag{1.55}$$

lo que corresponde a una deformación polinómica de la álgebra su(3).

Por sencillez, centrémonos primero en la transición permitida $1 \leftrightarrow 3$ y notemos que en todo subespacio invariante tridimensional $\mathcal{H}^{(N_a,N_b)}$ el estado $|1; N_a, N_b - 1\rangle$ desempeña el papel de *vacío* ya que

$$\hat{X}_-^{13}|1; N_a, N_b - 1\rangle = 0. \tag{1.56}$$

En este subespacio el operador $\hat{X}_z^{13}$ es diagonal y podemos obtener una descomposición polar similar a (1.22), a saber

$$\hat{X}_-^{13} = \sqrt{\hat{X}_-^{13}\hat{X}_+^{13}}\ \hat{E}^{13}. \tag{1.57}$$

El operador $\hat{E}^{13}$ solución de (1.57) puede ser expresado en $\mathcal{H}^{(N_a,N_b)}$ como

$$\begin{aligned}\hat{E}^{13} = {} & |1; N_a, N_b - 1\rangle\langle 3; N_a - 1, N_b - 1| - |3; N_a - 1, N_b - 1\rangle\langle 1; N_a, N_b - 1| \\ & + |2; N_a - 1, N_b\rangle\langle 2; N_a - 1, N_b|.\end{aligned} \tag{1.58}$$

De manera evidente, un razomamiento semejante para $2 \leftrightarrow 3$ da lugar al operador $\hat{E}^{23}$ como

$$\begin{aligned}\hat{E}^{23} = {} & |2; N_a - 1, N_b\rangle\langle 3; N_a - 1, N_b - 1| - |3; N_a - 1, N_b - 1\rangle\langle 2; N_a - 1, N_b| \\ & + |1; N_a, N_b - 1\rangle\langle 1; N_a, N_b - 1|.\end{aligned} \tag{1.59}$$

Centrémonos en la fase relativa entre en modo a y la transición dipolar atómica $1 \leftrightarrow 3$. La función de distribución de probabilidad de un estado descrito por la matriz de densidad $\hat{\varrho}(t)$ se define como

$$P(N_a, N_b, \phi_r^{13}, t) = \mathrm{Tr}[\hat{\varrho}(t)\ |\phi_r^{13}\rangle\langle\phi_r^{13}|], \tag{1.60}$$

donde los estados $|\phi_r^{13}\rangle$ son los autoestados de (1.58) y el subíndice r corre sobre los tres posibles autovalores, 0, y $\pm\pi/2$. Esta expresión puede ser interpretada como una distribución de probabilidad para la fase relativa y los operadores de excitación $\hat{N}_a$ y $\hat{N}_b$. A partir de esta función, podemos deducir la función para la fase relativa como la distribución marginal

$$P(\phi_r^{13}, t) = \sum_{N_a,N_b=0}^{\infty} P(N_a, N_b, \phi_r^{13}, t). \tag{1.61}$$

Hemos evaluado esta distribución numéricamente para los tres valores permitidos de la fase relativa en el caso en él que los átomos se encuentran inicialmente en el estado fundamental $|1\rangle$ y los modos a y b están en un estado coherente con número medio de fotones $\bar{n}_a$ y $\bar{n}_b$, respectivamente. Para simplificar los cálculos, hemos utilizado el tiempo adimensional

$$\tau = \frac{g_a t}{2\pi\sqrt{\bar{n}_a}}, \tag{1.62}$$

en todas las figuras y hemos supuesto que $g_a = g_b$, ya que esta restricción no limita nuestro estudio.

En la figura 1.2 hemos representado una situación típica de campo débil, en la que el número de excitaciones del sistema es pequeño, esto es $\bar{n}_a \sim \bar{n}_b \sim 1$. El dibujo muestra un comportamiento casi oscilatorio.

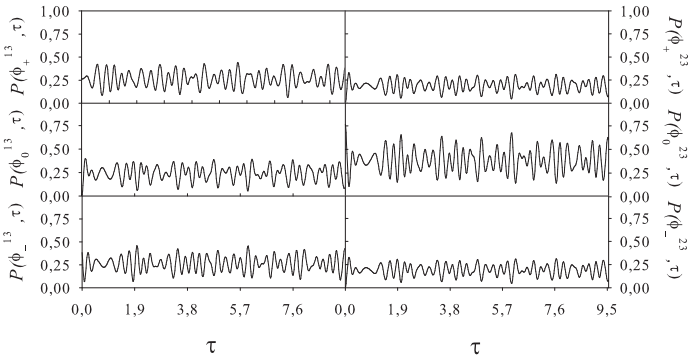

Fig. 1.2. Función de distribución de probabilidad para los seis valores permitidos para la fase relativa como función del tiempo rescalado τ en el caso de campo débil con $\bar{n}_a = \bar{n}_b = 1$.

Quizá es más interesante el caso de la dinámica de campo fuerte, este se da cuando el número de excitaciones en el sistema es grande y por tanto $\bar{n}_a$ o $\bar{n}_b$, o bien los dos, son grandes. En la figura 1.3 hemos representado las probabilidades de la fase relativa para número de fotones $\bar{n}_a = 50$ y (a) $\bar{n}_b = 0.5$, y (b) $\bar{n}_b = 50$, con el átomo inicialmente en el estado fundamental $|1\rangle$. Cuando $\bar{n}_b = 0.5$, la distribución $P(\phi_0^{13}, t)$ (que es la de encontrar un átomo en el nivel $|2\rangle$) es casi despreciable, mientras que $P(\phi_\pm^{13}, t)$ muestra colapsos y resurgimientos. Esto se puede interpretar físicamente de la siguiente manera: la transición $1 \leftrightarrow 3$ es intensa debido al proceso estimulado por el modo a en el que no hay tansferencia de población al nivel $|2\rangle$, lo que origina una oscilación regular del dipolo $1 \leftrightarrow 3$ con los correspondiente colapsos y resurgimientos en la fase relativa. La probabilidad de encontrar el átomo en el nivel $|1\rangle$, $P(\phi_0^{23}, t)$, tiende a $1/2$ (excepto en los resurgimientos), lo cual confirma que la transición $1 \leftrightarrow 3$ está casi saturada.

A medida que $\bar{n}_b$ crece, la posición de los colapsos y resurgimientos cambia, de acuerdo a estimaciones stándar. Cuando $\bar{n}_a = \bar{n}_b = 50$, $P(\phi_0^{13}, t)$ esta centrado en $1/4$,

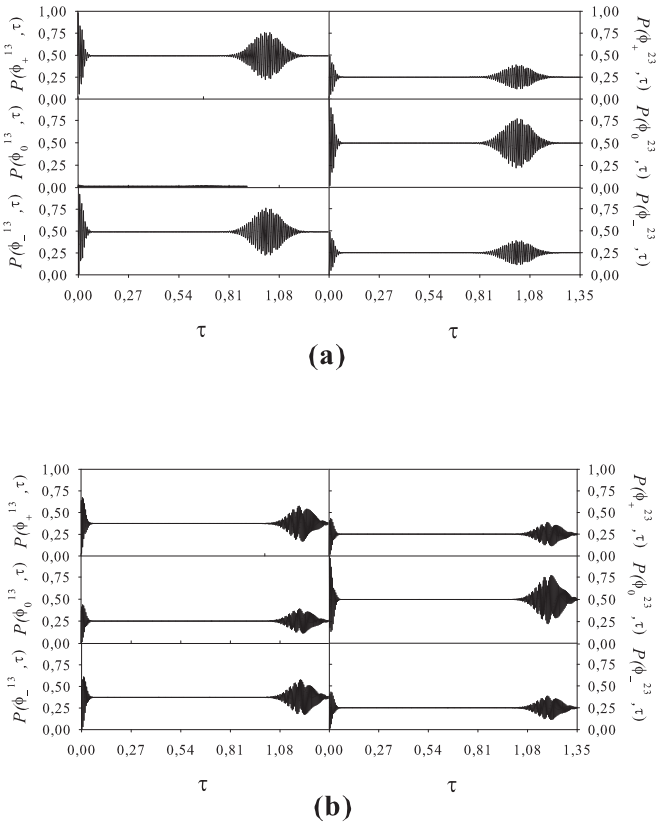

Fig. 1.3. Función de distribución de probabilidad para los seis valores permitidos para la fase relativa como función del tiempo rescalado τ en el límite de campo fuerte con: a) $\bar{n}_a = 50$, $\bar{n}_b = 0.5$ y b) $\bar{n}_a = 50$, $\bar{n}_b = 50$.

mientras que $P(\phi_0^{23}, t)$ esta centrado en $1/2$, esto muestra que las poblaciones tienden a estar equidistribuidas porque ahora también la transición $2 \leftrightarrow 3$ está también saturada.

1.3 Rotaciones pequeñas y hamiltonianos efectivos

Con el fin de hacer la discursión lo más autocontenida posible e introducir las ideas subyacentes al método, empecemos con el sencillo ejemplo de una partícula de espín

s en un campo magnético. El hamiltoniano de este sistema tiene la forma siguiente

$$\hat{H} = \omega \hat{S}_3 + g(\hat{S}_+ + \hat{S}_-), \tag{1.63}$$

donde g es la constante de acoplamiento y los operadores $\hat{S}_3$, $\hat{S}_+$, y $\hat{S}_-$ constituyen una representación $(2s+1)$-dimensional de la álgebra su(2).

El hamiltoniano (1.63) pertenece a la clase de los llamados hamiltonianos lineales y admite una solución exacta. Una manera conveniente de obtener la solución es aplicar la rotación

$$\hat{U} = \exp\left[\alpha(\hat{S}_+ - \hat{S}_-)\right], \tag{1.64}$$

y recordando que $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \ldots$, el hamiltoniano rotado, que es unitariamente equivalente al original, se transforma en

$$\hat{H} = \hat{U}\hat{H}\hat{U}^\dagger = [\omega\cos(2\alpha) + 2g\sin(2\alpha)]\hat{S}_3 + \frac{1}{2}[2g\cos(2\alpha) - \omega\sin(2\alpha)](\hat{S}_+ + \hat{S}_-). \tag{1.65}$$

La idea central es elegir ahora el parámetro α que permita cancelar los términos no diagonales en la ecuación (1.65). Esto puede hacerse tomando

$$\tan(2\alpha) = \frac{2g}{\omega}, \tag{1.66}$$

y el hamiltoniano transformado se reduce entonces a

$$\hat{H}_{\text{eff}} = \omega\sqrt{1 + \frac{4g^2}{\omega^2}}\ \hat{S}_3. \tag{1.67}$$

Ya que este hamiltoniano efectivo es diagonal, el problema dinámico está completamente resuelto. El hecho crucial es que cuando $g \ll \omega$ podemos aproximar la ecuación (1.66) por $\alpha \simeq g/\omega$, y entonces la ecuación (1.64) puede ser sustituida por

$$\hat{U} \simeq \exp\left[\frac{g}{\omega}(\hat{S}_+ - \hat{S}_-)\right]. \tag{1.68}$$

Esta rotación pequeña, aproximadamente (esto es, hasta términos de segundo orden en g/ω) diagonaliza el hamiltoniano (1.63), dando lugar al hamiltoniano efectivo

$$\hat{H}_{\text{eff}} = \hat{U}\hat{H}\hat{U}^\dagger \simeq \left(\omega + 2\frac{g^2}{\omega}\right)\hat{S}_3 \tag{1.69}$$

lo que de manera evidente coincide con la solución exacta (1.67) despues de expandir en una serie de g^2/ω^2. Una aplicación directa de la teoría estándar de perturbaciones independientes del tiempo a la ecuación (1.63) conduce inmediatamente a los mismos autovalores y autoestados que da el hamiltoniano (1.69) en el mismo orden de aproxi-

mación. No obstante, este método es completamente operacional y permite evitar el tedioso trabajo de calcular las correcciones sucesivas como sumas sobre todos los estados accesibles.

A partir del ejemplo anterior, vayamos un paso más allá tratando el caso más general de un sistema que admite algunas integrales de movimiento $\hat{N}_j$ y cuyo hamiltoniano de interacción puede ser escrito como

$$\hat{H}_{\text{int}} = \Delta\ \hat{X}_3 + g(\hat{X}_+ + \hat{X}_-), \tag{1.70}$$

donde g es una constante de acoplamiento y Δ es un parámetro que representa habitualmente la desintonía entre frecuencias de subsistemas diferentes (aunque no es así necesariamente). Los operadores $\hat{X}_\pm$ y $\hat{X}_3$ mantienen la primera relación de conmutación de su(2), $[\hat{X}_3, \hat{X}_\pm] = \pm\hat{X}_\pm$, sin embargo la segunda aparece modificada de la siguiente manera

$$[\hat{X}_+, \hat{X}_-] = P(\hat{X}_3), \tag{1.71}$$

donde $P(\hat{X}_3)$ es una función polinómica del operador diagonal $\hat{X}_3$ con coeficientes que quizá dependan de las integrales de movimiento $\hat{N}_j$. Estas relaciones de conmutación corresponden a las ya conocidas deformaciones polinómicas de su(2).

Supongamos ahora que por razones físicas (dependiendo del modelo considerado) se cumple la condición

$$\varepsilon = \frac{g}{\Delta} \ll 1. \tag{1.72}$$

Entonces, está claro que (1.70) es *casi* diagonal en la base que diagonaliza $\hat{X}_3$. De hecho, una perturbación stándar muestra inmediatamente que las correcciones, a primer orden, introducidas sobre los autovalores de $\hat{X}_3$ por la parte no diagonal $g(\hat{X}_+ + \hat{X}_-)$ son cero y aquellas de segundo orden son proporcionales a ε. Por lo tanto, aplicamos la siguiente transformación unitaria a (1.70) (la cual, de hecho, es una rotación no lineal pequeña)

$$\hat{U} = \exp[\varepsilon(\hat{X}_+ - \hat{X}_-)]. \tag{1.73}$$

Después de algunos cálculos conseguimos, hasta orden ε^2, el hamiltoniano efectivo

$$\hat{H}_{\text{eff}} = \Delta\ \hat{X}_3 + \frac{g^2}{\Delta} P(\hat{X}_3). \tag{1.74}$$

Esta técnica también proporciona una valiosa herramienta con la que obtener correcciones a los autoestados de (1.70). En efecto, resulta sencillo probar que estos autoestados pueden ser aproximados por

$$|\Psi_m\rangle = \hat{U}^\dagger |m\rangle, \tag{1.75}$$

donde $|m\rangle$ son cada uno de los autoestados de $\hat{X}_3$ y $\hat{U}$ es la rotación pequeña correspondiente. Debido a que $\hat{U}$ y $|m\rangle$ no dependen del tiempo, el operador $\hat{U}$ puede ser

aplicado a $|m\rangle$ como un desarrollo en ε. Por ejemplo, el autoestado $|\Psi_m\rangle$ hasta orden ε^2 toma la forma

$$|\Psi_m\rangle = \left[1 - \varepsilon\ (\hat{X}_+ - \hat{X}_-) - \frac{\varepsilon^2}{2}\ (1 + 2\hat{X}_+\hat{X}_- - \hat{X}_+^2 - \hat{X}_-^2)\right] |m\rangle. \tag{1.76}$$

Esta representación es especialmente ventajosa cuando el espacio de estados del modelo es un espacio de representación de la álgebra su(2) deformada que se construye de la manera habitual por la acción del operador creación $\hat{X}_+$; esto es, $|m\rangle \sim \hat{X}_+^m|0\rangle$, donde $|0\rangle$ es el vector de pesos mínimo que cumple la condición $\hat{X}_-|0\rangle = 0$.

Este procedimiento general muestra que todas las transiciones no resonantes pueden ser adiabáticamente eliminadas pudiendo trabajar con un hamiltoniano efectivo que contiene únicamente transiciones (cuasi) resonantes. El efecto de los términos no resonantes se reduce a un efecto Stark dinámico (que puede tener una forma bastante complicada). Evidentemente, las transformaciones que generan hamiltonianos efectivos también cambian las autofunciones, no obstante las correcciones son de orden ε e independientes del tiempo.

Como ejemplo relevante, apliquemos nuestro método al ya conocido modelo de Dicke.

Suponiendo ahora que se cumple el límite dispersivo; esto es,

$$|\Delta| \gg A\lambda\sqrt{\bar{n}+1}, \tag{1.77}$$

donde $\bar{n}$ es el número medio de fotones del campo, si aplicamos el método considerando los operadores del álgebra deformada (1.29) sobre el hamiltoniano definido en la ecuación (1.25), éste se transforma en el hamiltoniano efectivo

$$\hat{H}_{\text{eff}} = \Delta\ \hat{S}_3 + \frac{\lambda^2}{\Delta}[\hat{S}_3^2 - 2(\hat{a}^\dagger\hat{a} + 1)\hat{S}_3 - \hat{C}], \tag{1.78}$$

donde

$$\hat{C} = \frac{A}{2}\left(\frac{A}{2} + 1\right)\hat{\mathbb{1}}, \qquad \mu = \frac{\lambda^2}{\Delta}. \tag{1.79}$$

Siendo $\hat{C}$ el operador de Casimir para su(2).

1.4 Grado de polarización para campos bimodales

Consideremos una onda plana propagándose en la dirección z, cuyo campo eléctrico se encuentra en el plano xy. Los parámetros de Stokes valen

$$\begin{aligned} S_0 &= |\mathcal{E}_H|^2 + |\mathcal{E}_V|^2\,, \qquad S_1 = 2\ \mathrm{Re}(\mathcal{E}_H^*\mathcal{E}_V)\,, \\ S_2 &= 2\ \mathrm{Im}(\mathcal{E}_H^*\mathcal{E}_V)\,, \qquad S_3 = |\mathcal{E}_H|^2 - |\mathcal{E}_V|^2\,, \end{aligned} \tag{1.80}$$

donde los subíndices H y V indican las componentes horizontal y vertical del campo linealmente polarizado. Resulta sencillo mostrar que $S_1^2 + S_2^2 + S_3^2 = 1$. Esto quiere decir que los parámetros de Stokes de cualquier onda plana monocromática están sobre la esfera de Poincaré. Si las amplitudes del campo fluctúan, el *grado de polarización* se define como

$$\mathcal{P}_{\rm cl} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \,. \tag{1.81}$$

Desde el punto de vista cuántico, el campo puede ser descrito completamente por los operadores de amplitud $\hat{a}_H$ y $\hat{a}_V$. Las relaciones de conmutación de estos operadores son stándar:

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk} \,, \qquad j,k \in \{H,V\} \,. \tag{1.82}$$

Los operadores de Stokes se definen como las contrapartes cuánticas de las variables clásicas (1.80), a saber

$$\begin{aligned} \hat{S}_0 &= \hat{a}_H^\dagger \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V \,, \qquad \hat{S}_1 = \hat{a}_H^\dagger \hat{a}_V + \hat{a}_V^\dagger \hat{a}_H \,, \\ \hat{S}_2 &= i(\hat{a}_H \hat{a}_V^\dagger - \hat{a}_H^\dagger \hat{a}_V) \,, \qquad \hat{S}_3 = \hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V \,, \end{aligned} \tag{1.83}$$

y sus valores medios son precisamente los parámetros de Stokes $(\langle \hat{S}_0 \rangle, \langle \hat{\mathbf{S}} \rangle)$, donde $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$. Usando la relación (1.82), se puede observar inmediatamente que satisfacen las relaciones de conmutación

$$[\hat{\mathbf{S}}, \hat{S}_0] = 0 \,, \qquad [\hat{S}_1, \hat{S}_2] = 2i\hat{S}_3 \,, \tag{1.84}$$

y permutaciones ciclícas.

En términos matemáticos, una tranformación de polarización es cualquier tranformación generada por los operadores $\hat{\mathbf{S}}$. Deberíamos notar que el operador $\hat{S}_2$ es el generador infinitesimal de las rotaciones alrededor de la dirección de propagación, mientras que $\hat{S}_3$ es el generador infinitesimal de los cambios en la diferencia de fase entre los modos. Como indica la ecuación (1.84), estos dos operadores son suficientes para generar todas los transformaciones SU(2), lo cual en términos experimentales significa que dichas transformaciones se pueden lograr con una combinación de moduladores de fase y rotadores

La definición stándar de grado de polarización, en función de los operadores de Stokes es

$$\mathcal{P}_{\rm sc} = \frac{\sqrt{\langle \hat{\mathbf{S}} \rangle^2}}{\langle \hat{S}_0 \rangle} = \frac{\sqrt{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2}}{\langle \hat{S}_0 \rangle} \,, \tag{1.85}$$

donde el subíndice sc indica que se trata de una definición semiclásica, ya que imita la forma de (1.81). Aún cuando éste proporciona una imagen muy intuitiva, tiene varios

defectos que dan lugar a extraños conceptos, tales como el de estados cuánticos con polarización "oculta".

Hoy en día existe consenso en considerar la luz despolarizada como los estados del campo que continuan invariantes bajo cualquier transformación SU(2). Cualquier estado que satisfaga esta condición de invariancia también cumplirá la definición clásica de estado despolarizado, no obstante el razonamiento inverso no es cierto. Se ha mostrado que el operador de densidad de tales estados despolarizados puede ser escrito siempre como

$$\hat{\sigma} = \bigoplus_{N=0}^{\infty} \lambda_N \hat{\mathbb{1}}_N \,, \tag{1.86}$$

donde N denota la variedad de excitación en la cual hay un número exacto N de fotones en el campo. Todos los coeficientes λ_N son reales y no negativos y son consistentes con la condición de traza unidad que los operadores de densidad deben satisfacer

$$\sum_{N=0}^{\infty} (N+1)\lambda_N = 1 \,. \tag{1.87}$$

Ahora se puede cuantificar el grado de polarización como

$$\mathbb{P}(\hat{\varrho}) \propto \inf_{\hat{\sigma}\in\mathcal{U}} D(\hat{\varrho},\hat{\sigma}) \,, \tag{1.88}$$

donde $\mathcal{U}$ denota el conjunto de estados despolarizados de la forma (1.86) y $D(\hat{\varrho},\hat{\sigma})$ es cualquier medida de distancia (no necesariamente una métrica) entre las matrices densidad $\hat{\varrho}$ y $\hat{\sigma}$, tal que $\mathbb{P}(\hat{\varrho})$ satisface algunos requerimientos motivados por preceptos físicos y matemáticos. La constante de proporcionalidad en la ecuación (1.88) debe ser elegida de tal manera que $\mathbb{P}$ esté normalizada a la unidad, esto es, $\sup_{\hat{\varrho}} \mathbb{P}(\hat{\varrho}) = 1$.

Para nuestro problema, imponemos las dos condiciones siguientes:

(C1) $\mathbb{P}(\hat{\varrho}) = 0$ si y sólo si $\hat{\varrho}$ es despolarizado.
(C2) Las transformaciones unitarias qué conservan la energía $\hat{U}_E$ dejan $\mathbb{P}(\hat{\varrho})$ invariante; esto es, $\mathbb{P}(\hat{\varrho}) = \mathbb{P}(\hat{U}_E \hat{\varrho} \hat{U}_E^{\dagger})$.

Hay autores que exigen que $D(\hat{\varrho},\hat{\sigma})$ sea una métrica. Esto requiere tres propiedades adicionales:

1. Definida positiva: $D(\hat{\varrho},\hat{\sigma}) \geq 0$ y $D(\hat{\varrho},\hat{\sigma}) = 0$ si y sólo si $\hat{\varrho} = \hat{\sigma}$.
2. Simetría: $D(\hat{\varrho},\hat{\sigma}) = D(\hat{\sigma},\hat{\varrho})$.
3. Desigualdad triangular: $D(\hat{\varrho},\hat{\tau}) \leq D(\hat{\varrho},\hat{\sigma}) + D(\hat{\sigma},\hat{\tau})$.

Estas propiedades son bastante razonables, ya que la mayoría de las distancias utilizadas en mecánica cuántica están basadas en un producto escalar que automáticamente las cumple.

Para un analisis detallado consideraremos la métrica de Hilbert-Schmidt

$$D_{\mathrm{HS}}(\hat{\varrho},\hat{\sigma}) = ||\hat{\varrho}-\hat{\sigma}||^2_{\mathrm{HS}} = \mathrm{Tr}[(\hat{\varrho}-\hat{\sigma})^2]\,, \tag{1.89}$$

la cual ha sido estudiada previamente en contextos de enredo. De acuerdo con la estrategia general establecida en la definición (1.88), para un estado dado $\hat{\varrho}$ deberíamos encontrar el estado despolarizado que minimiza la distancia

$$D_{\mathrm{HS}}(\hat{\varrho},\hat{\sigma}) = \mathrm{Tr}(\hat{\varrho}^2) + \mathrm{Tr}(\hat{\sigma}^2) - 2\,\mathrm{Tr}(\hat{\varrho}\hat{\sigma})\,. \tag{1.90}$$

Con todo esto en mente, podemos definir el grado de polarización de Hilbert-Schmidt como

$$\mathbb{P}_{\mathrm{HS}}(\hat{\varrho}) = \mathrm{Tr}(\hat{\varrho}^2) - \sum_{N=0}^{\infty} \frac{p_N^2}{N+1}\,, \tag{1.91}$$

el cual viene determinado no sólo por la pureza $0 < \mathrm{Tr}(\hat{\varrho}^2) \leq 1$, sino también por la distribución del número de fotones p_N.

La distancia Hilbert-Schmidt no es monótonamente decreciente bajo cualquier mapa completamente positivo que preserve la traza. Esto ha motivado que la comunidad científica de información cuántica haya identificado la fidelidad como una aproximación alternativa. Por ello, propondremos como segundo candidato de distancia la fidelidad

$$F(\hat{\varrho},\hat{\sigma}) = [\mathrm{Tr}(\hat{\sigma}^{1/2}\,\hat{\varrho}\,\hat{\sigma}^{1/2})^{1/2}]^2\,. \tag{1.92}$$

Un manera habitual de transformar la fidelidad en una métrica es mediante la métrica de Bures

$$D_{\mathrm{B}}(\hat{\varrho},\hat{\sigma}) = 2[1-\sqrt{F(\hat{\varrho},\hat{\sigma})}]\,. \tag{1.93}$$

Ya que a mayor fidelidad $F(\hat{\varrho},\hat{\sigma})$, la distancia de Bures $D_{\mathrm{B}}(\hat{\varrho},\hat{\sigma})$ es más pequeña, podemos definir el grado de polarización de Bures como

$$\mathbb{P}_{\mathrm{B}}(\hat{\varrho}) = 1 - \sup_{\hat{\sigma}\in\mathcal{U}} \sqrt{F(\hat{\varrho},\hat{\sigma})}\,. \tag{1.94}$$

Una definición alternativa sería $1-\sup_{\hat{\sigma}\in\mathcal{U}} F(\hat{\varrho},\hat{\sigma})$, la cual surge de manera natural en el contexto de computación cuántica. Está claro que estas definiciones ordenan los estados $\hat{\varrho}$ de la misma manera. Desafortunadamente no hemos encontrado la expresión general del estado despolarizado $\hat{\sigma}$ que da un valor máximo de la fidelidad. Esta tarea debe ser realizada caso por caso.

2 Discrete quantum systems: a cursory look

This chapter describes quantum systems with d levels (also known as *qudits* in modern quantum information). Our purpose is to establish a basic framework to deal with the different systems we are going to study in this thesis. Fisrt, we consider the two-level case (*qubit*), paying special attention to the physical meaning of the operators associated to these systems, and treating both one simple qubit and a set of identical noninteracting qubits. We also disscuss possible approaches to describe the phase of these systems. The second section covers the description of the three-level case (*qutrit*), the associated operators and the phases.

2.1 Two-level systems

2.1.1 General considerations: Bloch sphere

The most general pure state of a two-level system (such as a spin 1/2, a two-level atom, the polarization of a photon, etc) is represented by the linear combination

$$|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle , \tag{2.1}$$

where $|1\rangle$ and $|2\rangle$ are a basis (usually called, the *computational* basis) in the Hilbert space. This vector can be described by the spinor

$$|\psi\rangle = \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} . \tag{2.2}$$

Because $|\psi\rangle$ is normalized to unity, the coefficients can be written as

$$c_1 = \sin(\vartheta/2) e^{i\varphi_1}, \qquad c_2 = \cos(\vartheta/2) e^{i\varphi_2}. \tag{2.3}$$

Furthermore, since any state is defined up to a global phase, these coefficients can be recast as

$$c_1 = \sin(\vartheta/2), \qquad c_2 = \cos(\vartheta/2) e^{i\varphi} \tag{2.4}$$

where $\varphi = \varphi_2 - \varphi_1$.

Any obervable in the bidimensional Hilbert space reduces to a 2×2 complex matrix that can be always expanded in terms of Pauli matrices (together with the identity)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 2| + |2\rangle\langle 1|, \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i(|1\rangle\langle 2| - |2\rangle\langle 1|), \tag{2.5}$$

$$\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |2\rangle\langle 2| - |1\rangle\langle 1|, \qquad \hat{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| + |2\rangle\langle 2|, \tag{2.6}$$

in the form

$$\hat{o} = o_1\hat{\sigma}_1 + o_2\hat{o}_2 + o_3\hat{\sigma}_3 + o_0\hat{\mathbb{1}}, \tag{2.7}$$

In other words, $(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ and $\hat{\mathbb{1}}$ constitute a basis for the operators actuing in this space, and the coefficients can be obtained as

$$o_j = \mathrm{Tr}(\hat{o}\ \hat{\sigma}_j). \tag{2.8}$$

It is often more convenient to work with ladder operators

$$\hat{\sigma}_+ = \frac{1}{2}(\hat{\sigma}_1 + i\hat{\sigma}_2) = |2\rangle\langle 1|, \qquad \hat{\sigma}_- = \frac{1}{2}(\hat{\sigma}_1 - i\hat{\sigma}_2) = |1\rangle\langle 2|, \tag{2.9}$$

which fullfill the conmutation relations

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_3, \qquad [\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm, \tag{2.10}$$

characteristic of the su(2) algebra.

The average values of Pauli matrices in the state (2.1) are

$$\begin{aligned} \langle\hat{\sigma}_1\rangle &= \sin\vartheta\cos\varphi, \\ \langle\hat{\sigma}_2\rangle &= \sin\vartheta\sin\varphi, \\ \langle\hat{\sigma}_3\rangle &= \cos\vartheta. \end{aligned} \tag{2.11}$$

In consequence, $\langle\hat{\boldsymbol{\sigma}}\rangle$ can be seen as a vector on the unit sphere, also known as Bloch sphere. Each point in these sphere corresponds to a pure state. The north pole corresponds to the state $|2\rangle$ and the south pole to $|1\rangle$, while the phase φ in these poles becomes ill defined.

This formalism can be easily extended to mixed states described by a density matrix $\hat{\varrho}$. In this case, one can express $\hat{\varrho}$ in terms of the Bloch vector $\mathbf{r} = (x_1, x_2, x_3)$ as

$$\hat{\varrho} = \frac{1}{2}(\hat{\mathbb{1}} + \mathbf{r}\cdot\hat{\boldsymbol{\sigma}}), \tag{2.12}$$

where $\mathbf{r} = 2\,\mathrm{Tr}(\hat{\varrho}\hat{\boldsymbol{\sigma}})$, in such way that

$$\begin{aligned} x_1 &= 2\ \mathrm{Re}(\varrho_{12}), \\ x_2 &= 2\ \mathrm{Im}(\varrho_{12}), \\ x_3 &= \varrho_{22} - \varrho_{11}. \end{aligned} \tag{2.13}$$

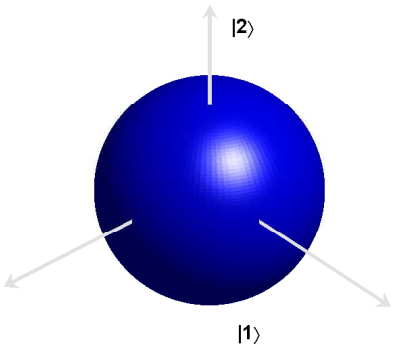

Fig. 2.1. Bloch sphere

It is easy to show that

$$|\mathbf{r}|^2 = 2\ \mathrm{Tr}(\hat{\varrho}^2) - 1 \leq 1, \tag{2.14}$$

where the equality holds only for pure states. Hence, mixed states can be pictured as points inside the unit ball. In particular, the maximal mixed state is precisely the origin

Alternatively, one can employ spherical coordinates and

$$\begin{aligned} \varrho_{11} &= \frac{1}{2}(1 - r\cos\vartheta), \\ \varrho_{12} &= \frac{1}{2} r \sin\vartheta e^{i\varphi}, \\ \varrho_{22} &= \frac{1}{2}(1 + r\cos\vartheta), \end{aligned} \tag{2.15}$$

so pure states correspond to $r = 1$.

2.1.2 Systems of qubits

The next step is the analysis of a collection of N identical qubits. The Hilbert space is now $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2 \cong \mathbb{C}^{2N}$, and the natural basis is constituted by the vectors

$\{|1\rangle_1 \otimes ... \otimes |1\rangle_N$, ..., $|1\rangle_1 \otimes ... \otimes |2\rangle_N$, ..., $|2\rangle_1 \otimes ... \otimes |2\rangle_N\}$, where the subscript j denote the jth system.

The state

$$|2\rangle_1 ... |2\rangle_m |1\rangle_{m+1} ... |1\rangle_N, \tag{2.16}$$

has a degeneration

$$\frac{N!}{m!(N-m)!}, \tag{2.17}$$

because permutations of subsystems do not modify the state. Since qubits are indistinguishable, the state should be described by a normalized linear combination of all the vectors of the form (2.16); i.e.,

$$|S, m\rangle = \sqrt{\frac{m!(N-m)!}{N!}} \sum |2\rangle_1 ... |2\rangle_m |1\rangle_{m+1} ... |1\rangle_N, \tag{2.18}$$

where the sum runs over all possible permutations. Once the collective states are defined, collective operators take the form

$$\hat{\mathbf{S}} = \frac{1}{2} \sum_{j=1}^{N} \hat{\boldsymbol{\sigma}}^j \quad \hat{S}_\pm = \sum_{j=1}^{N} \hat{\sigma}_\pm^j = \hat{S}_1 \pm i\hat{S}_2, \tag{2.19}$$

where $\hat{\boldsymbol{\sigma}}^j$ are the Pauli matrices corresponding to the jth qubit. In the collective basis (2.18)

$$\begin{aligned} \hat{S}_+ \, |S, m\rangle &= \sqrt{(m+1)(N-m)} \, |S, m+1\rangle, \\ \hat{S}_- \, |S, m\rangle &= \sqrt{m(N-m+1)} \, |S, m-1\rangle, \\ \hat{S}_3 \, |S, m\rangle &= (m - N/2) \, |S, m\rangle, \end{aligned} \tag{2.20}$$

where $(m - N/2)$ define the *inversion* of the system, i.e. the difference between excited and not excited qubits. The operators $\hat{\mathbf{S}}$ fulfill the conmutation relations of the su(2) algebra. Moreover, the state $|S, m\rangle$ is an eigenstate of the operator $\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2$, with eigenvalue

$$\hat{S}^2 |S, m\rangle = \frac{N}{2} \left(\frac{N}{2} + 1 \right) |S, m\rangle, \tag{2.21}$$

so, the states $|S, m\rangle$ form a basis of the $(2S+1)$-dimensional irreducible representation of the su(2) algebra with spin $S = N/2$. The vectors $|S, m\rangle$ are usually known as Dicke states.

2.1.3 Phase for a qubit

In a number of applications the phase of the qubit turns to play an essential role. The Bloch sphere picture suggest to identify this phase with φ in equation (2.4). However,

this phase φ appears as a pure state parameter instead of a variable associated with a *bona fide* Hermitian operator. To bypass this drawback, we observe that

$$\langle \hat{\sigma}_- \rangle = \sin(\vartheta/2)\cos(\vartheta/2)e^{i\varphi}, \tag{2.22}$$

so $\langle \hat{\sigma}_- \rangle$ defines (excepts for constant factors) the exponential $e^{i\varphi}$ over the Bloch sphere. It seems thus appropriate to define the complex exponential of the qubit phase $\hat{E}$ as the solution of the polar decomposition

$$\hat{\sigma}_- = \sqrt{\hat{\sigma}_-\hat{\sigma}_+}\ \hat{E}. \tag{2.23}$$

After this equation is solved, a phase operator $\hat{\phi}$ can be obtained by $\hat{E} = e^{i\hat{\phi}}$. In the standard basis, the solution of (2.23) reads

$$\hat{E} = |1\rangle\langle 2| + e^{i2\varphi_0}|2\rangle\langle 1|, \tag{2.24}$$

where φ_0 is a undefined matrix element that appears due to the unitarity requirement. Although the main features of this operator are largely independent of φ_0, its eigenvectors and eigenvalues depend on φ_0. For the sake of concreteness, we can make a definite choice by imposing further conditions. For instance, according to equation (2.22), the complex conjugation of the wave function should reverse the sign of $\hat{\phi}$. This leads to $e^{i2\varphi_0} = -1$ and therefore

$$\hat{E} = |1\rangle\langle 2| - |2\rangle\langle 1|, \tag{2.25}$$

with eigenvalues $\pm\pi/2$ and eigenvectors

$$|\phi_\pm\rangle = \frac{1}{2}(|1\rangle \pm i|2\rangle). \tag{2.26}$$

To any function $f(\phi)$ we can thus associate the operator

$$f(\hat{\phi}) = \sum_{\pm} |\phi_\pm\rangle\ f(\phi_\pm)\ \langle\phi_\pm|, \tag{2.27}$$

with average value

$$\langle f(\hat{\phi})\rangle = \sum_{\pm} f(\phi_\pm)P(\phi_\pm), \tag{2.28}$$

where $P\,(\phi_\pm)$ is the probability distribution

$$P(\phi_\pm) = \mathrm{Tr}(\hat{\varrho}\ |\phi_\pm\rangle\langle\phi_\pm|). \tag{2.29}$$

Note that this phase can take only two values due to the dimension of the Hilbert space. This seems anti-intuitive and one may think it preferable to describe the qubit

phase by a positive operator valued measure (POVM), taking continuous values in a 2π interval, even though this cannot lead to an operator description (see Appendix A). This is the approach we examine in the following. Now the shifting property associated with the qubit phase is

$$e^{i\phi'\hat{\sigma}_3}\ \hat{\Delta}(\phi)\ e^{-i\phi'\hat{\sigma}_3} = \hat{\Delta}(\phi+\phi'). \tag{2.30}$$

The most general POVM fulfilling this property and the statistical conditions of a POVM, that we can see in Apendix A, is of the form

$$\hat{\Delta}_\gamma(\phi) = \frac{1}{2\pi}(\hat{\mathbb{1}} + \gamma e^{i\phi}\hat{\sigma}_+ + \gamma e^{-i\phi}\hat{\sigma}_-), \tag{2.31}$$

here $\gamma \leq 1$ is a real number.

The probability distribution induced by this POVM is

$$P(\phi) = \frac{1}{2\pi}(1 + c\ e^{i\phi} + c^* e^{-i\phi}), \tag{2.32}$$

where $c = \langle 1|\hat{\varrho}|2\rangle\ \gamma$. Note that

$$\langle\hat{\sigma}_1\rangle = \frac{1}{\gamma}\int_{2\pi} d\phi\ \cos\phi\ P(\phi), \qquad \langle\hat{\sigma}_2\rangle = \frac{1}{\gamma}\int_{2\pi} d\phi\ \sin\phi\ P(\phi),$$
$$\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = \hat{\mathbb{1}}/4, \tag{2.33}$$

that is, $P(\phi)$ contains the complete statistics of $\hat{\sigma}_1$ and $\hat{\sigma}_2$: in particular, it contains the whole statistics of the qubit and not only of its phase.

Supose that qubit is described by two differents POVMs, labeled by γ_1 and γ_2, such that $\gamma_1 < \gamma_2$. If one take the dispersion D as a natural uncertity measure of the phase

$$D^2_{\gamma_j} = 1 - \left|\int_{2\pi} d\phi\ e^{i\phi} P_{\gamma_j}(\phi)\right|^2 = 1 - |c_{\gamma_j}|^2, \tag{2.34}$$

one has $D_{\gamma_1} \geq D_{\gamma_2}$: this means that $P_{\gamma_1}(\phi)$ is broader than $P_{\gamma_2}(\phi)$ if $\gamma_1 < \gamma_2$. Furthermore, one can easily demostrate that

$$\hat{\Delta}_{\gamma_1}(\phi) = \frac{1}{2\pi}\int_{2\pi} d\phi' \left\{1 + \frac{\gamma_1}{\gamma_2}[e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}]\right\}\hat{\Delta}_{\gamma_2}(\phi'), \tag{2.35}$$

so both POVMs contain the same information; since one of them can be expressed as a linear combination of the other.

A relevant feature of this approach based on POVMs is that it provides a description where any value for ϕ is allowed. Nevertheless, this continuous range of variation

is not completely effective in the sense that the values of $P(\phi)$ at every point ϕ cannot be independent, and we can find relations between them irrespective of the qubit state. In other words, all $\hat{\Delta}_\gamma(\phi)$ cannot be linearly independent because the Hilbert space is two dimensional and the algebra of operators is four dimensional.

The general $P(\phi)$ depends on the complex parameter c . This c can be determined by the value of $P(\phi)$ at two ϕ points not differing by π. Nevertheless, more manageable expressions emerge if we use three points instead of two, such as $\phi_r = 2\pi r/3$ (with $r = -1, 0, 1$). We have

$$c = \frac{2\pi}{3} \sum_{r=0,\pm1} P(\phi_r)\ e^{-i\phi_r}, \tag{2.36}$$

which allows us to express $P(\phi)$ as

$$P(\phi) = \frac{1}{3} \sum_{r,s=0,\pm1} P(\phi_r)\ e^{is(\phi-\phi_r)}, \tag{2.37}$$

and so the knowledge of the three values $P(\phi_r)$ gives $P(\phi)$ at any other point ϕ.

This effective discreteness allows us to compute the mean values of any function $f(\phi)$ in a way very similar to equation (2.28)

$$\langle f(\phi)\rangle = \frac{2\pi}{3} \sum_{r=0,\pm1} \tilde{f}(\phi_r)\ P(\phi_r), \tag{2.38}$$

where $\tilde{f}$ is related to f by

$$\int_{2\pi} d\phi\ e^{i\ell\phi} \tilde{f}(\phi) = \int d\phi\ e^{i\ell\phi} f(\phi), \qquad \ell = 0, \pm1, \tag{2.39a}$$

$$\int_{2\pi} d\phi\ e^{i\ell\phi} \tilde{f}(\phi) = 0, \qquad |\ell| = \pm2, \pm3, \ldots; \tag{2.39b}$$

Equations (2.39) altogether imply

$$\int_{2\pi} d\phi\ P(\phi)\ \tilde{f}(\phi) = \int_{2\pi} d\phi\ P(\phi)\ f(\phi), \tag{2.40}$$

for any $P(\phi)$. Discreteness is then also at the heart of this formalism.

To conclude we just quote that a reasonable choice of POVM could be

$$\hat{\Delta}(\phi) = |\phi\rangle\langle\phi|, \tag{2.41}$$

with

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}}(|1\rangle + e^{i\phi}|2\rangle). \tag{2.42}$$

While the operator $\hat{E}$ corresponds to a selection of an orthogonal basis from the set $|\phi\rangle$, this POVM does not privilege any $|\phi\rangle$ and all of them play the same role.

The generalization to N qubits is not very difficult by taking into account that the operators $\hat{S}_{\pm}$ and $\hat{S}_3$ constitute a $(2S+1)$-dimensional irreducible representation of the algebra su(2) and, in consequence, the phase take $2S+1$ different values. For the description in the terms of POVMs, the discreteness emerges too, but now one need to know the values of $P(\phi)$ in $2S+1$ independient points. In other words, the general form of $P(\phi)$ admits only $2S+1$ different frequencies.

2.1.4 Complementarity in the Bloch sphere

One can approach the qubit phase from a different perspective. The idea is that phase is complementary to amplitude. In physical terms, this means that if one of these variables is known with precission, the measure of the conjugate variable gives all the possible outputs with the same probability.

The notion of complementary variables can be appropiately formulated, from a mathematical point of view, in terms of mutually unbiased basis. Two different orthonormal bases $\mathcal{A}$ and $\mathcal{B}$ are said to be mutually unbiased if a system prepared in an eigenstate of any element of $\mathcal{A}$ (such as $|a\rangle$) has a uniform probability distribution of being found in any element of $\mathcal{B}$ (such as $|b\rangle$):

$$|\langle b|a\rangle|^2 = \frac{1}{d}, \quad \forall |a\rangle \in \mathcal{A},\ \forall |b\rangle \in \mathcal{B}, \tag{2.43}$$

Here d is the dimension of space, and in the limit $d \to \infty$, is the defining property of the eigenvectors of position and momentum (2.43).

Given an operator $\hat{A}$, one can assign it the Bloch vector, $\mathbf{n} = (\cos\varphi_A \sin\vartheta_A, \sin\varphi_A \sin\vartheta_A, \cos\vartheta_A)$ in the form

$$\hat{A} = \mathbf{n}_A \cdot \hat{\boldsymbol{\sigma}} = \hat{R}(\vartheta_A, \varphi_A)\ \hat{\sigma}_3\ \hat{R}^{-1}(\vartheta_A, \varphi_A), \tag{2.44}$$

with

$$\hat{R}\,(\vartheta, \varphi) = \exp\left[i\ \vartheta/2\,(\cos\varphi\ \hat{\sigma}_1 - \sin\varphi\ \hat{\sigma}_2)\right], \tag{2.45}$$

which is the displacement operator over the sphere surface. The condition of complementarity between $\hat{A}$ and a generic operator $\hat{B}$ [expressed also in the form (2.44)] can be written as

$$\mathbf{n}_A \cdot \mathbf{n}_B = 0. \tag{2.46}$$

That is, the subspace spanned by $\mathbf{n}_A$ in the Bloch sphere is orthogonal to the one spanned by $\mathbf{n}_B$. In this way, we get a one-parameter family of operators

$$\hat{B} = \mathbf{n}_B \cdot \hat{\boldsymbol{\sigma}}, \tag{2.47}$$

where the vector $\mathbf{n}_B$ fulfills (2.46), which can be written as

$$\cot\vartheta_B = -\tan\vartheta_A \cos\left(\varphi_B - \varphi_A\right). \tag{2.48}$$

In particular, if one take amplitude by $\hat{\sigma}_3$, (so $\mathbf{n}_A = \mathbf{u_3}$) the complementary operators are determined by all the $\mathbf{n}_B$'s in the equator:

$$\hat{E}_{\varphi_0} = \cos\varphi_0\ \hat{\sigma}_1 - \sin\varphi_0\ \hat{\sigma}_2 = \begin{pmatrix} 0 & e^{i\varphi_0} \\ e^{-i\varphi_0} & 0 \end{pmatrix}. \tag{2.49}$$

where φ_0 represents a reference phase. This, in fact, agrees with the exponential of the phase operator $\hat{E}$ obtained via a polar decomposition in (2.24) once we set $\varphi_0 = \pi/2$.

One can see that the transformation from the axis **3** in the sphere to the axis **1**, this is

$$\hat{\sigma}_1 = \hat{F}\hat{\sigma}_3\hat{F}^\dagger. \tag{2.50}$$

is given by the finite-dimensional Fourier transform

$$\hat{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.51}$$

On the contrary, the motions in the **12** are generated by the diagonal operator

$$\hat{V} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \tag{2.52}$$

because $\hat{\sigma}_2 = V^\dagger\hat{\sigma}_1 V$. In this spirit, one can define a one-parameter family of complementary operators to the amplitude as

$$\hat{E}_{\varphi_0} = e^{i\varphi_0\hat{\sigma}_3}\ \hat{\sigma}_1\ e^{-i\varphi_0\hat{\sigma}_3}. \tag{2.53}$$

which also is agree with (2.24). The principal features of this description are again independent of the reference phase φ_0.

2.2 Three-level systems

2.2.1 Bloch sphere for a qutrit

In this section, we generalize the previous results to the case of a three-level system (*qutrit*). In the computational basis a pure state of the qutrit is represented by the linear combination

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle, \tag{2.54}$$

that can be described as

$$|\psi\rangle = \begin{pmatrix} c_3 \\ c_2 \\ c_1 \end{pmatrix}. \tag{2.55}$$

The normalization condition, together with the choice of a global phase, allows us to express these coefficients as

$$c_1 = \sin(\xi/2)\cos(\vartheta/2), \quad c_2 = \sin(\xi/2)\sin(\vartheta/2)e^{i\varphi_{12}}, \quad c_3 = \cos(\xi/2)e^{i\varphi_{13}}, \tag{2.56}$$

and these coordinates vary over the ranges $0 \leq \vartheta, \xi \leq \pi$, and $0 \leq \varphi_{12}, \varphi_{13} \leq 2\pi$. To deal with operators acting on the three-dimensional qutrit Hilbert space, we use Gellmann matrices:

$$\hat{\lambda}_1 = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix} 0\,-i\,0 \\ i\;\;0\;\;0 \\ 0\;\;0\;\;0 \end{pmatrix}, \quad \hat{\lambda}_3 = \begin{pmatrix} 1\;\;0\;\;0 \\ 0\,-1\,0 \\ 0\;\;0\;\;0 \end{pmatrix}, \quad \hat{\lambda}_4 = \begin{pmatrix} 0\,0\,1 \\ 0\,0\,0 \\ 1\,0\,0 \end{pmatrix},$$
$$\hat{\lambda}_5 = \begin{pmatrix} 0\,0\,-i \\ 0\,0\;\;0 \\ i\,0\;\;0 \end{pmatrix}, \quad \hat{\lambda}_6 = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}, \quad \hat{\lambda}_7 = \begin{pmatrix} 0\,0\;\;0 \\ 0\,0\,-i \\ 0\,i\;\;0 \end{pmatrix}, \quad \hat{\lambda}_8 = \begin{pmatrix} 1\,0\;\;0 \\ 0\,1\;\;0 \\ 0\,0\,-2 \end{pmatrix}. \tag{2.57}$$

which are traceless operators of su(3) and satisfy

$$\hat{\lambda}_i\hat{\lambda}_j = 2/3\delta_{ij}\hat{\mathbb{1}} + d_{jkl}\hat{\lambda}_l + if_{jkl}\hat{\lambda}_l. \tag{2.58}$$

The coefficients f_{jkl} the structure constants of the Lie algebra, given by the commutators of the generators, and are completely antisymmetric in the three indices. The coefficients d_{jkl} are determined by the anti-commutators of the generators and are completely symmetric. We use latin indices, ranging from 1 to 8, to label these generators, while greek indices will run over the values 0 to 8.

By supplementing the eight generators with the operator

$$\hat{\lambda}_0 = \sqrt{\frac{2}{3}}\;\hat{\mathbb{1}}, \tag{2.59}$$

we obtain a basis for the space of linear operators in the qutrit Hilbert space, satisfying

$$\mathrm{Tr}(\hat{\lambda}_\alpha\hat{\lambda}_\beta) = 2\delta_{\alpha\beta}. \tag{2.60}$$

In consequence, a density operator can be expanded uniquely as

$$\hat{\varrho} = \frac{1}{3}\;c_\alpha\hat{\lambda}_\alpha, \tag{2.61}$$

where the (real) expansion coefficients c_α are given by

$$c_\alpha = \frac{3}{2}\,\mathrm{Tr}(\hat{\varrho}\ \hat{\lambda}_\alpha). \tag{2.62}$$

Normalization implies that $c_0 = \sqrt{3/2}$, so the density operator takes the form

$$\hat{\varrho} = \frac{1}{3}(\hat{\mathbb{1}} + \mathbf{c}\cdot\hat{\boldsymbol{\lambda}}). \tag{2.63}$$

Using equation (2.64) we obtain

$$\hat{\varrho}^2 = \frac{1}{9}\left(\hat{\mathbb{1}} + \frac{2}{3}\mathbf{c}\cdot\mathbf{c}\right)\hat{\mathbb{1}} + \frac{1}{3}\hat{\boldsymbol{\lambda}}\cdot\left(\frac{2}{3}\mathbf{c} + \frac{1}{3\sqrt{3}}\mathbf{c}*\mathbf{c}\right). \tag{2.64}$$

where the product $*$ is given by

$$(\mathbf{c}*\mathbf{d})_j = d_{jkl}\ c_k d_l. \tag{2.65}$$

For a pure state, $\hat{\varrho}^2 = \hat{\varrho}$, so we must have $\mathbf{c}\cdot\mathbf{c} = 3$ and $\mathbf{c}*\mathbf{c} = \sqrt{3}\mathbf{c}$. Defining the eight-dimensional unit vector $\mathbf{n} = \mathbf{c}/\sqrt{3}$, we find that any pure state of a qutrit can be written as

$$\hat{\varrho} = \frac{1}{3}(\hat{\mathbb{1}} + \sqrt{3}\mathbf{n}\cdot\hat{\boldsymbol{\lambda}}), \tag{2.66}$$

where $\mathbf{n}$ satisfies

$$\mathbf{n}\cdot\mathbf{n} = 1, \qquad \mathbf{n}*\mathbf{n} = \mathbf{n}. \tag{2.67}$$

Equation (2.68) implies in this case

$$n_j = \frac{\sqrt{3}}{2}\,\mathrm{Tr}(\hat{\varrho}\hat{\lambda}_j) = \frac{\sqrt{3}}{2}\langle\psi|\hat{\lambda}_j|\psi\rangle. \tag{2.68}$$

We can obtain expressions for $\mathbf{n}$ in the local coordinates (2.56). The pure states lie on the unit sphere in the eight-dimensional Hilbert space. In fact this vector can be understood as the Bloch vector for the qutrit, but not all operators on the unit sphere are pure states. For example, of the unit vectors $\mathbf{u}_i$ with $i = 0, ..., 8$, only $-\mathbf{u}_8$ satisfies the star-product condition (2.68) and thus is the unit vector for a pure state. The unit vectors that do not satisfy the star-product condition do not specify any state, pure or mixed, for they all give operators that have negative eigenvalues. The star-product condition (2.68) places three constraints for a pure state, thus reducing the number of real parameters required to specify a pure state from the seven parameters needed to specify an arbitrary eight-dimensional unit vector to four parameters, which can be taken to be the four coordinates of equation (2.56).

It is useful to notice that

$$|\langle\psi|\psi'\rangle|^2 = \mathrm{Tr}(\hat{\varrho}\hat{\varrho}') = \frac{1}{3}(\hat{\mathbb{1}} + 2\ \mathbf{n}\cdot\mathbf{n}'). \tag{2.69}$$

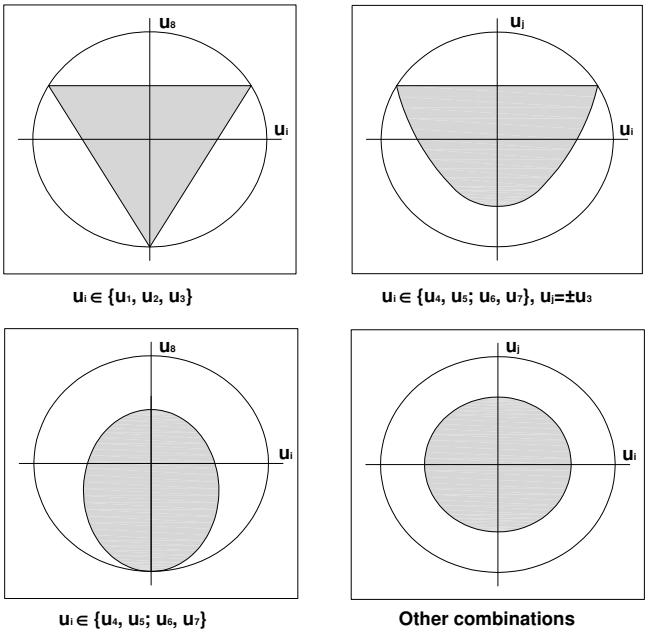

Fig. 2.2. Geometry of the 4-dimensional sphere for a qutrit represented as 2-dimensional sections: Large circles are sections of the ball $|\mathbf{n}| = \sqrt{3}/2$. Grey parts are the domain of the Bloch vector.

Orthonormal pure states have unit vectors $\mathbf{n}$ that satisfy $\mathbf{n} \cdot \mathbf{n}' = 1/2$, and are thus $2\pi/3$ apart. The states in an orthonormal basis have unit vectors that lie in a plane at the vertices of an equilateral triangle. The density operators that are diagonal in the orthonormal basis are the operators on the triangle or in its interior.

Sometimes it is useful to work with the operators

$$\hat{S}^{ij} = |j\rangle\langle i|, \tag{2.70}$$

which can be expressed in terms of the Gellmann matrices as

$$\begin{aligned}
\hat{S}^{21} &= |1\rangle\langle 2| = \frac{1}{2}(\hat{\lambda}_1 + i\hat{\lambda}_2), \qquad & \hat{S}^{12} &= |2\rangle\langle 1| = \frac{1}{2}(\hat{\lambda}_1 - i\hat{\lambda}_2), \\
\hat{S}^{31} &= |1\rangle\langle 3| = \frac{1}{2}(\hat{\lambda}_4 + i\hat{\lambda}_5), \qquad & \hat{S}^{13} &= |3\rangle\langle 1| = \frac{1}{2}(\hat{\lambda}_4 - i\hat{\lambda}_5), \\
\hat{S}^{32} &= |2\rangle\langle 3| = \frac{1}{2}(\hat{\lambda}_6 + i\hat{\lambda}_7), \qquad & \hat{S}^{23} &= |3\rangle\langle 2| = \frac{1}{2}(\hat{\lambda}_6 - i\hat{\lambda}_7).
\end{aligned} \tag{2.71}$$

and their conmutation are

$$[\hat{S}^{ij}, \hat{S}^{kl}] = \delta_{il}\hat{S}^{kj} - \delta_{kj}\hat{S}^{il}. \tag{2.72}$$

Obviously, the three diagonal operators $\hat{S}^{ii}$ measure *populations*, while the off-diagonal ladder operators $\hat{S}^{ij}$ represent *transitions* from level i to j. To emphasize this idea, one can alternatively define raising and lowering operators by

$$\begin{cases} \hat{S}_+^{ij} = \hat{S}^{ij} & \text{if } \ j > i, \\ \\ \hat{S}_-^{ji} = \hat{S}^{ij} & \text{if } \ j < i, \end{cases} \tag{2.73}$$

in complete analogy with the operators $\hat{\sigma}_\pm$ for the qubit.

Because of the trivial constraint $\hat{S}^{11} + \hat{S}^{22} + \hat{S}^{33} = \hat{\mathbb{1}}$, only two populations can vary independently. For this reason, it is customary to introduce two independent traceless operators [which constitute the maximal Abelian or Cartan subalgebra of su(3)]

$$\hat{S}_z^{12} = \frac{1}{2}(\hat{S}^{22} - \hat{S}^{11}), \qquad \hat{S}_z^{23} = \frac{1}{2}(\hat{S}^{33} - \hat{S}^{22}), \tag{2.74}$$

that measure inversions between the corresponding levels. In atomic systems, the selection rules usually rule out one of the transitions and therefore the two independent inversions are automatically fixed. For a general qutrit, these inversions can be arbitrarily chosen.

2.2.2 Phases for a qutrit

We expect the operators $\hat{S}_z^{12}$ and $\hat{S}_z^{23}$ to be conjugate to the corresponding (independent) phases of the qutrit. Note that $(\hat{S}_\pm^{12}, \hat{S}_z^{12})$ and $(\hat{S}_\pm^{23}, \hat{S}_z^{23})$ correspond to the two qubits $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ that exist in the qutrit. However, these two qubits are not independent, since equation (2.74) imposes highly nontrivial coupling between them. At the operator level, in view of (2.24) it seems appropriate to define

$$\hat{S}_-^{12} = \sqrt{\hat{S}_-^{12}\hat{S}_+^{12}}\ \hat{E}^{12}. \tag{2.75}$$

Here $\hat{E}^{12} = \exp(i\hat{\phi}^{12})$, $\hat{\phi}^{12}$ being the Hermitian operator representing the phase of the qubit $1 \leftrightarrow 2$. The unitary solution of equation (2.75) is given, up to an overall phase, by

$$\hat{E}^{12} = |1\rangle\langle 2| + e^{i\varphi_0^{12}}\ |2\rangle\langle 1| - e^{-i\varphi_0^{12}}\ |3\rangle\langle 3|, \tag{2.76}$$

where the undefined factor $e^{i\varphi_0^{12}}$ appears due to the unitarity requirement of $\hat{E}^{12}$. For the sake of concreteness, we can make again a definite choice: the complex conjugation of the wavefunction reverse the sign of $\hat{\phi}^{12}$, which immediately leads to the condition $e^{i\varphi_0^{12}} = -1$. We conclude then that a unitary phase operator that preserves the polar decomposition of equation (2.76) can be represented as

$$\hat{E}^{12} = |1\rangle\langle 2| - |2\rangle\langle 1| + |3\rangle\langle 3|. \tag{2.77}$$

The eigenstates of $\hat{\phi}^{12}$ are

$$\begin{aligned} |\phi_0^{12}\rangle &= |3\rangle, \\ |\phi_\pm^{12}\rangle &= \frac{1}{\sqrt{2}}(|2\rangle \pm i|1\rangle), \end{aligned} \tag{2.78}$$

with eigenvalues of 0 and $\pm\pi/2$, respectively. This is a remarkable result. It shows that the eigenvectors $|\phi_\pm^{12}\rangle$ look like the standard ones for a qubit. However, the "spectator" level $|3\rangle$ is an eigenstate of this operator, which introduces drastic changes. In other words, the phase of the qubit $1 \leftrightarrow 2$ "feels" the state $|3\rangle$.

An analogous reasoning for the transition $2 \leftrightarrow 3$ gives the corresponding operator $\hat{E}^{23}$

$$\hat{E}^{23} = |2\rangle\langle 3| - |3\rangle\langle 2| + |1\rangle\langle 1|, \tag{2.79}$$

with eigenvectors

$$\begin{aligned} |\phi_0^{23}\rangle &= |1\rangle, \\ |\phi_\pm^{23}\rangle &= \frac{1}{\sqrt{2}}(|3\rangle \pm i|2\rangle), \end{aligned} \tag{2.80}$$

and the same spectrum as before.

Finally, for the operator $\hat{E}^{13}$ one must be careful, because it connects the lowest to the highest vector. In fact, the polar decomposition in this case gives as a unitary solution

$$\hat{E}^{13} = a|3\rangle\langle 2| - b^*|3\rangle\langle 1| + b|2\rangle\langle 2| - a^*|2\rangle\langle 1| + |1\rangle\langle 3|, \tag{2.81}$$

with the condition $|a|^2 + |b|^2 = 1$. There are also nonunitary solutions to the polar decomposition, but they lack the interest to describe a phase observable in our context.

On physical grounds, we argue that the state $|2\rangle$ should be a "spectator" for the transition $1 \leftrightarrow 3$. Thus we impose $\hat{E}^{13}|2\rangle \propto |2\rangle$, which is only possible if $a = 0$ and we have that

$$\hat{E}^{13} = |1\rangle\langle 3| - |3\rangle\langle 1| + |2\rangle\langle 2|, \tag{2.82}$$

with eigenvectors

$$\begin{aligned} |\phi_0^{13}\rangle &= |2\rangle, \\ |\phi_\pm^{23}\rangle &= \frac{1}{\sqrt{2}}(|3\rangle \pm i|1\rangle). \end{aligned} \tag{2.83}$$

We observe that

$$\hat{E}^{12}\hat{E}^{23\dagger} \neq \hat{E}^{13}, \tag{2.84}$$

which clearly displays the quantum nature of this phase.

The same reasons for introducing a POVM for the qubit phases can be applied to the qutrit. If we take into account that

$$\begin{aligned} e^{i\phi\hat{S}_z^{12}} &= e^{-i\phi/2}|1\rangle\langle 1| + e^{i\phi/2}|2\rangle\langle 2| + |3\rangle\langle 3|, \\ e^{i\phi\hat{S}_z^{23}} &= e^{-i\phi/2}|2\rangle\langle 2| + e^{i\phi/2}|3\rangle\langle 3| + |1\rangle\langle 1|, \end{aligned} \tag{2.85}$$

and argue that phase-shift operators must be 2π periodic, we impose that any POVM $\hat{\Delta}(\phi_{12}, \phi_{23})$ for the qutrit should satisfy

$$\begin{aligned} e^{i2\phi\hat{S}_z^{12}}\ \hat{\Delta}(\phi_{12}, \phi_{23})\ e^{-i2\phi\hat{S}_z^{12}} &= \hat{\Delta}(\phi_{12} + \phi, \phi_{23}), \\ e^{i2\phi\hat{S}_z^{23}}\ \hat{\Delta}(\phi_{12}, \phi_{23})\ e^{-i2\phi\hat{S}_z^{23}} &= \hat{\Delta}(\phi_{12}, \phi_{23} + \phi). \end{aligned} \tag{2.86}$$

The general POVM fulfilling these requirements is of the form

$$\begin{aligned} \hat{\Delta}(\phi_{12}, \phi_{23}) = \frac{1}{(2\pi)^2}\{\hat{\mathbb{1}} + [\gamma_{12}e^{i(2\phi_{12}-\phi_{23})}|2\rangle\langle 1| + \gamma_{23}e^{i(2\phi_{23}-\phi_{12})}|3\rangle\langle 2| \\ + \gamma_{13}e^{i(\phi_{12}+\phi_{23})}|3\rangle\langle 1|] + \texttt{c.h.}\}, \end{aligned} \tag{2.87}$$

where h.c. denotes Hermitian conjugate, γ_{ij} are real numbers and ϕ_{ij} is the relative phase between states $|i\rangle$ and $|j\rangle$. If we choose the γ_{ij} different, say $\gamma_{12} = 1$ and the other two below the unity, then the expectation value of this POVM could reach the value zero for the superposition states $(|1\rangle + \exp(i\varphi)|2\rangle)/\sqrt{2}$. However, for superpositions of states $|1\rangle$ and $|3\rangle$ or $|2\rangle$ and $|3\rangle$, the expectation values of the POVM would always be greater than zero. Since there is no physical reason to assign special relevance to one specific superposition, we assume that the POVM must be symmetric with respect to the states, which leads to

$$\gamma \equiv \gamma_{12} = \gamma_{23} = \gamma_{13}. \tag{2.88}$$

Moreover, we make henceforth the choice $\gamma = 1$ because only for this choice can the POVM attain the expectation value zero for some particular state. In contrast with the result formulated in terms of operators, now there are only two relevant phases in the qutrit description: the third can be inferred from the other two, as in the classical description.

The proposed POVM provides qutrit phases where any values of ϕ_{12} and ϕ_{23} are allowed. However, note that the probability density induced by this POVM can be written as

$$P(\phi_{12},\phi_{23}) = \frac{1}{(2\pi)^2}\langle(1 + \{c_{12}\exp[i(2\phi_{12} - \phi_{23})] + c_{23}\exp[i(2\phi_{23} - \phi_{12})] + c_{13}\exp(i(\phi_{12} + \phi_{23}))\} + \texttt{c.c.}\rangle), \tag{2.89}$$

where $c_{ij} = \langle i|\hat{\varrho}|j\rangle\ \gamma$ and c.c. denotes complex conjugation. Therefore, this continuous range of variation is not effective in the sense that the values of $P(\phi_{12},\phi_{23})$ at every point (ϕ_{12},ϕ_{23}) are not independent, and we can find relations between them irrespective of the qutrit state. In other words, the complex parameters c_{ij} can be determined by the values of $P(\phi_{12},\phi_{23})$ at six points. Discreteness is also inevitably at the heart of the qutrit phase.

2.2.3 Complementarity for a qutrit

We can also analize the qutrit phase from the perspective of complementarity. To start, we consider the matrices

$$\hat{X} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \hat{Z} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.90}$$

with $\omega = \exp(2\pi i/3)$. Note that $\hat{Z}$ corresponds to $\hat{\sigma}_3$ with $\hat{Z}|j\rangle = \omega^j|j\rangle$, while $\hat{X}$ corresponds to $\hat{\sigma}_1$ with $\hat{Z}|j\rangle = |j+1\rangle$, where the operations must be understood $mod 2\pi/3$. These matrices obey the finite-dimensional version of the Weyl commutation relations:

$$\hat{Z}\hat{X} = \omega\hat{X}\hat{Z}. \tag{2.91}$$

and we can generate four classes of disjoint traceless operators, each containing two commuting operators:

$$\begin{aligned} &\mathcal{A}_{\chi 0} = \{\hat{Z},\hat{Z}^2\}, \quad \mathcal{B}_{\chi 1} = \{\hat{X},\hat{X}^2\}, \\ &\mathcal{C}_{\chi 2} = \{\hat{X}\hat{Z},(\hat{X}\hat{Z})^2\}, \quad \mathcal{D}_{\chi 3} = \{\hat{X}\hat{Z}^2,(\hat{X}\hat{Z}^2)^2\}. \end{aligned} \tag{2.92}$$

The eigenvectors of the operators in each one of these classes form mutually unbiased bases. The two Cartan operators, each associated with the independent inversions

$$\hat{h}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{h}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.93}$$

where $\hat{h}_1$ and $\hat{h}_2$ (that correspond to $2\hat{S}_z^{23}$ and $2\hat{S}_z^{12}$ in equation (2.75), respectively) can be easily expressed as linear combinations of $\hat{Z}$ and $\hat{Z}^2$. Note that $\hat{X}$ and $\hat{X}^2$

correspond to two different physical situations: in the computational basis $\hat{X}$ acts as $\hat{X}|j\rangle = |j+1\rangle$, while $\hat{X}^2|j\rangle = |j+2\rangle$.

Thus, we have two families of commuting phase operators

$$\begin{aligned} \hat{E}_{\varphi_{01},\varphi_{02}} &= e^{i(\varphi_{01}\hat{h}_1+\varphi_{02}\hat{h}_2)}\ \hat{X}\ e^{-i(\varphi_{01}\hat{h}_1+\varphi_{02}\hat{h}_2)} \\ \hat{E}^2_{\varphi_{01},\varphi_{02}} &= e^{i(\varphi_{01}\hat{h}_1+\varphi_{02}\hat{h}_2)}\ \hat{X}^2\ e^{-i(\varphi_{01}\hat{h}_1+\varphi_{02}\hat{h}_2)}. \end{aligned} \tag{2.94}$$

The expressions for $\hat{E}$ is

$$\hat{E}_{\varphi_{01},\varphi_{02}} = \begin{pmatrix} 0 & e^{i(\varphi_{01}-2\varphi_{02})} & 0 \\ 0 & 0 & e^{-i(2\varphi_{01}-\varphi_{02})} \\ e^{i(\varphi_{01}+\varphi_{02})} & 0 & 0 \end{pmatrix}, \tag{2.95}$$

where, again, φ_{01} y φ_{02} are reference phases, and coincides in this particular case with $\hat{E}^2$.

The phase operator (2.77) calculated as a polar decomposition does not coincide with the operator obtained by complementarity. This is due to the fact the phase operators from a polar decomposition use the idea of transitions. Indeed, while the perspective in terms of polar decomposition transitions leads to phase operators that fulfil the requirements of complementarity only for that between pairs of states involved in each transition, the perspective of complementarity fulfill the requirements of complementarity in the overall Hilbert state.

3 Relative phase in atom-field interactions

The interaction of an atomic system with a radiation field is a keystone of quantum optics. Needless to say, it is impossible to obtain exact solutions to this problem and some approximations must be used; the most common being that the radiation field is quasimonochromatic and its frequency coincides almost exactly with one of the transition frequencies of the atoms (supposed identical and with no direct interaction between them).

The two-level atom is the natural consequence of this hypothesis. The Jaynes-Cummings model (JCM) consisting of a single-mode radiation field interacting with the two-level atom is one of the few exactly solvable quantum-mechanical models in quantum optics. In the framework of this model many nonclassical effects of atom-field interactions, such as Rabi oscillations, collapse-revival phenomena, or sub-Poissonian statistics and squeezing of the radiation field have been predicted. Generalizations of the JCM can be divided into two group. One possible generalization consists in taking into account the multi-atom system (for instance the so-called Dicke model). Another generalization, deals with adding other levels . For instance, the three-level atom has been introduced to study two-photon excitations, coherent population trapping, and two-photon lasers.

In the semiclassical version of these models, correlations are safely ignored and the field is interpreted to be a purely classical electric field (Cives-Esclop et al. 1999; Kumar and Mehta 1970; Stroud and Jaynes 1970). Such an approximation has proven to be very successful and has the virtue of reducing the problem to the exclusive knowledge of the atomic dynamics, which is studied in terms of the Bloch vector.

For some phenomena, such as spontaneous emission by a fully excited atomic system, the quantization of the field is required. Then, one must take care also of the evolution of the field amplitudes, but the atomic dynamics is still explained in terms of inversion and dipole quadratures.

The natural way of understanding the resonant behavior of these models is in terms of the relative phase between the field and the atomic dipole (Ashcroft and Mermin 1996). While the quantum quadratures are well the operator for this relative phase has resisted a quantum description. At this respect, we think that, in spite of its maturity and success, these models are apparently incomplete since it lacks a satisfactory

description in terms of this relative phase, indispensable to compare with the classical world.

When focusing on the relative phase between two subsystems, we have two ways of proceeding: One is to start from previous descriptions of the field and dipole phases and manage them until we get the probability distribution for their difference. Alternatively, we can focus directly on the relative phase variable, trying to define the corresponding operator without any previous assumption about either the field or dipole phase descriptions. We analyze this problem in great detail in this chapter.

In section 2.1 we analyze the interaction of two-level atoms with a quantum single-mode field and the role played by the relative phase in these systems. First we study the interaction of a two-level atom with quantum field and in the following we extend the study to a collection of identical two-level atoms. In section 2.2 we analize the interaction of a three-level atom with a quantum two-mode field considering, once again, the role played by the relative phase.

3.1 Two-level atoms interacting with a quantum field

3.1.1 Jaynes-Cummings model

The Jaynes-Cummings model describes the interaction of a two-level atom, with resonance frecuency ω_0, with a single-mode radiation field of frecuency ω in a cavity. In the electric-dipole and rotating wave approximation the Hamiltonian of this model can be expressed in the form (in units $\hbar = 1$)

$$\hat{H} = \omega\ \hat{a}^\dagger \hat{a} + \omega_0\ \hat{\sigma}_3 + g(\hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+), \tag{3.1}$$

where g is the atom-field coupling constant, and $\hat{\sigma}_+$, $\hat{\sigma}_-$ and $\hat{\sigma}_3$ are the Pauli operators acting on atomic states. The photon creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, with commutator $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbb{1}}$, act on photon number states $|n\rangle_{\rm f}$, eigenstates of the photon number operator $\hat{a}^\dagger \hat{a}$,

$$\hat{a}^\dagger\ |n\rangle_{\rm f} = \sqrt{n+1}|n+1\rangle_{\rm f}, \quad \hat{a}\ |n\rangle_{\rm f} = \sqrt{n}|n\rangle_{\rm f}. \tag{3.2}$$

The Hamiltonian (3.1) can be rewritten as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}, \tag{3.3}$$

with

$$\begin{aligned} \hat{H}_0 &= \omega\ \hat{N}, \\ \hat{H}_{\rm int} &= \Delta\ \hat{\sigma}_3 + g(\hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+), \end{aligned} \tag{3.4}$$

where $\hat{N} = \hat{a}^\dagger \hat{a} + \hat{\sigma}_3$ is the total excitation number, and $\Delta = \omega_0 - \omega$ is the detuning between the field and atom frecuencies.

It is straightforward to check that

$$[\hat{N}, \hat{H}] = 0, \tag{3.5}$$

so the problem can be can be diagonalized in the subspaces with fixed $\hat{N}$.

The allowed values for $\hat{N}$, when atoms are initially in the ground state, are $n - 1/2$ with $n = 0, 1, ..., \infty$. The corresponding subspaces $\mathcal{H}_n$ are spanned by the common eigenvectors of $\hat{\sigma}_3$ and $\hat{a}^\dagger \hat{a}$:
$\{|-, n\rangle \equiv |-\rangle \otimes |n\rangle_{\rm f}, |+, n-1\rangle \equiv |+\rangle \otimes |n-1\rangle_{\rm f}\}$ for $n > 0$ and $|-, 0\rangle \equiv |+\rangle \otimes |n\rangle_{\rm f}$ for $n = 0$, denoting by $|-\rangle$ and $|+\rangle$ the ground and excited levels of the atom.

For simplicity we restrict henceforth our attention to case of exact resonance between the atomic and the field frequency $\omega_0 = \omega \equiv \omega$. The eigenstates in each one of the subspaces $\mathcal{H}_n$ are

$$\begin{aligned} |\Psi_0^{(0)}\rangle &= |-, 0\rangle \quad n = 0, \\ |\Psi_\pm^{(n)}\rangle &= \frac{1}{\sqrt{2}}(|-, n\rangle \pm |+, n-1\rangle) \quad n > 0. \end{aligned} \tag{3.6}$$

and the respective eigenvalues

$$\begin{aligned} h_0^{(0)} &= -\omega/2, \\ h_\pm^{(n)} &= [(n - \omega/2)\omega \pm g\sqrt{n}]. \end{aligned} \tag{3.7}$$

A first interesting and simple example is the case where the initial state of the system is the product of the atom in its ground state and the field in a number state $|\Psi(0)\rangle = |-, n\rangle$ with $n > 0$. At later times t, the state is given by

$$|\Psi(t)\rangle = \cos\left(g\sqrt{n}\ t\right)\ |-, n\rangle - i\,\mathrm{sen}\left(g\sqrt{n}\ t\right)\ |+, n-1\rangle, \tag{3.8}$$

and the population inversion $\hat{W} = |+\rangle\langle+| - |-\rangle\langle-| = 2\ \hat{\sigma}_3$ is then

$$\langle \hat{W} \rangle = -\cos(2\ g\sqrt{n}\ t). \tag{3.9}$$

Other interesting example is when the initial state of the field is a coherent state $|\alpha\rangle$ and the atom is in its ground state. At later times t population inversion is given by

$$\langle \hat{W} \rangle = -\sum_{n=0}^{\infty} Q_n \cos(2\ g\sqrt{n}t) = \mathrm{Re}\left[\sum_{n=0}^{\infty} Q_n e^{-i2\ g\sqrt{n}t}\right]. \tag{3.10}$$

where Q_n being the Poissonian weighting factor of the coherent state.

Oscillations which occur with photon number states, do not persist indefinitely when the field is initially prepared in a coherent state. Instead, evolution displays collapses and revivals of these oscillations.

3.1.2 Relative-phase for the Jaynes-Cummings model

In the spirit of our previous chapter, we describe the atom-field relative phase in terms of a polar decomposition of the complex amplitudes. To this end, let us introduce the operators

$$\begin{gathered} \hat{X}_+ = \hat{a}\ \hat{\sigma}_+, \qquad \hat{X}_- = \hat{a}^\dagger\ \hat{\sigma}_-, \\ \hat{X}_3 = \hat{\sigma}_3. \end{gathered} \tag{3.11}$$

These operators maintain the first commutation relation of su(2) in (1.10), $[\hat{X}_3, \hat{X}_\pm] = \pm\hat{X}_\pm$, but the second one is modified in the following way:

$$[\hat{X}_-, \hat{X}_+] = P(\hat{X}_3), \tag{3.12}$$

where $P(\hat{X}_3)$ is a second-order polynomial of the operator $\hat{X}_3$. This is a typical example of a polynomial deformation of the algebra su(2). Without embarking us in mathematical subtleties, the essential point for our purposes here is that one can develop a theory in a very close analogy with the standard su(2) algebra, that we analyzed the in previous chapter. In particular, it is clear that the state $|-, n\rangle$ plays the role of a *ground state*, since

$$\hat{X}_-|-, n\rangle = 0. \tag{3.13}$$

Then, we can build invariant subspaces, as in the usual theory of the angular momentum, since

$$|+, n-1\rangle \propto\ \hat{X}_+|-, n\rangle. \tag{3.14}$$

One can check that

$$\hat{X}_+^2|-, n\rangle = 0, \tag{3.15}$$

confirming that the number of accessible states is 2.

In consequence, the whole space of the system can be split as the direct sum $\mathcal{H} = \oplus_{n=0}^{\infty}\mathcal{H}_n$ of subspaces invariant under the action of the operators $(\hat{X}_+, \hat{X}_-, \hat{X}_3)$, and each one of them having a fixed number of excitations. These independent subspaces do not overlap in the evolution, in such a way that if the initial state belongs to one of them, it will remain in that subspace for all the evolution.

In each one of these invariant subspaces the operator $\hat{X}_3$ is diagonal, while $\hat{X}_+$ and $\hat{X}_-$ are ladder operators represented by finite-dimensional matrices (in analogy with $\hat{\sigma}_\pm$). This suggests to introduce a polar decomposition in the form

$$\hat{X}_- = \sqrt{\hat{X}_+\hat{X}_-}\ \hat{E} \tag{3.16}$$

where the *radial* operator $\sqrt{\hat{X}_+\hat{X}_-}$ is diagonal in the basis $\{|-,n\rangle, |+,n-1\rangle\}$, and

$$[\hat{X}_3, \hat{E}] = \hat{E}. \tag{3.17}$$

Once the polar decomposition (3.16) has been solved in each one of these subspaces, obtaining the operators $\hat{E}^{(n)}$, the solution for the whole space is

$$\hat{E} = \sum_{n=0}^{\infty} \hat{E}^{(n)}, \tag{3.18}$$

from which a Hermitian relative-phase operator $\hat{\phi}$ can be defined as $\hat{E} = e^{i\hat{\phi}}$. The solutions are

$$\begin{aligned} \hat{E}^{(0)} &= |-,0\rangle\langle -,0|, \\ \hat{E}^{(n)} &= |-,n\rangle\langle +,n-1| - |+,n-1\rangle\langle -,n|, \end{aligned} \tag{3.19}$$

with eigenvectors

$$\begin{aligned} |\phi_0^{(0)}\rangle &= |-,0\rangle, \quad \text{for } n=0, \\ |\phi_\pm^{(n)}\rangle &= \frac{1}{2}(|-,n\rangle \pm i|+,n-1\rangle), \quad \text{for } n>0, \end{aligned} \tag{3.20}$$

and eigenvalues

$$\begin{aligned} \phi_0^{(0)} &= 0, \\ \phi_\pm^{(n)} &= \pm\frac{\pi}{2}. \quad \text{for } n>0, \end{aligned} \tag{3.21}$$

We have that $\hat{E}^{(n)\dagger} = \hat{E}^{(n)}$ for $n > 0$, $\hat{E}^{(0)\dagger} = \hat{E}^{(0)}$ for $n = 0$, and therefore $\cos\hat{\phi} = 0$ outside $\mathcal{H}_0$, and $\sin\hat{\phi} = 0$ for $\mathcal{H}_0$. Another striking feature of this result is that the relative phase can take only three values. This may be surprising since any value for the field phase seems allowed. The reasons for these behaviors are

the same as those discussed for the qubit phase. Because the operator splits into components acting on two-dimensional subspaces $\mathcal{H}_n$ (one dimensional for $\mathcal{H}_0$), the previous features can be ascribed to the particular dimension of the atomic space. This is supported by the fact that this operator behaves properly when considering classical limits for either the atom or the field.

Another relevant point is that $\hat{E}$ cannot be written as a product of phase exponentials for each system. This relative phase is not the difference of absolute phases, and it does not have the usual mathematical properties of a difference. This is not exclusive of this formalism, and it also arises in other relative-phase approaches (Torgerson and Mandel 1996).

For any state, the information one can reap using a measurement of some observable is given by the statistical distribution of the measurement outcomes. For the relative phase, it seems natural to define the probability distribution function of a state, described by the density matrix $\hat{\varrho}$, as

$$\begin{aligned} P(0,\phi_0^{(0)},t) &= \langle\phi_0^{(0)}|\hat{\varrho}(t)|\phi_0^{(0)}\rangle \\ P(n,\phi_\pm^{(n)},t) &= \langle\phi_\pm^{(n)}|\hat{\varrho}(t)|\phi_\pm^{(n)}\rangle. \end{aligned} \tag{3.22}$$

From it, we can derive the distribution for the relative phase as

$$P(\phi,t) = \sum_{n=0} P(n,\phi,t). \tag{3.23}$$

For the previous example, where the initial state of the system is the product of the atom in its ground state and the field in a number state $|\Psi(0)\rangle = |-,0\rangle$ with $n > 0$, the probability of finding the sistem with the relative phases 0, $-\pi/2$ or $\pi/2$, at later t times is

$$\begin{aligned} P_{n=0}(0,t) &= 0, \\ P_n(-\pi/2,t) = \cos^2\left(g\sqrt{n}t - \frac{\pi}{4}\right), &\qquad P_n(\pi/2,t) = \sin^2\left(g\sqrt{n}t - \frac{\pi}{4}\right). \end{aligned} \tag{3.24}$$

This gives the mean value

$$\langle\hat{E}\rangle = -i\sin(2\ g\sqrt{n}t), \tag{3.25}$$

and so $\langle\cos\hat{\phi}\rangle = 0$.

The close relation (simply a time translation) between the evolution of the relative phase (3.25) and the population inversion (3.9) of the previous example allows us to expect similar collapses and revivals for the relative phase as those experienced by the population inversion for the coherent state. When the initial state is $|-\rangle|\psi\rangle$, with $|\psi\rangle$ an arbitrary field state, we have that the mean value of the exponential is

$$\langle \hat{E} \rangle = Q_0 - i \sum_{n=1}^{\infty} Q_n \sin(2\ g\sqrt{n}t), \tag{3.26}$$

where Q_n is the field photon-number distribution. Equivalently we have

$$\langle \sin \hat{\phi} \rangle = \sum_{n=0}^{\infty} Q_n \sin(2\ g\sqrt{n}t) = \mathrm{Im}\left[\sum_{n=0}^{\infty} Q_n e^{-i2\ g\sqrt{n}t} \right], \tag{3.27}$$

and $\langle \cos \hat{\phi} \rangle = Q_0$. This relative-phase evolution can be compared with the population inversion (3.10)

Next, we outline a plausible physical interpretation of the similarity between relative phase and population inversion. The interaction Hamiltonian (3.4) is proportional to $\cos\phi$. In classical terms, the dipole energy is maximum or minimum either when $\sin\phi = 0$ or when the field quadratures or the atomic dipole components vanish. In the quantum case, for the initial state $|-, n\rangle$, the population inversion has maximum or minimum values precisely when $\langle \cos \hat{\phi} \rangle = 0$. This relation holds very approximately when the initial state is of the field is a coherent state.

If the atom is initially in its ground state, the mean value of the atomic dipole operator $\hat{\mathbf{d}}$ vanishes and therefore so does the interaction Hamiltonian. Since in the resonant case the interaction Hamiltonian is a constant of the motion, we would expect $\cos\phi = 0$ and $\mathrm{sen}\,\phi = \pm 1$ at all later times. The relative phase is effectively uniform at $t = 0$ due to the randomness of the dipole phase. Due to the quantum fluctuations, the condition $\mathrm{sen}\,\phi = \pm 1$ cannot be established instantaneously, whereas this is possible classically. Nevertheless, the trend to satisfy this phase relation can be observed in the initial stages of the evolution, before the quantum evolution displays its complexity.

On the other hand, we have that $\langle \cos \hat{\phi} \rangle = Q_0$, and so $\arg\langle \hat{E} \rangle \approx \pm\pi/2$ will occur only provided that $\langle \sin \hat{\phi} \rangle \gg Q_0$.

Therefore expressions (1.27) and (1.28) show that the previously discussed relationship between relative phase and population inversion extends to the quantum case.

3.1.3 Dicke model

The Dicke model describes the interaction of a collection of A identical two-level atoms with a quantum single-mode field in a lossless cavity. The spatial dimensions of the atomic system are smaller than the wavelength of the field, so all the atoms feel the same field. The model neglects the dipole-dipole interaction between atoms (i.e., their wavefunctions do not overlap in the evolution).

The Hamiltonian for this model reads as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{int}}, \tag{3.28}$$

with

$$\hat{H}_0 = \omega\ \hat{N},$$
$$\hat{H}_{\rm int} = \Delta\ \hat{S}_3 + g\left(\hat{a}^\dagger \hat{S}_- + \hat{a}\hat{S}_+\right). \tag{3.29}$$

Here

$$\hat{N} = \hat{a}^\dagger \hat{a} + \hat{S}_3 \tag{3.30}$$

is the excitation number operator.

For simplicity we restrict again to the case of exact resonance between the atomic and the field frequencies $\omega_0 = \omega \equiv \omega$. Since the field mode is described in the usual Fock space $|n\rangle_{\rm f}$, the natural bare basis for the total system is $|n, m\rangle \equiv |n\rangle_{\rm f} \otimes |m\rangle_{\rm a}$. However, it is straightforward to check that

$$[\hat{H}_0, \hat{H}_{\rm int}] = 0, \tag{3.31}$$

so both are constants of motion. The Hamiltonian $\hat{H}_0$ (or, equivalently, the excitation number $\hat{N}$) determines the total energy stored by the radiation field and the atomic system, which is conserved by the interaction. This means that the appearance of m excited atoms requires the annihilation of m photons. This allows us to factor out $\exp(-i\hat{H}_0 t)$ from the evolution operator and drop it altogether. Hence, we can relabel the total basis as

$$|N, m\rangle \equiv |N - m\rangle_{\rm f} \otimes |m\rangle_{\rm a}. \tag{3.32}$$

With this terminology $|N, m-1\rangle$ means $|N-(m-1)\rangle_{\rm f} \otimes |m-1\rangle_{\rm a}$.

In such a basis, the interaction Hamiltonian, for a fixed value of N, is represented by the tridiagonal matrix

$$\hat{H}_{\rm int}^{(N)} = g \begin{pmatrix} 0 & h_0 & 0 & . & \dots \\ h_0 & 0 & h_1 & 0 & \dots \\ 0 & h_1 & 0 & h_2 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}, \tag{3.33}$$

with

$$h_m = \sqrt{(m+1)(N-m)(A-m)}. \tag{3.34}$$

The dimension of this matrix depends on whether $A > N$ or $A < N$, which are situations essentially different and must be handled separately.

Let us assume that $A > N$ and initially all the atoms are unexcited. Then, $m = 0$ and the conservation of the number of excitations implies that only the states (3.32) with $0 \le m \le N$ take part in the dynamics. Thus, the dimension of the subspace is $N + 1$.

On the contrary, when $A < N$ the number of initial photons is greater than the number of atoms and only the states (3.32) with $0 \leq m \leq A$ are involved in the evolution. The dimension is now $A + 1$

It is easy to check that, due to the properties of the tridiagonal matrices, the eigenvalues are distributed symmetrically with respect to zero, with one eigenvalue equal to zero if there are an odd number of them (Tanaś et al. 1991).

To find the state evolution we shall need the following matrix elements of the evolution operator

$$C^N_{m'm}(t) = \langle N, m'| \exp[-i\hat{H}^{(N)}_{\rm int}t]|N, m\rangle, \tag{3.35}$$

which can be written as

$$C^N_{m'm}(t) = \sum_{J=0}^{\mathcal{D}} U_{mJ}U^\dagger_{m'J} \exp[-i\varepsilon^{(N)}_J t], \tag{3.36}$$

where $\hat{U}$ is the unitary matrix that diagonalizes the Hamiltonian and $\varepsilon^{(N)}_I$ are the corresponding eigenvalues. In what follows we shall use the convention of denoting the dimension of the Hamiltonian matrix $\hat{H}^{(N)}_{\rm int}$ by $\mathcal{D} + 1$, that is,

$$\mathcal{D} = \min(N, A). \tag{3.37}$$

Now, let us assume that the initial field is taken to be in a coherent state $|\alpha\rangle_{\rm f}$ and that the atomic state is initially prepared in an atomic coherent state $|\zeta\rangle_{\rm a}$ (Arecchi et al. 1972; Perelomov 1986); i.e.,

$$|\Psi(0)\rangle = |\alpha\rangle_{\rm f} \otimes |\zeta\rangle_{\rm a}, \tag{3.38}$$

where

$$|\alpha\rangle_{\rm f} = \sum_n Q_n \, |n\rangle_{\rm f}, \tag{3.39}$$

Q_n being the Poissonian weighting factor of the coherent state (with zero phase) with mean number of photons $\bar{n}$

$$Q_n = \sqrt{e^{-\bar{n}}\frac{\bar{n}^n}{n!}}; \tag{3.40}$$

and

$$|\zeta\rangle_{\rm a} = \frac{1}{(1 + |\zeta|^2)^{A/2}} \sum_{m=0}^{A} \sqrt{\frac{A!}{m!(A-m)!}}\zeta^m |m\rangle_{\rm a} \equiv \sum_{m=0}^{A} A_m \, |m\rangle_{\rm a}, \tag{3.41}$$

where the parameter ζ is normally rewritten in terms of the spherical angles as

$$\zeta = -\tan(\vartheta/2)e^{-i\varphi}. \tag{3.42}$$

In other words, the initial state can be rewritten, taking into account (3.32), as

$$|\Psi(0)\rangle = \sum_{N,m} Q_{N-m} A_m |N, m\rangle. \tag{3.43}$$

With this initial condition the resulting state can be recast as

$$|\Psi(t)\rangle = \exp(-i\hat{H}_{\rm int}t)|\Psi(0)\rangle = \sum_{N=0}^{\infty} \sum_{m',m=0}^{\mathcal{D}} Q_{N-m} A_m \; C^N_{m'm}(t) \; |N, m'\rangle. \tag{3.44}$$

If the initial state is not of the same form, but it has a decomposition with different amplitudes A_m or Q_n, equation (3.44) is still valid when the appropriate coefficients are taken.

3.1.4 Relative-phase for the Dicke model

In the spirit of the previous sections, we shall describe the relative phase between a system of atoms and the field in terms of a polar decomposition of the complex amplitudes. To this end, we introduce the operators

$$\begin{gathered} \hat{X}_+ = \hat{a}\hat{S}_+, \qquad \hat{X}_- = \hat{a}^\dagger \hat{S}_-, \\ \hat{X}_3 = \hat{S}_3. \end{gathered} \tag{3.45}$$

These operators maintain the first commutation relation of su(2) in (1.10), $[\hat{X}_3, \hat{X}_\pm] = \pm\hat{X}_\pm$, but the second one is modified in the following way:

$$[\hat{X}_-, \hat{X}_+] = P(\hat{X}_3), \tag{3.46}$$

where $P(\hat{X}_3)$ represents a second-order polynomial function of the operator $\hat{X}_3$. This is again a polynomial deformation of the algebra su(2). Now the state $|N, 0\rangle$ plays the role of a *vacuum state*, since

$$\hat{X}_-|N, 0\rangle = 0. \tag{3.47}$$

Then, we can construct invariant subspaces, as in the usual theory of angular momentum, by

$$|N, m\rangle = \frac{1}{\mathcal{N}} \; \hat{X}_+^m |N, 0\rangle, \tag{3.48}$$

where $\mathcal{N}$ is a normalization constant. One can check that

$$\hat{X}_+^{\mathcal{D}+1}|N, 0\rangle = 0, \tag{3.49}$$

confirming that the number of accessible states is $\mathcal{D}+1$.

In consequence, the whole space of the system splits again as the direct sum $\mathcal{H}=\oplus_{N=0}^{\infty}\mathcal{H}_N$ of subspaces invariant under the action of the operators $(\hat{X}_+,\hat{X}_-,\hat{X}_3)$, and each one of them having a fixed number of excitations.

In each one of these invariant subspaces the operator $\hat{X}_3$ is diagonal, while $\hat{X}_+$ and $\hat{X}_-$ are ladder operators represented by finite-dimensional matrices. This suggests to introduce a polar decomposition in the form (3.16)

$$\hat{X}_-=\sqrt{\hat{X}_+\hat{X}_-}\;\hat{E} \tag{3.50}$$

We can guarantee now that the operator $\hat{E}=e^{i\hat{\phi}}$, representing the exponential of the relative phase, is unitary and commutes with the excitation number

$$\begin{gathered}\hat{E}\hat{E}^\dagger=\hat{E}^\dagger\hat{E}=\hat{\mathbb{1}},\\ [\hat{E},\hat{N}]=0.\end{gathered} \tag{3.51}$$

Thus, we may rather study its restriction to each invariant subspace $\mathcal{H}_N$, we shall denote by $\hat{E}^{(N)}$. It is easy to check that the action of the operator $\hat{E}^{(N)}$ in each subspace is given by

$$\begin{aligned}\hat{E}^{(N)}|N,m\rangle&=|N,m-1\rangle,\\ \hat{E}^{(N)\dagger}|N,m\rangle&=|N,m+1\rangle.\end{aligned} \tag{3.52}$$

Obviously, the action of $\hat{E}^{(N)}$ and $\hat{E}^{(N)\dagger}$ becomes undefined on the marginal states $|N,\mathcal{D}\rangle$ and $|N,0\rangle$. Therefore, it is necessary to add some conventions for closing the actions of these operators on the subspace $\mathcal{H}_N$. By analogy once again with the usual su(2) algebra, we shall use standard cyclic conditions and impose (up to global phase factors)

$$\begin{aligned}\hat{E}^{(N)}|N,0\rangle&=|N,\mathcal{D}\rangle,\\ \hat{E}^{(N)\dagger}|N,\mathcal{D}\rangle&=|N,0\rangle.\end{aligned} \tag{3.53}$$

With these conditions, the operator $\hat{E}^{(N)}$ can be expressed as

$$\begin{aligned}\hat{E}^{(N)}=&\sum_{m=0}^{\mathcal{D}}|N,m\rangle\langle N,m+1|\\ &+e^{i(\mathcal{D}+1)\phi^{(N)}}|N,\mathcal{D}\rangle\langle N,0|,\end{aligned} \tag{3.54}$$

$\phi^{(N)}$ being an arbitrary phase. Note that the crucial extra term in this equation, which establishes the unitarity of $\hat{E}^{(N)}$, is precisely based on the finite number of states. Therefore, in each invariant subspace $\mathcal{H}_N$ there are $\mathcal{D}+1$ orthonormal states satisfying

$$\hat{E}^{(N)}|\phi_r^{(N)}\rangle = e^{i\phi_r^{(N)}}|\phi_r^{(N)}\rangle, \tag{3.55}$$

with $r = 0, \ldots, \mathcal{D}$. These states can be expressed as

$$|\phi_r^{(N)}\rangle = \frac{1}{\sqrt{\mathcal{D}+1}} \sum_{m=0}^{\mathcal{D}} e^{im\phi_r^{(N)}}|N,m\rangle, \tag{3.56}$$

and, by taking the same 2π window in each subspace, we have

$$\phi_r^{(N)} = \phi_0 + \frac{2\pi r}{\mathcal{D}+1}, \tag{3.57}$$

and ϕ_0 is a fiducial or reference phase shift that can be arbitrarily chosen. The expression for $\hat{E}$ on the whole space is

$$\hat{E} = \sum_{N=0}^{\infty} \hat{E}^{(N)} = \sum_{N=0}^{\infty} \sum_{r=0}^{\mathcal{D}} |\phi_r^{(N)}\rangle \; e^{i\phi_r^{(N)}} \; \langle\phi_r^{(N)}|. \tag{3.58}$$

In the limit $\mathcal{D} \gg 1$, the spectrum becomes dense, as it might be expected. But, on the opposite limit, one may be surprised to find that the state $|0,0\rangle$ is a relative-phase eigenstate (with arbitrary eigenvalue ϕ_0). While this may provide a convincing argument that the theory is unreasonable, it is not the case. The value of ϕ_0 will not lead to any contradictions, because any choice will lead to a consistent theory. Our choice of this parameter says nothing about Nature, it only makes a statement about our individual preference (Björk, Trifonov, Tsegaye and Söderholm 1998; Trifonov et al. 2000). Note as well, that the relative-phase eigenstates are maximally entangled states. This has the consequence that the relative-phase operator has no classical correspondence in the general case, not even for highly excited states.

Once again, the probability distribution function for the relative phase of a state is described by the density matrix $\hat{\varrho}$, as

$$P(N,\phi_r,t) = \langle\phi_r^{(N)}|\hat{\varrho}(t)|\phi_r^{(N)}\rangle. \tag{3.59}$$

However, for *physical states* (Barnett and Pegg 1989) (i.e., states for which finite moments of the number operator are bounded) this expression will converge to a simpler form involving a continuous probability density we shall write as

$$P(N,\phi,t) = \langle\phi^{(N)}|\hat{\varrho}(t)|\phi^{(N)}\rangle, \tag{3.60}$$

where the vectors $|\phi^{(N)}\rangle$ defined in (3.56) lie in the subspace $\mathcal{H}_N$ with total number of excitations N. In fact, this expression can be interpreted as a joint probability distribution for the relative phase and the number of excitations. From it, we can derive the distribution for the relative phase as

$$P(\phi,t)=\sum_{N=0}^{\infty}P(N,\phi,t), \tag{3.61}$$

while

$$P(N,t)=\int_{2\pi}d\phi\ P(N,\phi,t) \tag{3.62}$$

can be viewed as the probability distribution of having N excitations in the system. These factorizations are an obvious consequence of the fact that the relative phase and the excitation number are compatible.

For a general initial state as expressed in equation (3.43), and the evolution given by (3.44) we have

$$P(N,\phi,t)=\frac{1}{2\pi}\left|\sum_{m',m=0}^{\mathcal{D}}Q_{N-m}A_m\ C^N_{m'm}(t)e^{im'\phi}\right|^2, \tag{3.63}$$

and then we arrive at the total relative-phase probability distribution:

$$P(\phi,t)=\frac{1}{2\pi}\sum_{N=0}^{\infty}\left|\sum_{m',m=0}^{\mathcal{D}}Q_{N-m}A_m\ C^N_{m'm}(t)e^{im'\phi}\right|^2. \tag{3.64}$$

This is our basic and compact result which we use to analyze the evolution of the phase properties of the Dicke model.

In figure 3.1 we have numerically evaluated this distribution $P(\phi,t)$ as a function of ϕ and the rescaled adimensional time $\tau=gt$, for the case when all the atoms are initially unexcited and the field is in a coherent state with various values of the mean number of photons $\bar{n}$. In all the cases, when $\tau=0$ we have that $C^N_{m'm}(0)=\delta_{m'm}$ and therefore

$$P(\phi,t=0)=\frac{1}{2\pi}\sum_{N=0}^{\infty}\left|\sum_{m=0}^{\mathcal{D}}Q_{N-m}A_m\ e^{im\phi}\right|^2. \tag{3.65}$$

In particular, when all the atoms are initially unexcited only the coefficient A_0 survives and the previous expression reduces to

$$P(\phi,t=0)=\frac{1}{2\pi}. \tag{3.66}$$

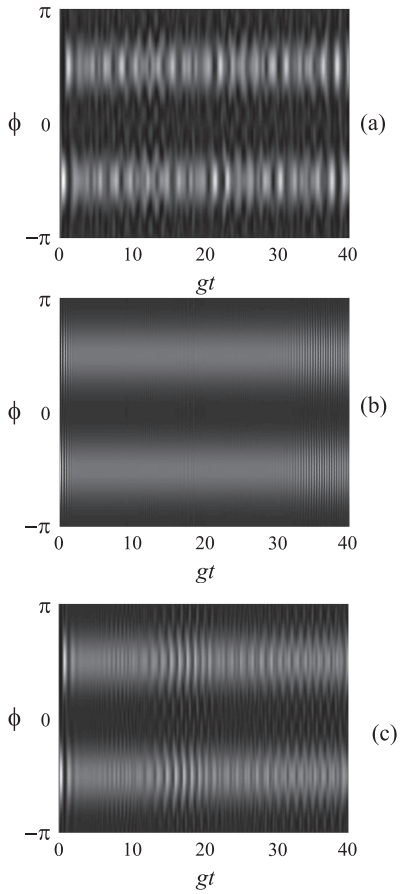

Fig. 3.1. Gray-level contour plot of the probability distribution $P(\phi, t)$ as a function of ϕ and the rescaled time gt for the case of $A = 5$ atoms initially unexcited and the field in a coherent state with the following values of the mean number of photons: a) $\bar{n} = 1$ (weak field), b) $\bar{n} = 50$ (strong field), and $\bar{n} = 5$ (intermediate field).

This flat distribution reflects the fact that the random phase of the dipole in such states induces a uniform distribution centred at ϕ_0. At this respect, it is interesting to notice that classically the Lorentz model at resonance predicts for the relative phase values of $\pm\pi/2$. It turns out that this is also a possible choice to fix the reference phase ϕ_0 in the quantum description. For simplicity, in all the graphics we have chosen ϕ_0 as the origin 0.

Two quite different behaviors are evident from these graphics. The first occurs in the weak-field region (Heidmann et al. 1985; Kozierowski et al. 1992, 1990) when the number of excitations in the system is much smaller than the number of the atoms, $N \ll A$. If, for simplicity, we assume that all the atoms are unexcited and the average number of photons in the initially coherent field is small, say $\bar{n} \sim 1$, then we can retain only the dominant terms in (3.64), getting

$$P(\phi,t) \simeq \frac{1}{2\pi}\{1+\bar{n}[|C^1_{00}(t)|^2+|C^1_{01}(t)|^2+2\mathrm{Re}(C^1_{00}(t)C^1_{01}(t)^* e^{i\phi})]\}e^{-\bar{n}}. \tag{3.67}$$

We see that, due to the periodic temporal dependence of the terms $C^N_{m'm}(t)$, this distribution is oscillatory for all times, which is corroborated numerically in figure 3.1.a.

The second (and perhaps more interesting) case corresponds to the strong-field region (Chumakov et al. 1994; Klimov and Chumakov 1995; Knight and Shore 1993; l. Drobn and Jex 1993; Retamal et al. 1997), when the initial number of photons is much larger than the number of atoms $A \ll N$. Then, following the ideas of (Chumakov et al. 1994; Klimov and Chumakov 1995; Retamal et al. 1997) one can show that the coefficients $C^N_{m'm}(t)$ can be approximated, up to order $A/\sqrt{\bar{n}}$, by

$$C^N_{m'm}(t) \simeq d^A_{m'm}(-\Omega_N t), \tag{3.68}$$

where

$$\Omega_N = 2g\sqrt{N-A/2+1/2}, \tag{3.69}$$

and $d^A_{m'm}$ are Wigner d functions,which are defined as the matrix elements for finite rotations by operators from SU(2) group representations

$$d^A_{m'm}(\vartheta) = d^A_{mm'}(\vartheta) = \langle m'|e^{i\vartheta\hat{S}_x}|m\rangle, \tag{3.70}$$

where $m, m' = 0, 1, \ldots, A$. The point now is that essentially only one subspace of dimension $A+1$ dominates the dynamics. Moreover, a calculation using the explicit form of these d functions, gives

$$P(\phi,t) = \frac{1}{2\pi}\sum_{N=0}^{\infty} Q_N^2 \left|\sum_{m=0}^{A}\sqrt{\frac{A!}{(A-m)!m!}}[\tan(\Omega_N t/2)]^m e^{im(\phi-\pi/2)}\right|^2 [\cos(\Omega_N t/2)]^{2A}, \tag{3.71}$$

where we have assumed that all the atoms are initially unexcited. When $A \gg 1$ and when oscillations are well resolved, one can perform an expansion of the square root getting

$$P(\phi,t) = \sqrt{\frac{A}{2\pi}}\sum_N \phi_N(t)\ \exp\left[-\frac{A}{2}(\phi-\pi/2+\delta_N)^2\right], \tag{3.72}$$

where $\phi_N(t)$ is a function of time of complicated structure that accounts for the collapses and revivals and that is of little interest for our purposes here, and $\delta_N =$

$\arg[\tan(\Omega_N t/2)]$. Now, it is clear that, since δ_N takes only the values 0 and π, the previous Gaussian distributions tend to have two peaks at $\phi = \pm\pi/2$, in agreement with the classical expectations. The presence of collapses and revivals are evident in figure 3.1.b, which confirms previous numerical and analytical evidence. The well-known nearly time-independent behavior in the time windows between collapse and revival is also clear. As we can see, the distribution tends to be randomized in the evolution, although keeping these two peaks at $\pm\pi/2$.

In the intermediate region, when $N \sim A$, the behavior is more complex, as shown in figure 3.1.c, and no analytical approximations are available.

For the particular case of the Jaynes-Cummings model one can diagonalize exactly the Hamiltonian in each subspace $\mathcal{H}_N$, obtaining the well-known dressed states (Cohen-Tannoudji et al. 1989), that turn to be trapping states (Gea-Banacloche 1991); i.e., the atomic population $\langle S_3(t)\rangle$ remains constant in spite of the existence of both the radiation field and atomic transitions (Cirac and Sánchez-Soto 1990). These states play a fundamental role, so it seems interesting to analyze the corresponding problem for the case of the Dicke model. In the strong-field limit one can make the replacement $a \to \alpha = \sqrt{\bar{n}}\, e^{i\vartheta}$ and the interaction Dicke Hamiltonian becomes proportional to the operator

$$\hat{H}_{\rm cl} = \left(e^{i\vartheta}\hat{S}_+ + e^{-i\vartheta}\hat{S}_-\right), \tag{3.73}$$

where the phase of the classical field has been chosen to coincide with the phase of the initial coherent state of the field. The semiclassical atomic states are defined now as eigenstates of $\hat{H}_{\rm cl}$ taking this phase as zero:

$$2\hat{S}_x|\underline{P}\rangle_{\rm a} = \Lambda_P|\underline{P}\rangle_{\rm a}, \tag{3.74}$$

with $\Lambda_P = A - 2P$ and $P = 0, 1, \ldots, A$.

Following (Chumakov et al. 1994; Klimov and Chumakov 1995; Retamal et al. 1997), we shall call *factorized states* those states for which the initial field is taken to be in a strong coherent state $|\alpha\rangle_{\rm f}$ and the atomic system is initially prepared in a semiclassical atomic state $|\underline{P}\rangle_{\rm a}$. For such states, the total wavefunction of the system can be approximately written as a product of its field and atomic parts

$$|\Psi(t)\rangle \simeq |\underline{P}(t)\rangle_{\rm a} \otimes |\alpha(t)\rangle_{\rm f} \tag{3.75}$$

with

$$\begin{aligned} |\underline{P}(t)\rangle_{\rm a} &= \exp\left[-i\frac{\Lambda_P(\hat{S}_3 + A/2)}{2\sqrt{\bar{n} - A/2 + 1/2}}gt\right]|\underline{P}\rangle_{\rm a} \\ |\alpha(t)\rangle_{\rm f} &= \exp\left[-i\Lambda_P\sqrt{a^\dagger a - A/2 + 1/2}\; gt\right]|\alpha\rangle_{\rm f}, \end{aligned} \tag{3.76}$$

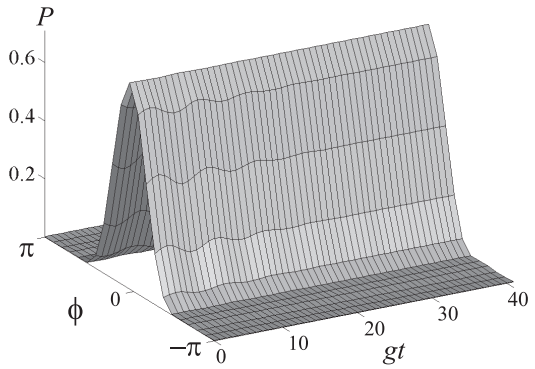

Fig. 3.2. Probability distribution function $P(\phi, t)$ as a function of ϕ and the rescaled time gt for the case of a factorized state with $A = 3$. The atomic coherent state has $\vartheta = \pi/2$ and $\varphi = 0$ and the field state has $\bar{n} = 20$.

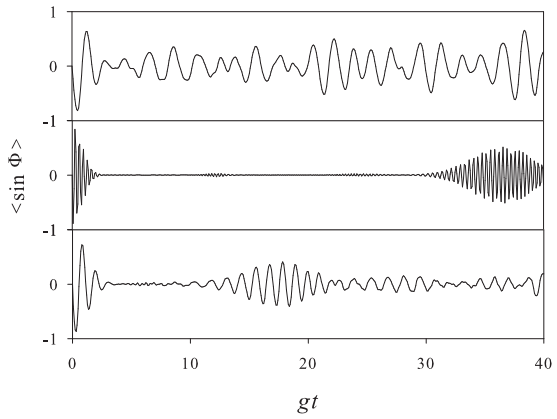

Fig. 3.3. Plots of $\langle \sin \phi \rangle$ versus gt for the same values of $\bar{n}$ as in figure 3.1 (from top to bottom).

and one can verify that they are also (approximately) trapping states.

For these states, one can find after a simple calculation,

$$P(\phi,t)=\frac{1}{A+1}\left|\sum_m {}_{\rm a}\langle m|\underline{P}\rangle_{\rm a} e^{im\phi}\right|^2 . \tag{3.77}$$

The probability distribution is time independent due to the factorization (3.75). From the arguments in (Chumakov et al. 1994; Klimov and Chumakov 1995; Retamal et al. 1997), one infers that this factorization holds up to times $gt\sim\sqrt{\bar{n}}$ (which can be very long times, in this limit) and with an accuracy in the coefficients of the order of $A/\sqrt{\bar{n}}$.

Moreover, using the properties of the semiclassical atomic states and assuming $A\gg 1$, one can replace the sum by an integral, obtaining finally

$$P(\phi,t)\simeq\sqrt{\frac{A}{2\pi}}\; e^{-A\phi^2/2}; \tag{3.78}$$

i.e., a Gaussian independent of time. In figure 3.2 we have plotted the probability distribution obtained from a numerical computation of equation (3.64), showing this quite remarkable behavior, except for the presence of very small (almost inappreciable) oscillations superimposed.

To gain more physical insight in these behaviors, in figure 3.3 we have plotted the evolution of the mean value of $\langle\sin\phi\rangle$ for various values of N, confirming the previous physical discussion.

To conclude, let us consider the Dicke model in the large-detuning limit; which is usually known as the dispersive limit. More specifically, we are in the case when

$$\Delta\gg g\sqrt{\bar{n}+1}A. \tag{3.79}$$

Then, following the procedure we shall develop in next chapter, the interaction Hamiltonian in equation (3.29) can be replaced by the effective Hamiltonian

$$\hat{H}_{\rm eff}=\Delta\;\hat{S}_3+\mu[\hat{S}_3^2-2(\hat{a}^\dagger\hat{a}+1)\hat{S}_3-\hat{C}], \tag{3.80}$$

where

$$\hat{C}=\frac{A}{2}\left(\frac{A}{2}+1\right)\hat{\mathbb{1}},\qquad \mu=\frac{\mu^2}{\Delta}. \tag{3.81}$$

The obvious advantage of this Hamiltonian is that it is diagonal and allows for a compact analytical expression for the coefficients $C^N_{m'm}(t)$ as

$$\begin{aligned}C^N_{m'm}(t)=\delta_{m'm}\exp\big(-it\{&\Delta(m-A/2)+\mu\,[2(N-m)+1]\,(m-A/2)\\&+\;\mu\left[C-(m-A/2)^2\right]\}\big).\end{aligned} \tag{3.82}$$

When the atoms are initially unexcited or excited (or, more generally, when $A_m=\delta_{mk}$) and for any initial state of field, we have

$$P(\phi, t) = \frac{1}{2\pi}, \tag{3.83}$$

for all the times.

For an arbitrary initial state of the atomic system and the field we get

$$P(\phi, t) = \frac{1}{2\pi} \sum_{N=0}^{\infty} \left| \sum_{m=0}^{A} Q_{N-m} A_m e^{-i f_m^N t} e^{im\phi} \right|^2 , \tag{3.84}$$

with

$$f_m^N = 2Nm\mu + [\Delta + \mu(2A+1)]m - 3\mu m^2. \tag{3.85}$$

Since (3.80) is quadratic in $\hat{S}_3$ and is, therefore, analogous to the Hamiltonian quadratic in the number operator of a single-mode field propagating through a Kerr medium, one could expect (Agarwal et al. 1997) that the evolution of coherent atomic states in the dispersive limit of the Dicke model leads to the generation of Schrödinger cat states. This superposition reaches the most pure form for initial number field states (in particular, the vacuum state minimizes the atomic entropy (Klimov and Saavedra 1998)).

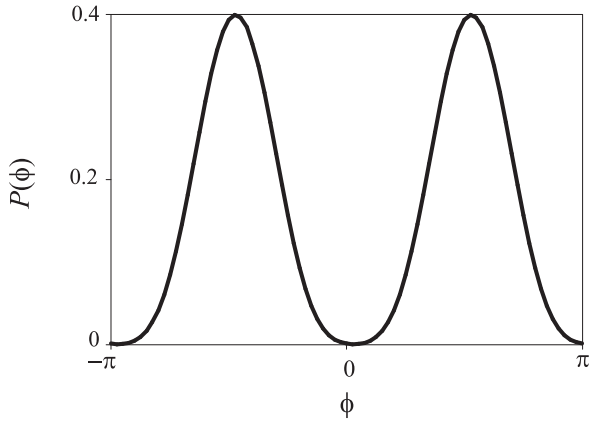

Fig. 3.4. Probability distribution function $P(\phi, t)$ as a function of ϕ for the time $\mu t = \pi/6$ for the case of an atomic coherent state ($A = 5$) with $\vartheta = \pi/2$ and $\varphi = 0$ and a field coherent state with $\bar{n} = 10$. The presence of the two humps corroborate the presence of a catlike state.

The situation with the relative-phase distribution is quite different. It is easy to see, for example, that if the field is prepared initially in a number state $|k\rangle_{\rm a}$ then $Q_n = \delta_{kn}$ and the relative phase distribution is flat. Nevertheless, for initial atomic

and field coherent states the relative-phase distribution splits for some special times into several humps. These catlike states, according to (3.85), appear at times $\tau = \mu t = \pi/6 \pmod{2\pi}$. To confirm analytically this behavior, we expand equation (3.84) when initially we have strong coherent states for both field and atoms, with $\bar{n} \gg A \gg 1$. By replacing once again the sum by integrals, one easily gets

$$P(\phi, t = \pi/6\lambda) = \sqrt{\frac{A}{8\pi}} \left\{ e^{-[\phi - \pi\phi_{\bar{n}}/(3\mu)]^2 A/2} + e^{-[\phi + \pi - \pi\phi_{\bar{n}}/(3\mu)]^2 A/2} \right\}, \tag{3.86}$$

where

$$\phi_{\bar{n}} = 2\bar{n}\mu + A + \mu(2A + 1), \tag{3.87}$$

and all the phases must be understood mod(2π). The two separated Gaussians indicates the presence of two humps and, therefore, the presence of catlike states. To further confirm this, in figure 3.4 we have computed numerically the distribution function $P(\phi, t = \pi/6\lambda)$ at the times predicted by the theory. The graphic clearly demonstrates the presence of the two-component state, according to our previous considerations.

3.2 Three-level atoms interacting with quantum fields

The natural way to continue is the analysis of the interaction of a three-level atom with a two-mode electromagnetic field.

3.2.1 Three-level atom coupled to a two-mode field

We wish to explore in some detail the phase properties of a three-level system. To be specific we shall consider a Λ configuration, as shown in figure 3.5, with energy levels $\omega_1 < \omega_2 < \omega_3$ and with allowed dipole transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, but not $1 \leftrightarrow 2$.

The Hamiltonian for this system as

$$\hat{H} = \hat{H}_{\rm a} + \hat{H}_{\rm f} + \hat{V}, \tag{3.88}$$

where

$$\begin{aligned} \hat{H}_{\rm a} &= \sum_i \omega_i \hat{S}^{ii}, \\ \hat{H}_{\rm f} &= \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b}, \\ \hat{V} &= g_a(\hat{a}\hat{S}_+^{13} + \hat{a}^\dagger \hat{S}_-^{13}) + g_b(\hat{b}\hat{S}_+^{23} + \hat{b}^\dagger \hat{S}_-^{23}). \end{aligned} \tag{3.89}$$

Here $\hat{H}_{\rm a}$ describes the dynamics of the free atom and $\hat{H}_{\rm f}$ represents the cavity modes of frequency ω_a and ω_b, with annihilation operators $\hat{a}$ and $\hat{b}$, respectively. Finally, in

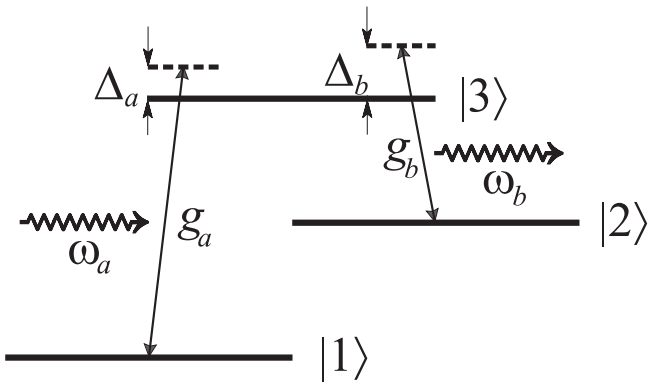

Fig. 3.5. The level scheme of a three-level Λ-type atom interacting with two single-mode quantum fields, coupling the two ground states to a common excited atomic state.

the interaction term $\hat{V}$, written in the dipole and rotating-wave approximations, we assume that the allowed transition $1 \leftrightarrow 3$ couples (quasi)resonantly to the mode a and the transition $2 \leftrightarrow 3$ couples to the mode b with coupling constants g_a and g_b that will be taken as real numbers.

The bare basis for the total system is $|i\rangle_{\rm a} \otimes |n_a, n_b\rangle_{\rm f}$, where $|n_a, n_b\rangle_{\rm f}$, is the usual two-mode Fock basis. However, one can check that the two excitation-number operators

$$\hat{N}_a = \hat{a}^\dagger \hat{a} - \hat{S}^{11} + \hat{\mathbb{1}}, \qquad \hat{N}_b = \hat{b}^\dagger \hat{b} - \hat{S}^{22} + \hat{\mathbb{1}}, \tag{3.90}$$

are conserved quantities. In consequence, we can rewrite the Hamiltonian (3.88) as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}, \tag{3.91}$$

with

$$\begin{aligned} \hat{H}_0 &= \omega_a \hat{N}_a + \omega_b \hat{N}_b + (\omega_3 - \omega_a - \omega_b)\hat{\mathbb{1}}, \\ \hat{H}_{\rm int} &= -\Delta_a \hat{S}^{11} - \Delta_b \hat{S}^{22} + g_a(\hat{a}\hat{S}^{13}_{+} + \hat{a}^\dagger \hat{S}^{13}_{-}) + g_b(\hat{b}\hat{S}^{23}_{+} + \hat{b}^\dagger \hat{S}^{23}_{-}). \end{aligned} \tag{3.92}$$

The detunings are defined as

$$\Delta_a = \omega_{31} - \omega_a, \qquad \Delta_b = \omega_{32} - \omega_b, \tag{3.93}$$

with $\omega_{ij} = \omega_i - \omega_j$. It is straightforward to check that

$$[\hat{H}_0, \hat{H}_{\rm int}] = 0. \tag{3.94}$$

Therefore, both the free Hamiltonian $\hat{H}_0$ and the interaction Hamiltonian $\hat{H}_{\rm int}$ are constants of motion. $\hat{H}_0$ determines the total energy stored in the system, which is conserved by the interaction. This allows us to factor out $\exp(-i\hat{H}_0 t)$ from the evolution operator and drop it altogether. Thus, the problem can be reduced to study the restriction of $\hat{H}_{\rm int}$ to each subspace $\mathcal{H}^{(N_a,N_b)}$ with fixed values of the pair of excitation numbers (N_a, N_b). In each one of these subspaces $\mathcal{H}^{(N_a,N_b)}$ there are three basis vectors that can be written as

$$|i; n_a = N_a - \mu_i, n_b = N_b - \nu_i\rangle, \tag{3.95}$$

where the values of μ_i and ν_i are defined as

$$(\mu_1, \mu_2, \mu_3) = (0, 1, 1), \qquad (\nu_1, \nu_2, \nu_3) = (1, 0, 1). \tag{3.96}$$

Note that when $N_a = 1$ and $N_b = 0$ or $N_a = 0$ and $N_b = 1$ some states may have negative photon occupation number and must be eliminated. In the subspace $\mathcal{H}^{(N_a,N_b)}$, $\hat{H}_{\rm int}$ is represented by the 3×3 matrix

$$\hat{H}_{\rm int}^{(N_a,N_b)} = \begin{pmatrix} 0 & g_b\sqrt{N_b} & g_a\sqrt{N_a} \\ g_b\sqrt{N_b} & -\Delta_b & 0 \\ g_a\sqrt{N_a} & 0 & -\Delta_a \end{pmatrix}. \tag{3.97}$$

Let us assume that at $t = 0$ the atomic wave function can be written as the superposition

$$|\Psi(0)\rangle = \sum_i c_i \; |i\rangle_{\rm a} \otimes |\alpha_a, \alpha_b\rangle_{\rm f} \tag{3.98}$$

with

$$|\alpha_a, \alpha_b\rangle = \sum_{n_a, n_b = 0}^{\infty} Q_{n_a} Q_{n_b} \; |n_a, n_b\rangle_{\rm f}. \tag{3.99}$$

At a later time t the state vector for the atom-field system in the interaction picture can be expressed as

$$|\Psi(t)\rangle = \sum_{N_a, N_b = 0}^{\infty} \sum_{i,j=1}^{3} Q_{N_a - \mu_j} \; Q_{N_b - \nu_j} \; c_j \mathcal{U}_{ij}^{(N_a,N_b)}(t) \; |i, N_a - \mu_i, N_b - \nu_i\rangle, \tag{3.100}$$

where $\mathcal{U}_{ij}^{(N_a,N_b)}(t)$ are the matrix elements of the evolution operator in the subspace $\mathcal{H}^{(N_a,N_b)}$

$$\mathcal{U}_{ij}^{(N_a,N_b)}(t) = \langle i; N_a - \mu_i, N_b - \nu_i | \exp[-i\hat{H}_{\rm int}^{(N_a,N_b)} t] | j; N_a - \mu_j, N_b - \nu_j\rangle, \tag{3.101}$$

which can be calculated exactly as can be seen, e.g. in (Yoo and Eberly 1985). This state describes completely the system evolution and will be the basis for our phase analysis in the following.

3.2.2 The role of atom-field relative phase

Our objetive is to describe the atom-field relative phase by resorting to a polar decomposition of the corresponding complex amplitudes. To this end, let us define the operators

$$\begin{aligned} \hat{X}_{+}^{13} &= \hat{a}\ \hat{S}_{+}^{13}, \qquad \hat{X}_{z}^{13} = \hat{S}_{z}^{13}; \\ \hat{X}_{+}^{23} &= \hat{b}\ \hat{S}_{+}^{23}, \qquad \hat{X}_{z}^{23} = \hat{S}_{z}^{23}. \end{aligned} \tag{3.102}$$

These operators satisfy most of the usual su(3) commutation relations with one of these recast as

$$[\hat{X}_{+}^{13}, \hat{X}_{-}^{23}] = -\hat{Y}_{+}^{12}, \tag{3.103}$$

where

$$\hat{Y}_{+}^{12} = -\hat{a}\hat{b}^{\dagger}\hat{S}_{+}^{12}. \tag{3.104}$$

However, some of them must be modified in the following way

$$\begin{aligned} [\hat{X}_{+}^{13}, \hat{X}_{-}^{13}] &= \hat{N}_a(\hat{\mathbb{1}} - 2\hat{S}^{11} - \hat{S}^{22}), \\ [\hat{X}_{+}^{23}, \hat{X}_{-}^{23}] &= \hat{N}_b(\hat{\mathbb{1}} - 2\hat{S}^{22} - \hat{S}^{11}), \\ [\hat{Y}_{+}^{12}, \hat{Y}_{-}^{12}] &= \hat{N}_a\hat{N}_b(\hat{S}^{11} - \hat{S}^{22}), \end{aligned} \tag{3.105}$$

which corresponds to a polynomial deformation of the algebra su(3).

For simplicity, let us focus first on the allowed transition $1 \leftrightarrow 3$ and notice that in every three-dimensional invariant subspace $\mathcal{H}^{(N_a,N_b)}$ the state $|1; N_a, N_b - 1\rangle$ plays the role of a *vacuum state* since

$$\hat{X}_{-}^{13}|1; N_a, N_b - 1\rangle = 0. \tag{3.106}$$

In this subspace the operator $\hat{X}_{z}^{13}$ is diagonal and we can work out again a polar decomposition similar to (1.76), namely

$$\hat{X}_{-}^{13} = \sqrt{\hat{X}_{-}^{13}\hat{X}_{+}^{13}}\ \hat{E}^{13}, \tag{3.107}$$

where the operator $\sqrt{\hat{X}_{-}^{13}\hat{X}_{+}^{13}}$ is diagonal and

$$\begin{aligned} \hat{E}^{13}\hat{E}^{13\dagger} &= \hat{E}^{13\dagger}\hat{E}^{13} = \hat{I}, \\ [\hat{E}^{13}, \hat{N}_a] &= [\hat{E}^{13}, \hat{N}_b] = 0. \end{aligned} \tag{3.108}$$

The first equation ensures that the operator $\hat{E}^{13} = e^{i\hat{\phi}^{13}}$, representing the exponential of the relative phase between the field and the dipole $1 \leftrightarrow 3$, is unitary. The second one guarantees that we may study its restriction to each invariant subspace $\mathcal{H}^{(N_a,N_b)}$.

Much in the same way as we did in chapter 1 [see equation (1.77)], the operator $\hat{E}^{13}$ solution of (3.107) can be expressed in $\mathcal{H}^{(N_a,N_b)}$ as

$$\begin{aligned}\hat{E}^{13} = {} & |1; N_a, N_b - 1\rangle\langle 3; N_a - 1, N_b - 1| - |3; N_a - 1, N_b - 1\rangle\langle 1; N_a, N_b - 1| \\ & + |2; N_a - 1, N_b\rangle\langle 2; N_a - 1, N_b|. \end{aligned} \tag{3.109}$$

As one would expect, $\hat{E}^{13}$ it acts as a ladder-like operator

$$\begin{aligned}\hat{E}^{13}|3; N_a - 1, N_b - 1\rangle = |1; N_a, N_b - 1\rangle, \\ \hat{E}^{13}|2; N_a - 1, N_b\rangle = |2; N_a - 1, N_b\rangle,\end{aligned} \tag{3.110}$$

and thus has eigenvectors

$$\begin{aligned}|\phi_0^{13}\rangle = |2; N_a - 1, N_b\rangle, \\ |\phi_\pm^{13}\rangle = \frac{1}{\sqrt{2}}(|3, N_a - 1, N_b - 1\rangle \pm i|1; N_a, N_b - 1\rangle),\end{aligned} \tag{3.111}$$

while the eigenvalues of $\hat{\phi}^{13}$ are 0 and $\pm\pi/2$, respectively.

Obviously, a similar reasoning for the transition $2 \leftrightarrow 3$ gives the corresponding operator $\hat{E}^{23}$ as

$$\begin{aligned}\hat{E}^{23} = {} & |2; N_a - 1, N_b\rangle\langle 3; N_a - 1, N_b - 1| - |3; N_a - 1, N_b - 1\rangle\langle 2; N_a - 1, N_b| \\ & + |1; N_a, N_b - 1\rangle\langle 1; N_a, N_b - 1|, \end{aligned} \tag{3.112}$$

with eigenvectors

$$\begin{aligned}|\phi_0^{23}\rangle = |1; N_a, N_b - 1\rangle, \\ |\phi_\pm^{23}\rangle = \frac{1}{\sqrt{2}}(|3, N_a - 1, N_b - 1\rangle \pm i|2; N_a - 1, N_b\rangle),\end{aligned} \tag{3.113}$$

and the same eigenvalues as before. The states (3.111) and (3.113) are the basis for our subsequent analysis of the dynamics of the relative phase.

3.2.3 Relative-phase distribution function

Once again, let us first focus on the relative phase between the field mode a and the dipole transition $1 \leftrightarrow 3$. According to (3.22) it seems natural to define the probability distribution function of a state described by the density matrix $\hat{\varrho}(t)$ as

$$P(N_a, N_b, \phi_r^{13}, t) = \mathrm{Tr}[\hat{\varrho}(t)\ |\phi_r^{13}\rangle\langle\phi_r^{13}|], \tag{3.114}$$

where the vectors $|\phi_r^{13}\rangle$ are given in equation (3.111) and the subscript r runs the three possible eigenvalues 0, and $\pm\pi/2$. This expression can be interpreted as a joint probability distribution for the relative phase and the excitation operators $\hat{N}_a$ and $\hat{N}_b$. From this function, we can derive the distribution for the relative phase as the marginal distribution

$$P(\phi_r^{13}, t) = \sum_{N_a, N_b=0}^{\infty} P(N_a, N_b, \phi_r^{13}, t). \tag{3.115}$$

For a general state as in equation (3.100), one has

$$P(N_a, N_b, \phi_r^{13}, t) = |\langle\phi_r^{13}|\Psi(t)\rangle|^2, \tag{3.116}$$

which, through direct calculation, gives

$$P(\phi_0^{13}, t) = \sum_{N_a, N_b=0}^{\infty} \left| \sum_{j=1}^{3} Q_{N_a-\mu_j} Q_{N_b-\nu_j}\ c_j\ \mathcal{U}_{2j}^{(N_a,N_b)}(t) \right|^2 , \tag{3.117}$$

$$P(\phi_\pm^{13}, t) = \sum_{N_a, N_b=0}^{\infty} \left| \sum_{j=1}^{3} Q_{N_a-\mu_j} Q_{N_b-\nu_j}\ c_j\ [\mathcal{U}_{3j}^{(N_a,N_b)}(t) \pm i\mathcal{U}_{1j}^{(N_a,N_b)}(t)] \right|^2 .$$

Much in the same way one also gets analogous results for the transition $2 \leftrightarrow 3$:

$$P(\phi_0^{23}, t) = \sum_{N_a, N_b=0}^{\infty} \left| \sum_{j=1}^{3} Q_{N_a-\mu_j} Q_{N_b-\nu_j}\ c_j\ \mathcal{U}_{1j}^{(N_a,N_b)}(t) \right|^2 , \tag{3.118}$$

$$P(\phi_\pm^{23}, t) = \sum_{N_a, N_b=0}^{\infty} \left| \sum_{j=1}^{3} Q_{N_a-\mu_j} Q_{N_b-\nu_j}\ c_j\ [\mathcal{U}_{3j}^{(N_a,N_b)}(t) \pm i\mathcal{U}_{2j}^{(N_a,N_b)}(t)] \right|^2 .$$

This is our basic and compact result to analyze the evolution of the relative phase.

We have numerically evaluated this distribution for the three allowed values of the relative phase for the case when the atom is initially in the ground state $|1\rangle$ and modes a and b are in a coherent state with a mean number of photons $\bar{n}_a$ and $\bar{n}_b$, respectively. For computational simplicity we have used the rescaled time

$$\tau = \frac{g_a t}{2\pi\sqrt{\bar{n}_a}}, \tag{3.119}$$

in all the plots and have assumed that $g_a = g_b$, which is not a serious restriction.

In figure 3.6 we have plotted a typical situation of a weak field, in which the number of excitations in the system is small, say $\bar{n}_a \sim \bar{n}_b \sim 1$. The pattern shows an almost oscillatory behavior, which can be easily understood if we retain only the two first terms in the sums over N_a and N_b and use the explicit form of the evolution operator $\mathcal{U}$. A relevant and general feature that is apparent from this figure is that the probabilities associated with ϕ_+^{ij} and ϕ_-^{ij} always oscillate out of phase, a point previously demonstrated for the case of the Jaynes-Cummings model (Luis and Sánchez-Soto 1997).

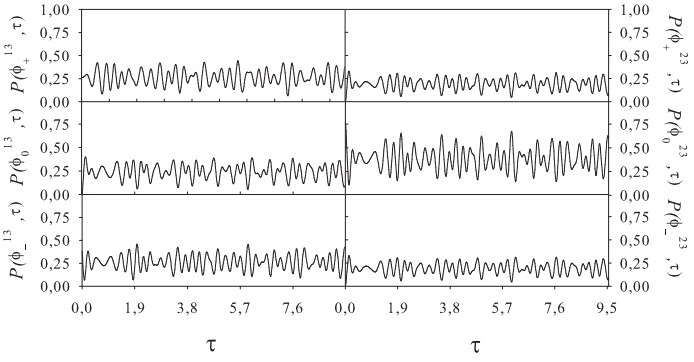

Fig. 3.6. The probability distribution function for the six allowed values of the relative phase as a function of the rescaled time τ in the case of a weak field with $\bar{n}_a = \bar{n}_b = 1$.

Perhaps more interesting is the case of strong-field dynamics, when the number of excitations in the system is large and so $\bar{n}_a$ or $\bar{n}_b$, or both, are large. In figure 3.7 we have plotted the relative-phase probabilities for $\bar{n}_a = 50$ and (a) $\bar{n}_b = 0.5$, and (b) $\bar{n}_b = 50$ photons, with the atom initially in the ground state $|1\rangle$. When $\bar{n}_b = 0.5$, the distribution $P(\phi_0^{13}, t)$ (which is the probability of finding the atom in the level $|2\rangle$) is almost negligible, while $P(\phi_\pm^{13}, t)$ show collapses and revivals. One may interpret this physically as follows: the transition $1 \leftrightarrow 3$ is so intense due to stimulated processes in mode a that there is no population transfer to level $|2\rangle$, which originates a regular oscillation of the dipole $1 \leftrightarrow 3$ with the corresponding collapses and revivals in the relative phase. The well-known (nearly) time-independent behavior in the time window between collapse and revival is also clear. The probability of finding the atom in the level $|1\rangle$, $P(\phi_0^{23}, t)$, tends to be $1/2$ (except at the revivals), which confirms that the transition $1 \leftrightarrow 3$ is almost saturated.

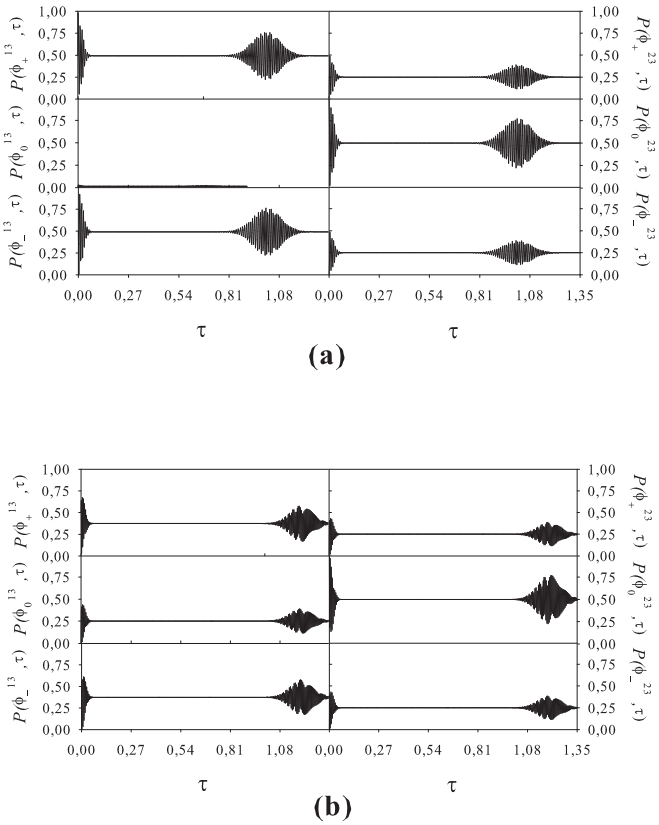

Fig. 3.7. The probability distribution function for the six allowed values of the relative phase as a function of the rescaled time τ in the strong-field limit with: a) $\bar{n}_a = 50$, $\bar{n}_b = 0.5$ and b) $\bar{n}_a = 50$, $\bar{n}_b = 50$.

When $\bar{n}_b$ grows, the position of the collapses and revivals changes, according to standard estimates (Yoo and Eberly 1985). When $\bar{n}_a = \bar{n}_b = 50$, $P(\phi_0^{13}, t)$ is centered at 1/4, while $P(\phi_0^{23}, t)$ is centered at 1/2, showing that the populations tend to be equidistributed because now the transition $2 \leftrightarrow 3$ is almost saturated too.

Very interesting physical phenomena arise when one considers coherent superpositions of atomic states, because it is then possible to cancel absorption or emission under certain conditions, i.e., the atom is effectively transparent to the incident field even in the presence of resonant transitions. A semiclassical analysis (Scully and Zubairy 1999), in which the fields are treated as c-numbers and described by the complex

Rabi frequencies $\Omega_a e^{-i\vartheta_a}$ and $\Omega_b e^{-i\vartheta_b}$ (note that ϑ_a and ϑ_b are the 'phases' of the respective fields), easily shows that when the initial atomic state is a superposition of the two lower levels of the form

$$|\Psi(0)\rangle_{\rm a} = \frac{1}{\sqrt{2}}(|1\rangle + e^{i\varphi}\ |2\rangle), \tag{3.120}$$

coherent trapping occurs whenever

$$\Omega_a = \Omega_b, \qquad \vartheta_a - \vartheta_b - \varphi = \pm\pi. \tag{3.121}$$

In other words, when these conditions are fulfilled the population is trapped in the lower states and there is no absorption.

To corroborate this behavior valid in the strong-field limit, let us note that, although the transition $1 \leftrightarrow 2$ is dynamically forbidden, one can still define phase eigenvectors for it:

$$\begin{aligned} |\phi_0^{12}\rangle &= |3, N_a - 1, N_b - 1\rangle, \\ |\phi_\pm^{12}\rangle &= \frac{1}{\sqrt{2}}(|2; N_a - 1, N_b\rangle \pm i|1; N_a, N_b - 1\rangle). \end{aligned} \tag{3.122}$$

In figure 3.8 we have plotted the probabilities $P(\phi_r^{12}, t)$ when the atom is initially in a trapped state like (3.120) with $\bar{n}_a = \bar{n}_b = 50$. We see that $P(\phi_0^{12}, t)$, which is the probability of finding the atom in the upper state, shows the remarkable behavior of trapping, except for the presence of very small superimposed oscillations. Note also that $P(\phi_\pm^{12}, t)$ also shows the same kind of behavior.

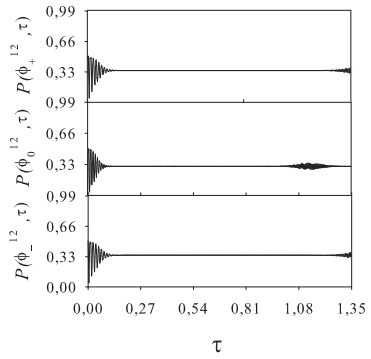

Fig. 3.8. The probability distribution function for the allowed values of the relative phase $\hat{\phi}^{12}$ as a function of the rescaled time τ for a trapped state with $\bar{n}_a = \bar{n}_b = 50$.

4 Small rotations and effective Hamiltonians

In most nonlinear optical phenomena, the fields and the atomic transitions are usually far from resonance (Schenzle 1981). In fact, no appreciable population redistribution is brought about the irradiation even of intense fields. The atomic polarization is then a fast variable controlled by the slow motion of the electromagnetic field amplitudes and follows its evolution adiabatically. Under this assumption we can eliminate the atomic degrees of freedom and are left with a small number of equations for the fields alone. These equations appear as suffering a nonlinear field-field interaction and can consequently be reinterpreted as the Heisenberg equations of motion for the field operators under the dynamics of an effective nonlinear Hamiltonian.

However, this adiabatic elimination (that was first used to study the parametric oscillator (Graham 1968; Graham and Haken 1968)) presents some drawbacks. First, it does not provide a general prescription for finding effective Hamiltonians, since the particular details strongly depend on the model considered. Second, it could become very cumbersome, and explicit but enormously complicated expressions for the different orders of approximation can be found in many original publications (Bloembergen 1990; Shen 1985). Third, and even worse, the procedure is not uniquely defined: depending on the term eliminated, the final Hamiltonian could be different (Klimov et al. 1999; Lugiato et al. 1990; Puri and Bullough 1988).

In consequence, it seems pertinent to find a setting to support this usual approach of effective Hamiltonians. To this end, we first note that most of the effective Hamiltonians in quantum optics contain cubic or higher terms in creation and annihilation operators. Among others, typical examples are kth harmonic generation, k-wave mixing, and generalized Dicke models. The key point for our purposes is the recent observation that the common mathematical structure underlying all these cases is a polynomial deformation of su(2), which arises as the dynamical symmetry algebra of the corresponding Hamiltonian. This nonlinear algebra has recently found an important place in quantum optics (Debergh 1997; Delgado et al. 2000; Karassiov 1992, 1994; Sunilkumar et al. 2000) because it allows us to handle the problems in very close analogy with the usual treatment of an angular momentum.

In (Klimov and Sánchez-Soto 2000) a Lie-like was devised method (Steinberg 1987) that allows one to get approximately effective diagonal Hamiltonians, provided the existence of this nonlinear su(2) dynamical algebra.

Since the su(3) algebra is the natural extension of su(2) three-level systems, when three-level systems interact with quantum fields, a nonlinear deformation of su(3) naturally emerges. In this chapter we apply our method to some nontrivial examples: a collection of two-level atoms dispersively interacting with a quantum field in a cavity; and Ξ and Λ configurations of three-level systems interacting with fields under different resonance conditions. Finally, Hamiltonians for multilevel systems under N-photon resonance conditions.

4.1 Effective Hamiltonians for nonlinear su(2) dynamics

4.1.1 Motivation of the method

To keep the discussion as self-contained as possible and to introduce the physical ideas underlying the method, let us start with the very simple example of a particle of spin s in a magnetic field. The Hamiltonian for this system has the following form

$$\hat{H} = \omega \hat{S}_3 + g(\hat{S}_+ + \hat{S}_-), \tag{4.1}$$

where g is the coupling constant and the operators $\hat{S}_3$, $\hat{S}_+$, and $\hat{S}_-$ constitute a $(2s+1)$-dimensional representation of the su(2) algebra, obeying the usual commutation relations.

The Hamiltonian (4.1) belongs to the class of the so-called linear Hamiltonians (Peřinova and Lukš 1994) and admits an exact solution. A convenient way of working out the solution is to apply the rotation

$$\hat{U} = \exp\left[\alpha(\hat{S}_+ - \hat{S}_-)\right], \tag{4.2}$$

and recalling that $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \ldots$, the rotated Hamiltonian, which is unitarily equivalent to the original one, becomes

$$\hat{H} = \hat{U}\hat{H}\hat{U}^\dagger = [\omega\cos(2\alpha) + 2g\sin(2\alpha)]\hat{S}_3 + \frac{1}{2}[2g\cos(2\alpha) - \omega\sin(2\alpha)](\hat{S}_+ + \hat{S}_-). \tag{4.3}$$

Now, the central idea is to choose the parameter α so as to cancel the nondiagonal terms appearing in equation (4.3). This can be accomplished by taking

$$\tan(2\alpha) = \frac{2g}{\omega}, \tag{4.4}$$

and the transformed Hamiltonian reduces then to

$$\hat{H}_{\text{eff}} = \omega\sqrt{1+\frac{4g^2}{\omega^2}}\ \hat{S}_3. \tag{4.5}$$

Since this effective Hamiltonian is diagonal in the angular momentum basis, the dynamical problem is completely solved. The crucial observation is that when $g \ll \omega$ we can approximate equation (4.4) by $\alpha \simeq g/\omega$, and then equation (4.2) can be substituted by

$$\hat{U} \simeq \exp\left[\frac{g}{\omega}(\hat{S}_+ - \hat{S}_-)\right]. \tag{4.6}$$

This small rotation approximately (i.e., up to second-order terms in g/ω) diagonalizes the Hamiltonian (4.1), giving rise to

$$\hat{H}_{\text{eff}} = \hat{U}\hat{H}\hat{U}^\dagger \simeq \left(\omega + 2\frac{g^2}{\omega}\right)\hat{S}_3 \tag{4.7}$$

which obviously coincides with the exact solution (4.5) after expanding in a series of g^2/ω^2. A direct application of the standard time-independent perturbation theory (Cohen-Tannoudji et al. 1992) to equation (4.1) leads immediately to the same eigenvalues and eigenstates than (4.7) in the same order of approximation. However, we stress that our method is fully operatorial and avoids the tedious work of computing the successive corrections as sums over all the accessible states.

4.1.2 Nonlinear small rotations

Having in mind the previous simple example, let us go one step further by treating the more general case of a system that admits some integrals of motion $\hat{N}_j$ and whose interaction Hamiltonian can be written as

$$\hat{H}_{\text{int}} = \Delta\ \hat{X}_3 + g(\hat{X}_+ + \hat{X}_-), \tag{4.8}$$

where g is the coupling constant and Δ is a parameter usually representing a detuning between frequencies of different subsystems (although it is not strictly necessary). The operators $\hat{X}_\pm$ and $\hat{X}_3$ maintain the first commutation relation of su(2), $[\hat{X}_3, \hat{X}_\pm] = \pm\hat{X}_\pm$, but the second one is modified in the following way

$$[\hat{X}_+, \hat{X}_-] = P(\hat{X}_3), \tag{4.9}$$

where $P(\hat{X}_3)$ is an arbitrary polynomial function of the diagonal operator $\hat{X}_3$ with coefficients perhaps depending on the integrals of motion $\hat{N}_j$. These commutation relations correspond once again to a polynomial deformation of su(2), as the ones treated in chapter 2.

Now, suppose that for some physical reasons (depending on the particular model under consideration) the condition

$$\varepsilon = \frac{g}{\Delta} \ll 1 \tag{4.10}$$

is fulfilled. Then, it is clear that (4.8) is *almost* diagonal in the basis that diagonalizes $\hat{X}_3$. In fact, a standard perturbation analysis immediately shows that the first-order corrections introduced by the nondiagonal part $g(\hat{X}_+ + \hat{X}_-)$ to the eigenvalues of $\hat{X}_3$ vanish and those of second order are proportional to ε. Thus, we apply the following unitary transformation to (4.8) (which, in fact, is a *small nonlinear rotation*)

$$\hat{U} = \exp[\varepsilon(\hat{X}_+ - \hat{X}_-)]. \tag{4.11}$$

After some calculations we get, up to order ε^2, the effective Hamiltonian

$$\hat{H}_{\text{eff}} = \Delta\ \hat{X}_3 + \frac{g^2}{\Delta} P(\hat{X}_3). \tag{4.12}$$

Then, the evolution (as well as the spectral) problem is completely solved in this approximation. Besides the advantage of having the effective Hamiltonian expressed in an operatorial form, the method has the virtue of generality, since it is valid for any model whose Hamiltonian could be written down in terms of the generators of a polynomial deformation of su(2).

Our technique also provides a valuable tool for obtaining corrections to the eigenstates of (4.8). Indeed, it is easy to realize that these eigenstates can be approximated by

$$|\Psi_m\rangle = \hat{U}^\dagger |m\rangle, \tag{4.13}$$

where $|m\rangle$ denotes an eigenstate of $\hat{X}_3$ and $\hat{U}$ is the corresponding small rotation. Since U and $|m\rangle$ do not depend on time, the operator $\hat{U}$ can be applied to $|m\rangle$ as an expansion in ε. For example, the eigenstate $|\Psi_m\rangle$ up to order ε^2 takes on the form

$$|\Psi_m\rangle = \left[1 - \varepsilon\ (\hat{X}_+ - \hat{X}_-) - \frac{\varepsilon^2}{2}\ (1 + 2\hat{X}_+\hat{X}_- - \hat{X}_+^2 - \hat{X}_-^2)\right] |m\rangle. \tag{4.14}$$

This representation is especially advantageous when the state space of the model is a representation space of the deformed su(2) algebra that is constructed in the usual way by the action of the raising operator $\hat{X}_+$; i.e., $|m\rangle \propto \hat{X}_+^m |0\rangle$, where $|0\rangle$ is a lowest weigth vector fulfilling the standard condition $\hat{X}_-|0\rangle = 0$.

This procedure shows that we can adiabatically eliminate all nonresonant transitions and work with an effective Hamiltonian containing only (quasi) resonant transitions. The effect of nonresonant terms reduces to a dynamical Stark shift (which can have a quite complicated form). Obviously, transformations generating effective Hamiltonians also change eigenfunctions, but the corrections are of order ε and do not depend on time. Such corrections correspond to low-amplitude transitions that take place in the case of nonresonant interactions. We shall elaborate on these topics in the next sections.

4.1.3 Dispersive limit of the Dicke model

As a relevant example, let us apply our method to the Dicke model, describing the interaction of a single-mode field of frequency ω with a collection of A identical two-level atoms with transition frequency ω_0. The model Hamiltonian is given in equation (2.2).

We assume now that the dispersive limit holds (Brune et al. 1996); i.e.,

$$|\Delta| \gg A\lambda\sqrt{\bar{n}+1}, \tag{4.15}$$

where $\bar{n}$ is the average number of photons in the field. If we apply our method, considering the deformed su(2) operators in equation (2.18), the small nonlinear rotation (4.11) transforms the interaction Hamiltonian (2.2) into

$$\hat{H}_{\rm eff} = \Delta\ \hat{S}_3 + \frac{\lambda^2}{\Delta}[\hat{S}_3^2 - 2(\hat{a}^\dagger\hat{a}+1)\hat{S}_3 - \hat{C}], \tag{4.16}$$

where

$$\hat{C} = \frac{A}{2}\left(\frac{A}{2}+1\right)\hat{\mathbb{1}}, \qquad \mu = \frac{\lambda^2}{\Delta}. \tag{4.17}$$

This effective Hamiltonian (4.16) was previously obtained in (Agarwal et al. 1997) by quite a different method (see also (Klimov and Saavedra 1998)).

4.2 Effective Hamiltonians for nonlinear su(3) dynamics

4.2.1 Three-level systems interacting with quantum fields

The method of small rotations can be applied to more complicated systems. In this section we focus on Hamiltonians that can be represented in terms of the su(3) algebra. This structure naturally arises when dealing with systems with three relevant levels. In this case three possible configurations (Ξ, V, and Λ) are admissible [see (Yoo and Eberly 1985) for details].

For definiteness, we shall consider the interaction of A identical three-level systems in a cascade or Ξ configuration (i.e., with associated energies $E_1 < E_2 < E_3$ and allowed dipole transitions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$, but not the $1 \leftrightarrow 3$) interacting with a single-mode quantum field of frequency ω. The Hamiltonian of this model is

$$\hat{H}_\Xi = \hat{H}_{\rm f} + \hat{H}_{\rm a} + \hat{H}_{\rm int}, \tag{4.18}$$

with

$$\begin{aligned}\hat{H}_{\rm f} &= \omega \hat{a}^\dagger \hat{a},\\ \hat{H}_{\rm a} &= E_1 \hat{S}^{11} + E_2 \hat{S}^{22} + E_3 \hat{S}^{33},\\ \hat{H}_{\rm int} &= g_{12}(\hat{a}\hat{S}_+^{12} + \hat{a}^\dagger \hat{S}_-^{12}) + g_{23}(\hat{a}\hat{S}_+^{23} + \hat{a}^\dagger \hat{S}_-^{23}).\end{aligned} \tag{4.19}$$

It is natural also to introduce the deformed su(3) algebra as

$$\begin{gathered}\hat{X}^{11} = \hat{S}^{11}, \qquad \hat{X}^{22} = \hat{S}^{22}, \qquad \hat{X}^{33} = \hat{S}^{33},\\ \hat{X}_+^{12} = \hat{a}\hat{S}_+^{12}, \qquad \hat{X}_+^{23} = \hat{a}\hat{S}_+^{23}.\end{gathered} \tag{4.20}$$

Then the Hamiltonian (4.18) can be recast as $\hat{H}_\Xi = \hat{H}_0 + \hat{H}_{\rm int}$, with

$$\begin{aligned}\hat{H}_0 &= \omega \hat{N}_\Xi,\\ \hat{H}_{\rm int} &= -\Delta_{12}\ \hat{X}^{11} + \Delta_{23}\ \hat{X}^{33} + g_{12}(\hat{X}_+^{12} + \hat{X}_-^{12}) + g_{23}(\hat{X}_+^{23} + \hat{X}_-^{23}),\end{aligned} \tag{4.21}$$

where we have used the conserved excitation number $\hat{N}_\Xi = \hat{a}^\dagger \hat{a} + \hat{S}^{33} - \hat{S}^{11}$ and the detunings are

$$\Delta_{12} = E_2 - E_1 - \omega, \qquad \Delta_{23} = E_3 - E_2 - \omega. \tag{4.22}$$

The operators $\hat{X}^{ij}$ satisfy the usual su(3) commutation relations (1.75), with some of them recast as

$$[\hat{X}_+^{12}, \hat{X}_+^{23}] = -\hat{Y}_+^{13}, \qquad [\hat{X}_-^{12}, \hat{X}_-^{23}] = \hat{Y}_-^{13}, \qquad [\hat{X}_+^{12}, \hat{X}_-^{23}] = 0, \tag{4.23}$$

where

$$\hat{Y}_+^{13} = \hat{a}^2 \hat{S}_+^{13}. \tag{4.24}$$

However, we have to modify some of them in the following way:

$$[\hat{X}_+^{ij}, \hat{X}_-^{ij}] = P(\hat{X}^{ii}, \hat{X}^{jj}), \qquad [\hat{Y}_+^{ij}, \hat{Y}_-^{ij}] = Q(\hat{X}^{ii}, \hat{X}^{jj}), \tag{4.25}$$

where $P(\hat{X}^{ii}, \hat{X}^{kk})$ and $Q(\hat{X}^{ii}, \hat{X}^{kk})$ are polynomials of the diagonal operators $\hat{X}^{ii}$ $(i = 1, 2, 3)$ and define, then, a polynomial deformation of su(3).

Effect of a far-off resonant level

The dynamics generated by su(3) is obviously richer than that of su(2), since a greater number of physical degrees of freedom are now available. To see how our method works, let us assume that one of the transitions, say the $2 \leftrightarrow 3$, is (quasi) resonant with the field; i.e.,

$$|\Delta_{23}| \ll A g_{23} \sqrt{\bar{n} + 1}, \tag{4.26}$$

while the transition $1 \leftrightarrow 2$ is far-off resonant

$$|\Delta_{12}| \gg A g_{12}\sqrt{\bar{n}+1}. \tag{4.27}$$

It is clear from our previous analysis that the small nonlinear rotation

$$\hat{U}_{12} = \exp[\varepsilon_{12}(\hat{X}_{+}^{12} - \hat{X}_{-}^{12})], \tag{4.28}$$

with $\varepsilon_{12} = g_{12}/\Delta_{12} \ll 1$, eliminates the interaction term $g_{12}(\hat{X}_{+}^{12} + \hat{X}_{-}^{12})$, representing the nonresonant transition $1 \leftrightarrow 2$. In this way, we obtain the effective Hamiltonian

$$\begin{aligned}\hat{H}_{\text{eff}} = {} & \Delta_{12}\ \hat{X}^{11} + \Delta_{23}\ \hat{X}^{33} + g_{23}(\hat{X}_{+}^{23} + \hat{X}_{-}^{23}) - \frac{g_{12}g_{23}}{\Delta_{12}}(\hat{Y}_{+}^{13} + \hat{Y}_{-}^{13}) \\ & + \frac{g_{12}^2}{\Delta_{12}} P(\hat{X}^{11}, \hat{X}^{22}) + \frac{g_{12}g_{23}}{\Delta_{12}}([\hat{X}_{+}^{12}, \hat{X}_{-}^{23}] + [\hat{X}_{+}^{23}, \hat{X}_{-}^{12}]).\end{aligned} \tag{4.29}$$

The remarkable point is that by eliminating the transition $1 \leftrightarrow 2$, we have generated an effective transition $1 \leftrightarrow 3$ (represented by the operators $\hat{Y}_{\pm}^{13}$), which was absent in the initial Hamiltonian. Nevertheless, this transition is also nonresonant due to conditions (4.26) and (4.27) and, accordingly, can be eliminated by the following transformation

$$\hat{U}_{13} = \exp[\varepsilon_{13}(\hat{Y}_{+}^{13} - \hat{Y}_{-}^{13})], \tag{4.30}$$

where the parameter ε_{13} must fulfill

$$\varepsilon_{13} = \frac{g_{12}g_{23}}{\Delta_{12}(\Delta_{12} + \Delta_{23})} \ll 1. \tag{4.31}$$

In this particular case the polynomial function $P(\hat{X}^{ii}, \hat{X}^{jj})$ is

$$P(\hat{X}^{11}, \hat{X}^{22}) = \hat{S}_{+}^{12}\hat{S}_{-}^{12} + \hat{a}^{\dagger}\hat{a}(\hat{S}^{22} - \hat{S}^{11}). \tag{4.32}$$

If initially level 1 is unpopulated, we have $\hat{S}^{11} = 0$ (which will be conserved, since there are no transitions to the level 1) and

$$P(\hat{X}^{11}, \hat{X}^{22}) = \hat{S}^{22}(\hat{a}^{\dagger}\hat{a} + 1). \tag{4.33}$$

In consequence, the effective Hamiltonian, taking into account the existence of a far-off resonant level, has the form

$$\hat{H}_{\text{eff}} = \Delta_{23}\ \hat{S}^{33} + g_{23}(\hat{a}\hat{S}_{+}^{23} + \hat{a}^{\dagger}\hat{S}_{-}^{23}) + \frac{g_{12}^2}{\Delta_{12}}\hat{S}^{22}(\hat{a}^{\dagger}\hat{a} + 1). \tag{4.34}$$

This means that the far-lying level produces a mark in the system in the form of a dynamical Stark-shift term. Due to this Stark shift, the initially nonresonant transition $2 \leftrightarrow 3$ becomes resonant in a subspace with some fixed photon number. This opens the possibility of separating n-photon field states from an initial coherent state interacting with a collection of three-level atoms just by choosing a suitable relation between the detunings and the interaction constants, and projecting to the second level at appropriate moments.

Two-photon resonance

Let us now envisage the different situation in which the two-photon resonance condition between levels 1 and 3 is fulfilled; i.e., $E_3 - E_1 = 2\omega$. This means that $\Delta_{12} = -\Delta_{23}$ and the transition generated by the operators $\hat{Y}_{\pm}^{13}$ cannot be removed. However, the term $g_{23}(\hat{X}_{+}^{23} + \hat{X}_{-}^{23})$, which generates (nonresonant) transitions between levels 2 and 3, can be eliminated by a transformation $\hat{U}_{23}$ analogous to (4.28) with $\varepsilon_{23} \ll 1$. The transformed Hamiltonian becomes then

$$\hat{H}_{\text{eff}} = -\Delta_{12}\ (\hat{X}^{11} + \hat{X}^{33}) - \frac{g_{12}g_{23}}{\Delta_{12}}(\hat{Y}_{+}^{13} + \hat{Y}_{-}^{13}) + \frac{g_{12}^2}{\Delta_{12}}P(\hat{X}^{11}, \hat{X}^{22}) - \frac{g_{23}^2}{\Delta_{12}}P(\hat{X}^{22}, \hat{X}^{33}), \tag{4.35}$$

where $P(\hat{X}^{11}, \hat{X}^{22})$ and $P(\hat{X}^{22}, \hat{X}^{33})$ are defined according to equation (4.32). Finally, if we further impose the absence of initial population in level 2, we obtain

$$\hat{H}_{\text{eff}} = \frac{g_{12}g_{23}}{\Delta_{12}}(\hat{a}^2\hat{S}_{+}^{13} + \hat{a}^{\dagger 2}\hat{S}_{-}^{13}) + (\hat{S}_3^{13} + A/2)\left[(g_{23}^2/\Delta_{12} - g_{12}^2/\Delta_{12})\hat{a}^\dagger\hat{a} + g_{23}^2/\Delta_{12}\right] + A\frac{g_{12}^2}{\Delta_{12}}\hat{a}^\dagger\hat{a}, \tag{4.36}$$

which is the effective two-photon Dicke Hamiltonian including the dynamical Stark shift obtained in (Puri and Bullough 1988) [see also (Klimov et al. 1999)].

4.2.2 The dispersive limit of the Λ configuration

Let us consider for a moment the case of the Λ configuration, in which the allowed dipole transitions are now $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, but not the $1 \leftrightarrow 2$. The Hamiltonian governing the evolution is still of the form (4.18), but now with

$$\hat{H}_{\text{int}} = g_{13}(\hat{a}\hat{S}_{+}^{13} + \hat{a}^\dagger\hat{S}_{-}^{13}) + g_{23}(\hat{a}\hat{S}_{+}^{23} + \hat{a}^\dagger\hat{S}_{-}^{23}). \tag{4.37}$$

By using the integral of motion $\hat{N}_\Lambda = \hat{a}^\dagger\hat{a} + \hat{S}^{33}$ we can rewrite

$$\hat{H}_{\text{int}} = -\Delta_{31}\ \hat{X}^{11} - \Delta_{32}\ \hat{X}^{22} + g_{13}(\hat{X}_{+}^{13} + \hat{X}_{-}^{13}) + g_{23}(\hat{X}_{+}^{23} + \hat{X}_{-}^{23}), \tag{4.38}$$

where, as in equation (4.20), we have introduced the deformed su(3) operators as

$$\hat{X}_{+}^{13} = \hat{a}\hat{S}_{+}^{13}, \qquad \hat{X}_{+}^{23} = \hat{a}\hat{S}_{+}^{23}. \tag{4.39}$$

These deformed generators satisfy a set of commutation relations similar to (4.23), but instead of (4.24) we must use

$$\hat{Y}_{+}^{12} = (\hat{S}^{33} - \hat{a}^{\dagger}\hat{a}), \tag{4.40}$$

which is the mathematical reason for the well-known different behaviours exhibited by Ξ and Λ configurations.

Let us focus on the dispersive regime, when

$$|\Delta_{13}| \gg A g_{13}\sqrt{\bar{n}+1}, \qquad |\Delta_{23}| \gg A g_{23}\sqrt{\bar{n}+1}. \tag{4.41}$$

Then a couple of small rotations eliminate the far-off resonant transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, obtaining

$$\begin{aligned}\hat{H}_{\text{eff}} = & -\Delta_{31}\ \hat{S}^{11} - \Delta_{32}\ \hat{S}^{22} \\ & + \frac{g_{13}^2}{\Delta_{13}}[(\hat{S}^{11}+1)\hat{S}^{33} + \hat{a}^{\dagger}\hat{a}(\hat{S}^{33} - \hat{S}^{11})] + \frac{g_{23}^2}{\Delta_{23}}[(\hat{S}^{22}+1)\hat{S}^{33} + \hat{a}^{\dagger}\hat{a}(\hat{S}^{33} - \hat{S}^{22})] \\ & + \frac{g_{13}g_{23}}{\Delta_{31}}(\hat{S}_{+}^{12} + \hat{S}_{-}^{12})(\hat{S}^{33} - \hat{a}^{\dagger}\hat{a}).\end{aligned} \tag{4.42}$$

Note that both $\hat{a}^{\dagger}\hat{a}$ and $\hat{S}^{33}$ are now integrals of motion. The two first terms correspond to trivial free atomic dynamics. The next two terms represent the standard dynamical Stark shift. Finally, the last term describes an effective interaction between levels 1 and 2. The remarkable point is that there is a population transfer (and not only phase transfer, as it could be expected from a dispersive interaction) between these two levels without exchange of photons. The intensity of the transition $1 \leftrightarrow 2$ depends on the difference between the population of level 3 and the photon number. Thus, no population transfer between levels 1 and 2 will occur in the sector where the number of photons is exactly equal to the initial population of the level 3. It is easy to observe that the (effective) transitions $1 \leftrightarrow 2$ are stronger when $\Delta_{31} = \Delta_{32}$; i.e., when the levels 1 and 2 have the same energy (Zeeman-like systems).

4.3 Effective Hamiltonians for nonlinear su(d) dynamics

4.3.1 Multilevel systems interacting with a quantum field

Having demonstrated the role played by nonlinear algebras in the systematic construction of effective Hamiltonians, we would like to pursue here a natural extension to multilevel systems. Specifically, we are interested in considering Hamiltonians that can be represented in terms of the su(d) algebra, which naturally arises when describing systems with d levels. Obviously, the evolution of a collection of A identical d-level systems (for definiteness, we assume a cascade configuration, such that $E_i < E_j$ for $i < j$) interacting with a single-mode quantum field can be modeled by a Hamiltonian as in equation (4.18) with

$$\begin{aligned}\hat{H}_{\rm f} &= \omega\ \hat{a}^\dagger\hat{a},\\ \hat{H}_{\rm a} &= \sum_{i=1}^{d} E_i\ \hat{S}^{ii},\\ \hat{H}_{\rm int} &= \sum_{i=1}^{d-1} g_i(\hat{a}\hat{S}_+^{ii+1} + \hat{a}^\dagger\hat{S}_-^{ii+1}).\end{aligned} \tag{4.43}$$

where $\hat{S}^{ii}$ $(i = 1,\ldots,d)$ are population operators of the ith energy level, and $\hat{S}_\pm^{ij}$ describe transitions between levels i and j. The operators $\hat{S}^{ij}$ form the u(d) algebra and satisfy the commutation relations

$$[\hat{S}^{ij},\hat{S}^{kl}] = \delta_{jk}\hat{S}^{il} - \delta_{il}\hat{S}^{kj}. \tag{4.44}$$

By introducing inversion-like operators

$$\hat{S}_3^{ii+1} = \frac{1}{2}(\hat{S}^{i+1i+1} - \hat{S}^{ii}), \tag{4.45}$$

then $(\hat{S}_\pm^{ij}, \hat{S}_3^{ii+1})$ turn out to be the su(d) algebra.

The Hamiltonian (4.43) admits the integral of motion

$$\hat{N} = \hat{a}^\dagger\hat{a} + \sum_{i=1}^{d-1}\mu_i\ \hat{S}_3^{ii+1}, \tag{4.46}$$

with $\mu_i = i(d-i)$. We also introduce the detunings by

$$\Delta_j = E_j - E_1 - (j-1)\ \omega, \tag{4.47}$$

and assume that Δ_j satisfy the following resonant condition

$$\Delta_d = 0, \tag{4.48}$$

which means that the field in a $(d-1)$-photon resonance with the atomic system; i.e., $E_d - E_1 = (d-1)\omega$. Thus, the Hamiltonian (4.43) can be recast as $\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}$, with

$$\begin{aligned}\hat{H}_0 &= \omega\hat{N} + A\ E\\ \hat{H}_{\rm int} &= \hat{h}_0 + \hat{V},\end{aligned} \tag{4.49}$$

where $E = (E_d + E_1)/2$ and

$$\hat{h}_0 = \sum_{i=1}^{d} \Delta_i \ \hat{S}^{ii}, \qquad \hat{V} = \sum_{i=1}^{d-1} g_i(\hat{a}\hat{S}_+^{ii+1} + \hat{a}^\dagger \hat{S}_-^{ii+1}). \tag{4.50}$$

From our previous experience it is clear that the operators

$$\hat{X}^{ii} = \hat{S}^{ii}, \qquad \hat{X}_+^{ij} = \hat{a}\hat{S}_+^{ij}, \qquad \hat{X}_-^{ij} = \hat{a}^\dagger \hat{S}_-^{ij}, \tag{4.51}$$

form a polynomial deformation of su(d). In consequence, and according to our general scheme, we can introduce the transformation

$$\hat{U} = \exp\left[\sum_{i=1}^{d-1} \varepsilon_i(\hat{X}_+^{ii+1} - \hat{X}_-^{ii+1})\right], \tag{4.52}$$

where

$$\varepsilon_i = \frac{g_i}{\Delta_{i+1} - \Delta_i} \tag{4.53}$$

are assumed to be small numbers, $\varepsilon_i \ll 1$, which means that the transitions are far from the one-photon resonance ($\Delta_{i+1} - \Delta_i = E_{i+1} - E_i - \omega \gg g_i$). Thus, all one-photon transitions are eliminated by (4.52) and the transformed Hamiltonian takes the form

$$\begin{aligned} \hat{H}_{\text{eff}}^{(1)} &= \hat{h}_0 + \hat{h}_{\text{d}} + \hat{h}_{\text{nd}} \\ &+ \sum_{\ell=1}^{d-2} \frac{\ell}{(\ell+1)!} \sum_{i=1}^{d-\ell-1} \lambda_i^{(\ell+1)} \left(\hat{a}^{\ell+1} \ \hat{S}_+^{ii+\ell+1} + \hat{a}^{\dagger \ell+1} \ \hat{S}_-^{ii+\ell+1}\right). \end{aligned} \tag{4.54}$$

Here, the effective interaction constants $\lambda_i^{(n)}$ can be obtained from the recurrence relation

$$\lambda_i^{(n+1)} = \varepsilon_{i+n}\lambda_i^{(n)} - \varepsilon_i \lambda_{i+1}^{(n)}, \tag{4.55}$$

with the initial term $\lambda_i^{(1)} = g_i$. It is easy to see that $\lambda_i^{(n+1)} \ll \lambda_i^{(n)}$.

The piece h_{d} contains only diagonal terms in the atomic operators and depends on the integral of motion $\hat{N}$ (or, equivalently, depends only on the photon-number operator $\hat{a}^\dagger \hat{a}$). This operator h_{d} appears naturally represented as an expansion in the small parameter ε_i whose first term is

$$\hat{h}_{\text{d}} = \sum_{i=1}^{d-1} g_i \varepsilon_i [\hat{a}^\dagger \hat{a}(\hat{S}^{i+1i+1} - \hat{S}^{ii}) + (\hat{S}^{ii} + 1)\hat{S}^{i+1i+1}]. \tag{4.56}$$

The essential point is that, given its structure, this diagonal part cannot be removed from the effective Hamiltonian (4.54). On the contrary, the operator $\hat{h}_{\text{nd}}$ contains only nondiagonal terms that can be eliminated by rotations of the type (4.52) unless some specific resonance conditions are fulfilled. In this respect, let us note that the price we pay for eliminating one-photon transitions is the generation of all possible k-photon transitions ($k = 2, ..., d-1$).

4.3.2 Three-photon resonance

Let us consider the particular case of four-level systems ($d = 4$) and suppose that there are no transitions in one- and two-photon resonance with the field. After eliminating one-photon transitions the transformed Hamiltonian (4.54) has the form

$$\hat{H}^{(1)}_{\text{eff}} = \hat{h}_0 + \hat{h}_{\text{d}} + \hat{h}_{\text{nd}} + \frac{1}{3}\lambda^{(3)}_1 \left(\hat{a}^3 \hat{S}^{14}_+ + \hat{a}^{\dagger 3} \hat{S}^{14}_-\right) + \frac{1}{2}\sum_{i=1}^{2} \lambda^{(2)}_i \left(\hat{a}^2 \hat{S}^{ii+2}_+ + \hat{a}^{\dagger 2} \hat{S}^{ii+2}_-\right), \quad (4.57)$$

where the interaction constants are defined, according to (4.53) and (4.55), as

$$\varepsilon_1 = \frac{g_1}{\Delta_2}, \qquad \varepsilon_2 = \frac{g_2}{\Delta_3 - \Delta_2}, \qquad \varepsilon_3 = -\frac{g_3}{\Delta_3},$$
$$\lambda^{(2)}_1 = g_1 g_2 \frac{2\Delta_2 - \Delta_3}{\Delta_2(\Delta_3 - \Delta_2)}, \qquad \lambda^{(2)}_2 = g_2 g_3 \frac{2\Delta_3 - \Delta_2}{\Delta_3(\Delta_2 - \Delta_3)}, \qquad \lambda^{(3)}_1 = \frac{3 g_1 g_2 g_3}{\Delta_3 \Delta_2}, \quad (4.58)$$

and the resonance condition $\Delta_4 = 0$ (that is, three-photon resonance $E_4 - E_1 = 3\omega$) has been imposed.

According to the general scheme, the term representing two-photon transitions in (4.57) can be removed using a transformation analogous to (4.52):

$$\hat{U}_2 = \exp\left[\frac{1}{2}\sum_{i=1}^{2} \alpha^{(2)}_i (\hat{a}^2 \hat{S}^{ii+2}_+ - \hat{a}^{\dagger 2} \hat{S}^{ii+2}_-)\right], \quad (4.59)$$

where

$$\alpha^{(2)}_i = \frac{\lambda^2_i}{\Delta_{i+2} - \Delta_i} \quad (4.60)$$

is a small parameter because there are no resonant two-photon transitions ($\Delta_{i+2} - \Delta_i = E_{i+2} - E_i - 2\omega \gg \lambda^{(2)}_i$) and thus $\alpha^{(2)}_j \ll \varepsilon_j$. It is worth noting that the transformation (4.59) does not introduce new terms of order ε^2 to the effective Hamiltonian.

The diagonal part in (4.57) is

$$\hat{h}_{\text{d}} = \sum_{i=1}^{3} g_i \varepsilon_i \left[\hat{a}^\dagger \hat{a} (\hat{S}^{i+1i+1} - \hat{S}^{ii}) + (\hat{S}^{ii} + 1)\hat{S}^{i+1i+1}\right], \quad (4.61)$$

while the nondiagonal term deserves a more careful analysis. Its explicit form is

$$\hat{h}_{\text{nd}} = \frac{1}{2}(\varepsilon_1 g_3 + \varepsilon_3 g_1)(\hat{S}^{12}_+ \hat{S}^{34}_- + \hat{S}^{34}_+ \hat{S}^{12}_-) + \frac{1}{2}(\varepsilon_1 g_2 + \varepsilon_2 g_1)(\hat{S}^{12}_+ \hat{S}^{23}_- + \hat{S}^{23}_+ \hat{S}^{12}_-). \quad (4.62)$$

It is clear that the first term in the above expression describes a resonant dipole-dipole interaction, under the condition $\Delta_2 = -\Delta_3$. On the other hand, the second term describes a resonant interaction whenever $2\Delta_2 = \Delta_3$, which is incompatible with the pervious one and the absence of one- and two-photon resonances. If no one of these conditions are fulfilled, the term $\hat{h}_{\rm nd}$ can be eliminated by the transformation

$$\hat{U}_2^{(2)} = \exp\left[\frac{1}{2}\sum_{i,j=1}^{3}\beta_{ij}(\hat{S}_+^{ii+1}\hat{S}_-^{jj+1} - \hat{S}_+^{jj+1}\hat{S}_-^{ii+1})\right], \tag{4.63}$$

where

$$\beta_{ij} = \frac{\varepsilon_i g_j}{\Delta_{i+1} - \Delta_i + \Delta_j - \Delta_{j+1}}. \tag{4.64}$$

Then, since $\hat{S}^{11} + \hat{S}^{22} + \hat{S}^{33} + \hat{S}^{44} = A$ and imposing the condition of the absence of initial population in levels 2 and 3, we obtain the effective Hamiltonian describing three-photon resonant transitions

$$\begin{aligned}\hat{H}_{\rm eff}^{(2)} &= \frac{g_1 g_2 g_3}{\Delta_2\Delta_3}(\hat{a}^3\hat{S}_+^{14} + \hat{a}^{\dagger 3}\hat{S}_-^{14}) \\ &\quad - (\hat{S}_3^{14} + A/2)[\hat{a}^\dagger\hat{a}(g_1^2/\Delta_2 - g_3^2/\Delta_3) + g_3^2/\Delta_3] + A\frac{g_1^2}{\Delta_2}\hat{a}^\dagger\hat{a}.\end{aligned} \tag{4.65}$$

The effective three-photon Hamiltonian (4.65) contains a dynamical Stark shift similar to that appearing in the two-photon case (4.36). Nevertheless, in the two-photon case the interaction term and the Stark shift are of the same order of magnitude, while in the three-photon case the interaction term is one order of magnitude less than the Stark shift. This would lead to essential differences in the evolution of some observables.

4.3.3 Multimode fields and resonance conditions

The formalism developed can also be used to treat the interaction of multifrequency fields with atomic systems. This type of interaction is much richer because some additional resonance conditions can be satisfied. To this end, let us consider a four-level system in a Ξ configuration interacting with two field modes of frequencies ω_a and ω_b, and annihilation operators $\hat{a}$ and $\hat{b}$, respectively. The Hamiltonian describing this interaction is still of the form (4.18) with

$$\begin{aligned}\hat{H}_{\rm f} &= \omega_a\ \hat{a}^\dagger\hat{a} + \omega_b\ \hat{b}^\dagger\hat{b}, \\ \hat{H}_{\rm a} &= \sum_{i=1}^{4} E_i\ \hat{S}^{ii}, \\ \hat{H}_{\rm int} &= \sum_{i=1}^{3}\left[(g_{ai}\hat{a} + g_{bi}\hat{b})\hat{S}_+^{ii+1} + (g_{ai}\hat{a}^\dagger + g_{bi}\hat{b}^\dagger)\hat{S}_-^{ii+1}\right],\end{aligned} \tag{4.66}$$

where g_{ai} and g_{bi} are the corresponding coupling constants. We assume that the following resonance conditions are satisfied:

$$E_4 - E_1 = 3\omega_b, \qquad E_3 - E_1 = 2\omega_b, \tag{4.67}$$

while all one-photon transitions are out of resonance. In such a case, the Hamiltonian (4.66) may describe, for example, a fifth-order process involving the absorption of three photons from one field and the stimulated emission of two photons of different frequency. The effective Hamiltonian describing *explicitly* these transitions can be obtained according to the general method. Now the integral of motion (4.46) takes the form

$$\hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + \sum_{i=1}^{3} \mu_i \hat{S}_3^{ii+1}, \tag{4.68}$$

where $\mu_i = i(4-i)$, and the Hamiltonian can be recast also as in (4.49) with $\hat{H}_{\rm int} = \hat{h}_0 + \hat{V}_a + \hat{V}_b$, where

$$\begin{gathered} \hat{h}_0 = \delta\ \hat{b}^\dagger \hat{b} + \sum_{i=1}^{4} \Delta_i \hat{S}^{ii}, \\ \hat{V}_a = \sum_{i=1}^{3} g_{ai} (\hat{X}_{a+}^{ii+1} + \hat{X}_{a-}^{ii+1}), \qquad \hat{V}_b = \sum_{i=1}^{3} g_{bi} (\hat{X}_{b+}^{ii+1} + \hat{X}_{b-}^{ii+1}). \end{gathered} \tag{4.69}$$

Here the polynomial deformation is defined by

$$\hat{X}_{a+}^{ij} = \hat{a}\hat{S}_+^{ij}, \quad \hat{X}_{a-}^{ij} = \hat{a}^\dagger \hat{S}_-^{ij}; \qquad \hat{X}_{b+}^{ij} = \hat{b}\hat{S}_+^{ij}, \quad \hat{X}_{b-}^{ij} = \hat{b}^\dagger \hat{S}_-^{ij}, \tag{4.70}$$

and $\delta = \omega_b - \omega_a$.

Now we can eliminate all the one-photon nonresonant transitions generated by means of two transformations (one for mode $\hat{a}$ the other for mode $\hat{b}$) identical to (4.52). The transformed Hamiltonian (up to order $1/\Delta^2$) is

$$\begin{aligned} \hat{H}_{\rm eff}^{(1)} = \hat{h}_0 + \hat{h}_{\rm d}^{(a)} + \hat{h}_{\rm d}^{(b)} + \frac{1}{2}\lambda_1^{(2)} \left(\hat{a}^2 \hat{S}_+^{13} + \hat{a}^{\dagger 2} \hat{S}_-^{13} \right) \\ + \frac{1}{3}\lambda_1^{(3)} \left(\hat{b}^3 \hat{S}_+^{14} + \hat{b}^{\dagger 3} \hat{S}_-^{14} \right) + \xi_2^{(ab)} \left(\hat{a}\hat{b}\hat{S}_+^{24} + \hat{a}^\dagger \hat{b}^\dagger \hat{S}_-^{24} \right), \end{aligned} \tag{4.71}$$

donde $\hat{h}_{\rm d}^{(a)}$ and $\hat{h}_{\rm d}^{(b)}$ are defined according to (5.49) for both modes and the coupling constants $\lambda_1^{(2)}$ and $\lambda_1^{(3)}$ are given by (4.58). The constant $\xi_2^{(ab)}$ is

$$\xi_2^{(ab)} = \frac{g_{a3} g_{b2}}{\Delta_4 - \Delta_3} - \frac{g_{b3} g_{a2}}{\Delta_3 - \Delta_2}. \tag{4.72}$$

This effective Hamiltonian deserves some comments. First of all, when the resonance condition

$$E_4 - E_2 = \omega_a + \omega_b \tag{4.73}$$

is fulfilled (which is compatible with the two- and three-photon resonance conditions), then the last term in the Hamiltonian describes resonant transitions between levels 2 and 4, as a result of the simultaneous absorption and emission of quanta from both modes, and take place only if levels 2 and 4 are populated. In absence of such an additional resonance condition, the Hamiltonian (4.71) describes two simultaneous and competing processes: a first-order process ($\lambda_1^{(2)} \sim 1/\Delta$) involving two-photon transitions between levels 1 and 3, and a second-order process ($\lambda_1^{(3)} \sim 1/\Delta^2$) of three-photon transitions between levels 1 and 4.

To conclude, we note that the method presented here is restricted to su(d) deformed algebras. In fact, many other phenomena can be modeled by Hamiltonians quite similar to (4.8), namely

$$\hat{H}_{\rm int} = a\ \hat{Y}_0 + g(\hat{Y}_+ + \hat{Y}_-) + \hat{C}, \tag{4.74}$$

where C is some integral of motion and a a constant. In these theories, the polynomial deformation is defined in the following fashion in the Cartan-Weyl basis:

$$[\hat{Y}_0, \hat{Y}_\pm] = \pm\hat{Y}_\pm, \qquad [\hat{Y}_-, \hat{Y}_+] = \Psi(\hat{Y}_0) = \Phi(\hat{Y}_0 + 1) - \Phi(\hat{Y}_0), \tag{4.75}$$

where $\Phi(\hat{Y}_0)$ are appropriate structure polynomials. A detailed study of the applications of these algebras for solving evolution problems in nonlinear quantum models may be found in (Karassiov 1994). For these model, the machinery of small rotations works well and constitutes the most systematic way of constructing effective Hamiltonians.

5 Relative phase for two-mode fields

In a classical framework, in order to analyze optical fenomena related with interference (such as, visibility of fringes) one needs to work with light beams with a well defined phase between them. The polarization properties of light are directly related to this type of analysis. In quantum optics it is essential to deal with new conditions which allow us to provide a complete analysis of the interference phenomena from a quantum point of view.

The outline of this chapter is as follows: in section 1 we derive a criterion for a two-mode field to have a well-defined relative phase. After that, we review the definition of the classical visibility of inteference fringes to continue taking a quantum-mechanical view of visibility, called "generalized visibility". In section 2, we study the polarization structure and the invariance properties of quantum two mode fields. Finally, in section 3, we connect the idea of distance with the problem of assessing the polarization characteristics of a quantum field, exploring a suitable definition of polarization degree that avoids at least some of the diffculties that previous approaches based on Stokes parameters encounter.

5.1 States with well-defined relative phase

5.1.1 Differential phase shifts

In this chapter we deal with two-mode fields, which will be denoted by subscripts H and V, indicating horizontal and vertical components, respectively.

To derive a criterion for a well-defined relative phase, we start by considering the action of differential phase shift; i.e., letting the two modes undergo free evolution for unequal times. The free evolution the system for a time τ is described by $\hat{U}(\tau) = \exp(-i\hat{H}_0\tau)$, where the Hamiltonian is $\hat{H}_0 = \omega\ \hat{n}$, and ω and $\hat{n}$ is the angular frequency and number operator, respectively. Thus the evolution operator for two modes undergoing free evolution for times τ_H and τ_V , respectively, becomes

$$\hat{H} = e^{-i\omega(\tau_H+\tau_V)\hat{N}/2}\ e^{i\varphi\hat{n}_{12}/2}, \quad \hat{N} = \hat{n}_H + \hat{n}_V, \quad \hat{n}_{12} = \hat{n}_H - \hat{n}_V, \tag{5.1}$$

where the differential equation phase shift is $\varphi = \omega(\tau_H - \tau_V)$. Since the operator $\exp[-i\omega(\tau_H + \tau_V)\hat{N}/2]$ in (5.1) gives the same phase shift for all states with fixed photon numbers, and (two-mode) states with different photon numbers are orthogonal and cannot interfere, this operator can be neglected in the context of interference. Hence the unitary differential phase shift operator is photon-number conserving and can be written

$$\hat{U}_{\rm PS}(\varphi) = \exp(i\varphi\ \hat{n}_{12}). \tag{5.2}$$

We now turn our attention to what constitutes a state with a well-defined relative phase. We take an operational approach and assign this property to any two-mode state on which (at least) two different relative phases can be encoded and read out with certainty. This definition avoids all complications with associating a well-defined relative-phase with some relative phase operator, or with the properties of the relative phase statistical distribution.

It is well known to encode either of two relative phases φ_H or φ_V so that they can be read out with certainty, i.e., be projected onto orthogonal meter eigenstates, the relation

$$\langle\xi|\hat{U}_{\rm PS}(\varphi_V - \varphi_H)|\xi\rangle = 0, \tag{5.3}$$

must be fulfilled. That is, some relative phase shift $\varphi = \varphi_V - \varphi_H$ must render the state $U_{\rm PS}(\varphi)|\xi\rangle$ orthogonal to $|\xi\rangle$. In our treatment, equation (5.3) constitutes thus the mathematical criterion for a state with a well-defined relative phase.

In the following, the basis states of the excitation manifold N will be denoted as $|N,k\rangle = |k\rangle_H \otimes |N-k\rangle_V$, $(k = 0, 1, \ldots, N)$ and the states $|\xi\rangle$ will be written as

$$|\xi\rangle = \sum_{N=0}^{\infty}\sum_{k=0}^{N} C_{N,k}\ |N,k\rangle. \tag{5.4}$$

Since $\hat{U}_{\rm PS}(\varphi)|0,0\rangle = 0$, it is clear that we must have $\langle 0,0|\xi\rangle = 0$ in order to satisfy equation (5.3). If one wishes to quantify the ability to distinguish between two relative phases, one can define the distinguishability D (in the maximum-likelihood estimation sense (Björk, Trifonov, Tsegaye and Söderholm 1998)) between the two relative phases φ_H and φ_V by

$$D = \sqrt{1 - 4p_H p_V |\langle\xi|\hat{U}_{\rm PS}(\varphi_H - \varphi_V)|\xi\rangle|^2}, \tag{5.5}$$

where p_H and p_V are the *a priori* probabilities of encoding the phase shifts φ_H and φ_V, respectively. This distinguishability limit is referred to as the Helstrom bound (Helstrom 1976)and is well known in estimation theory. We see that for any nonzero *a priori* probabilities p_H and p_V, unit distinguishability implies that equation (5.3) has to be fulfilled.

5.1.2 Classical visibility

Interference phenomena are directly related with the relative phase between modes. From an experimental point of view, this gives interference fringes. Classically, the visibility of these fringes give us an idea of the sharpeness of the relative phase. In particular, if we cannot observe fringes, relative phase is random.

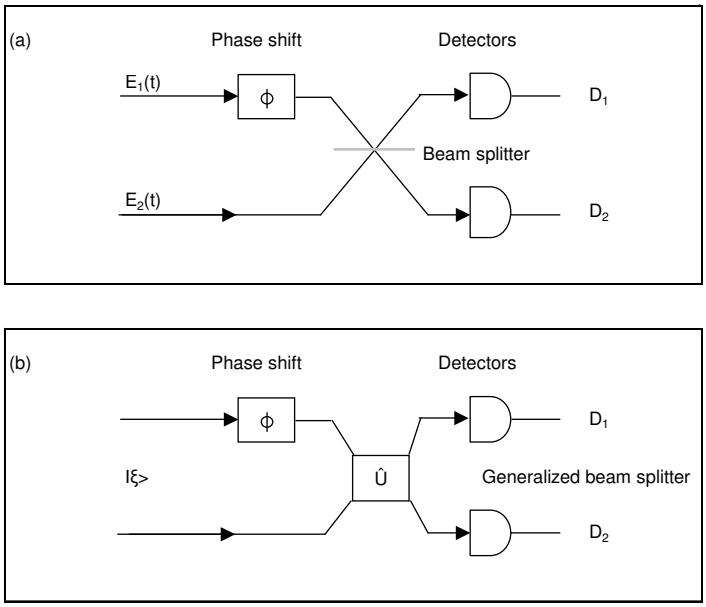

Fig. 5.1. (a) A schematic setup for a visibility experiment. (b) A quantum-mechanical generalized visibility experiment.

We consider a Mach-Zehnder interferometer despicted in figure 5.1, to avoid any unnecessary complication of more elaborated setups. Two fields interfere in a beam splitter and the relative phase between them can be varied by a phase shifter. After the beam splitter, the outgoing field intensities are monitored by two detectors. The visibility is defined in terms of the ensemble-averaged modulation of the measured intensities as a function of the relative-phase shift. To make a quantitative analysis of classical visibility, we consider the interference of two harmonic waves (sufficiently generally) described by

$$E_H(t) = \mathcal{E}_H \cos(\omega t + \vartheta), \qquad E_V(t) = \mathcal{E}_V \cos(\omega t), \tag{5.6}$$

where E_H and E_V are (real) electric field amplitudes. The phase shifter transforms E_H into

$$E_H(t) = \mathcal{E}_H \cos(\omega t + \vartheta + \varphi). \tag{5.7}$$

The beam splitter is described by the unitary transformation

$$U(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{5.8}$$

which assumes, without loss of generality, that reference planes have been chosen so that all coefficients are real. If the beam splitter transmittance is neither zero nor unity, the (time-averaged) intensity I detected by the two detectors is a sinousoidally varying function of the phase shift φ.

The visibility for each detector is defined

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}, \tag{5.9}$$

where $I_{\max}$ and $I_{\min}$ are the maximum and minimum intensities measured by the detector. In general, the visibility is not equal for the two detectors: to get unit visibility one must arrange so that $I_{\min} = 0$ while $I_{\max} \neq 0$. Also note that the intensity detected in each detector is a second-order correlation function. For the special case when $|\cos\alpha| = |\sin\alpha| = 1/\sqrt{2}$, the detector signals provide a measure of the complex degree of coherence between the two fields (Mandel and Wolf 1995).

Using equations (5.6), (5.8) and (5.9), it is straightforward to show that $I_{\min} = 0$ for both detector, implies a phase shift φ fulfilling $\vartheta + \varphi = 0$ or $\vartheta + \varphi = \pi$. In the first case, it is possible to get the intensity falling onto detector D_1 to be zero if $\tan\alpha = -\mathcal{E}_H/\mathcal{E}_V$. To make the intensity falling onto detector D_2 be zero, one must require $\tan\alpha = \mathcal{E}_V/\mathcal{E}_H$. If the phase φ is instead set to $\pi - \vartheta$, then detector D_1 sees zero intensity for $\tan\alpha = \mathcal{E}_H/\mathcal{E}_V$ and D_2 sees zero intensity for $\tan\alpha = -\mathcal{E}_V/\mathcal{E}_H$. In both cases, we can only get the visibility to equal unity in both beam splitter output modes if $\mathcal{E}_H = \pm\mathcal{E}_V$ and $|\cos\alpha| = |\sin\alpha| = 1/\sqrt{2}$.

In summary, from the classical viewpoint, in order to be able to get unit visibility in the sense just defined, the field amplitudes must be equal and the beam splitter needs to have equal transmission and reflection and equal to $1/\sqrt{2}$.

5.1.3 Quantum visibility

Let us now turn to the quantum mechanical situation depicted in figure 5.1b. One mode of the state $|\xi\rangle$ is phase shifted by a relative amount φ to the other. Subsequently, the state is transformed by a generalized beam splitter, described by the unitary transformation $\hat{U}$ (whose properties will be defined below) and finally the output modes of the generalized beam splitter are detected by two photon-counting

detectors. The generalized visibility is, in analogy with the classical version, defined in terms of the ensemble-averaged modulation of the detected photon number in the respective photon detectors. The main difference between the quantum and the classical setups is that the generalized beam splitter is typically a nonlinear beam splitter that has to match the impinging two-mode state. Different two-mode states require different generalized beam splitters. If it is possible to find two relative phase shift settings such that for one setting all the photons of state $|\xi\rangle$ impinge on detector D_1, and for the other setting all the photons impinge on detector D_2, then we define the state $|\xi\rangle$ to have unit generalized visibility. Comparing this definition with our operational definition of a state with a well-defined relative phase, one is lead to suspect that the two definitions are interrelated. Let us next cast generalized visibility in a mathematical framework. We begin by defining the quantum-mechanical visibility of the setup in figure 5.1b in analogy with the classical visibility

$$V = \frac{\langle \hat{n} \rangle_{\max} - \langle \hat{n} \rangle_{\min}}{\langle \hat{n} \rangle_{\max} + \langle \hat{n} \rangle_{\min}}, \tag{5.10}$$

where $\hat{n}$ is the number operator associated with the pertinent detector. To get unit generalized visibility in one of the output modes, we must have $\langle \hat{n} \rangle_{\min} = 0$ while $\langle \hat{n} \rangle_{\max} \neq 0$. This implies that for some suitable differential phase shift φ the vacuum state $|0\rangle$ must impinge on the detector, since this is the only single-mode state with $\langle \hat{n} \rangle = 0$. Now consider a two-mode quantum state $|\xi\rangle$, which in general is entangled. It is always possible to find a unitary and photon-number preserving transformation $\hat{U}$ such that

$$\hat{U}|\xi\rangle = |\psi\rangle \otimes |0\rangle, \tag{5.11}$$

i.e., the state $|\xi\rangle$ is transformed to a factorizable state with no excitation in one of the modes. In a classical visibility measurement, $\hat{U}$ is assumed to describe the action of a 50/50 beam splitter, but for a generalized visibility measurement we must only require that $\hat{U}$ is unitary and photon-number preserving. The latter requirement is reasonable as we want interference, rather than photon loss, to determine the generalized interference. A consequence of the photon-number conservation is that

$$\hat{U}|0,0\rangle = e^{i\varsigma}|0,0\rangle, \tag{5.12}$$

where ς is a real number. In order to get unit generalized visibility in the measurement depicted in the figure 5.1b it is necessary that if the state $|\xi\rangle$ is phase shifted by an appropriate amount φ, $\hat{U}$ must transform this new state to a state of the form $|0\rangle \otimes |\varphi\rangle$. Hence,

$$\hat{U}\hat{U}_{\mathrm{PS}}|\xi\rangle = |0\rangle \otimes |\varphi\rangle. \tag{5.13}$$

Equations (5.10) and (5.12) together with the requirement that $|\xi\rangle \neq |0,0\rangle$ are sufficient and necessary conditions to get unit generalized visibility in the output modes of a generalized beam splitter. To see what requirements these equations imply for

the state $|\xi\rangle$, let us compute the scalar product between the final states (5.11) and (5.13):

$$(\langle\psi| \otimes \langle 0|)(|0\rangle \otimes |\varphi\rangle) = \langle\psi|0\rangle\langle 0|\varphi\rangle. \tag{5.14}$$

Using the left hand sides of equations (5.11) and (5.12), the scalar product can be rewritten as

$$\langle\psi|0\rangle\langle 0|\varphi\rangle = \langle\xi|0,0\rangle\langle 0,0|\xi\rangle = \langle\xi|\hat{U}\hat{U}^{\dagger}\hat{U}_{\mathrm{PS}}(\varphi)|\xi\rangle = \langle\xi|\hat{U}_{\mathrm{PS}}(\varphi)|\xi\rangle, \tag{5.15}$$

where we have used the fact that vacuum is unaffected by a phase shift and equation (5.10) to arrive equation (5.14). A rearrangement now gives us our final result, delineating a necessary and sufficient condition to achieve unit generalized visibility as

$$\langle\xi|\hat{U}_{\mathrm{PS}}(\varphi)|\xi\rangle - |\langle\xi|0,0\rangle|^2 = 0. \tag{5.16}$$

That the condition is sufficient follows from that any state fulfilling equation (5.16) can also fulfill both equations (5.11) and (5.13), which is necessary and sufficient to get unit generalized visibility. That the condition is necessary follows from the fact that any state that does not fulfill the condition (5.16) can fulfill only one of equations (5.10) and (5.12), not both.

We see that in order to get unit generalized visibility there must exist a differential phase shift φ such that all the photon-number manifolds of the state $\hat{U}_{\mathrm{PS}}(\varphi)|\xi\rangle$ are rendered orthogonal to the manifolds of the initial $|\xi\rangle$. Comparing equations (5.3) and (5.16), one sees that the former condition is stronger than the latter in that every state fulfilling equation (5.3) will also fulfill equation (5.16), but a state satisfying equation (5.16) does not necessarily satisfy equation (5.3). The physical reason is that since visibility (both the classical and the generalized) is an ensemble-averaged quantity, the fact that $\hat{U}_{\mathrm{PS}}(\varphi)$ does nothing to the state $|0,0\rangle$ is not important to the generalized visibility. Null photon counts will contribute neither to the minima nor to the maxima (as φ is varied) of the photon interference pattern.

As long as the state $|\xi\rangle$ contains higher photon-number manifolds, and each of these states are simultaneously rotated to an orthogonal state for some differential phase shift φ, an interference pattern with unit generalized visibility can be observed. On the contrary, in order to predict a phase shift with certainty (i.e., for every individual detected state) the state must not contain any component of the vacuum state, since every time the state $\hat{U}_{\mathrm{PS}}(\varphi_H)$ or $\hat{U}_{\mathrm{PS}}(\varphi_V)$ collapses into the vacuum state it leads to an inconclusive result of which phase shift φ_H or φ_V was used. Hence, the relation (5.3) is sharper than the requirement of unit generalized visibility (5.16) in that every state fulfilling equation (5.3) can also display unit generalized visibility, while a perfect generalized visibility does not ensure that the relative phase of the state is precisely defined.

Visibility and coherence are intimately connected. Let us therefore briefly discuss the connection between generalized visibility and coherence. In terms of coherence

theory, the detectors in figure 5.1b measure a superposition of even order correlation functions. The explicit choice of $\hat{U}$ will determine the particular superposition (the fact that particle detectors are quadratic in the incident fields assures that no odd-order coherence functions are measured). If the two-mode state is in a photon-number eigenstate with a total of N photons, only the even-order correlation functions up to set $2N$ are measured. This is due to the fact that the interaction Hamiltonian realizing any particle number preserving, two-mode, unitary transformation $\hat{U}$ can be synthesized by a normally-ordered polynomial of order $2N$ in the creation and annihilation operators of the fields (Björk, Söderholm and Karlsson 1998). The coherent properties of a N-particle state is not simply given by the $2N$th order correlation function (Mandel and Wolf 1995). Yet, the criterion for when a two-mode state has a well-defined relative phase is surprisingly simple (5.3). A classical visibility measurement is a special case of a generalized visibility measurement where $\hat{U}$ has a particular form (expressed in creation and annihilation operators it contains only the linear term of each mode) so that no correlation functions higher than of the second order are measured.

Let us show that unit generalized visibility does not require any symmetry of the state $|\xi\rangle$ with the respect of permutation of modes. As demonstrated above this is necessary in a classical experiment. To show this we construct a simple example, e.g., the state

$$|\xi\rangle = \sqrt{\frac{3}{10}}(|N,0\rangle + |N,1\rangle) + \sqrt{\frac{2}{10}}|N,3\rangle, \tag{5.17}$$

where $N \geq 3$. This state has no symmetry with respect to the permutation of modes, and its average excitation in the second mode is much larger than its excitation in the first mode, if N is large. Yet, for $\varphi = \pm \arctan(\sqrt{15}/4) \simeq \pm 0.42\pi$ rad, equation (5.3) is satisfied. Hence, the state has a well-defined relative phase and can therefore display unit generalized visibility.

Other interesting example, consists of the eigenstate of the relative-phase operator (Luis and Sánchez-Soto 1993). The most general form of such an eigenstate in photon-number manifold N can be written (Luis and Sánchez-Soto 1993)

$$|\phi_r^{(N)}\rangle = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} e^{ik\phi_r^{(N)}} |N,k\rangle, \tag{5.18}$$

where $\phi_r^{(N)} = \phi_0^{(N)} + 2\pi r/(N+1)$, $(r = 1..N)$. Since the eigenstates are orthogonal, they will fulfill equation (5.3) above for $N \neq 0$ and $\varphi_V - \varphi_H = 2\pi k r/(N+1)$. Thus, these states have a well-defined relative phase. In spite of displaying nonunit classical visibility, they can display unit generalized visibility. To give a specific example of a unitary transformation which gives any state a unit generalized visibility, consider the unitary transformation

$$\hat{U} = \sum_{N=0}^{\infty} \sum_{r=0}^{N} |N, r\rangle\langle\varphi_r^{(N)}|. \tag{5.19}$$

In every manifold, the state $|\varphi_r^{(N)}\rangle$ is transformed into the number difference state $|N, r\rangle$.

We can conclude that generalized visiblility depends a lot of the generalized beam-splitter choice. The most of the beam splitters do not give us unit visibility, in spite of the state fulfills (5.3).

5.2 SU(2) invariance properties of two-mode fields

As we have seen above, classical visibility is not sufficient to know when a state is well-defined relative phase. Furthermore, we have shown that in a new definition of visibility (generalized visibility), we need to include higher-order correlations. In order to get this, we have introduced generalized beam splitters as unitary transformations preserving the total photon number. If we take this into account, it would be interesting to analize what is the highest precission we can get in a relative-phase measure.

In 1852, Stokes devised a systematic treatment of two-mode states by introducing the parameters that today bear his name. A great advantage with them is that they are easily measured. To introduce these parameters, we assume a monochromatic plane wave propagating in the z direction, whose electric field lies in the xy plane.

$$\begin{aligned} S_0 &= |\mathcal{E}_H|^2 + |\mathcal{E}_V|^2\,, \qquad S_1 = 2\ \mathrm{Re}(\mathcal{E}_H^* \mathcal{E}_V)\,, \\ S_2 &= 2\ \mathrm{Im}(\mathcal{E}_H^* \mathcal{E}_V)\,, \qquad S_3 = |\mathcal{E}_H|^2 - |\mathcal{E}_V|^2\,, \end{aligned} \tag{5.20}$$

It is easy to show that $S_1^2 + S_2^2 + S_3^2 = 1$. That is, the Stokes parameters of any monochromatic, non-stochastic plane wave lie on the surface of the Poincaré sphere. If the field amplitudes are fluctuating, then the *degree of classical polarization* is defined as

$$\mathcal{P}_{\mathrm{cl}} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}\,. \tag{5.21}$$

From a quantum point of view, a two-mode field that can be fully described by two complex amplitude operators, denoted by $\hat{a}_H$ and $\hat{a}_V$. The commutation relations of these operators are standard:

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\,, \qquad j, k \in \{H, V\}\,. \tag{5.22}$$

The Stokes operators are then defined as the quantum counterparts of the classical variables, namely (Alodjants and Arakelian 1999; Chirkin et al. 1993; Collett 1970; Jauch and Rohrlich 1976)

$$\hat{S}_0 = \hat{a}_H^\dagger \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V \,, \qquad \hat{S}_1 = \hat{a}_H^\dagger \hat{a}_V + \hat{a}_V^\dagger \hat{a}_H \,,$$
$$\hat{S}_2 = i(\hat{a}_H \hat{a}_V^\dagger - \hat{a}_H^\dagger \hat{a}_V) \,, \qquad \hat{S}_3 = \hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V \,, \tag{5.23}$$

and their mean values are precisely the Stokes parameters $(\langle \hat{S}_0 \rangle, \langle \hat{\mathbf{S}} \rangle)$, where $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$. Using the relation (5.22), one immediately gets that the Stokes operators satisfy the commutation relations of angular momentum:

$$[\hat{\mathbf{S}}, \hat{S}_0] = 0 \,, \qquad [\hat{S}_1, \hat{S}_2] = 2i\hat{S}_3 \,, \tag{5.24}$$

and cyclic permutations. The noncommutability of these operators precludes the simultaneous exact measurement of their physical quantities. Among other consequences, this implies that no field state (leaving aside the two-mode vacuum) can have definite nonfluctuating values of all the Stokes operators simultaneously. This is expressed by the uncertainty relation

$$(\Delta \hat{\mathbf{S}})^2 = (\Delta \hat{S}_1)^2 + (\Delta \hat{S}_2)^2 + (\Delta \hat{S}_3)^2 \geq 2\langle \hat{S}_0 \rangle \,. \tag{5.25}$$

Contrary to what happens in classical optics, the electric vector of a monochromatic quantum field never describes a definite ellipse (Luis 2002).

In mathematical terms, a linear polarization transformation is any transformation generated by the operators $\hat{\mathbf{S}}$. It is well known (Yurke et al. 1986) that the operator $\hat{S}_2$ is the infinitesimal generator of geometrical rotations around the direction of propagation, whereas $\hat{S}_3$ is the infinitesimal generator of differential phase shifts between the modes. As indicated by equation (5.24), these two operators suffice to generate all SU(2) transformations, which in experimental terms means that they can be accomplished with a combination of phase plates and rotators (Björk et al. 2002).

The standard definition of the degree of polarization is now (Born and Wolf 1980; Saastamoinen and Tervo 2004; Simon 1990)

$$\mathcal{P}_{\mathrm{sc}} = \frac{\sqrt{\langle \hat{\mathbf{S}} \rangle^2}}{\langle \hat{S}_0 \rangle} = \frac{\sqrt{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2}}{\langle \hat{S}_0 \rangle} \,, \tag{5.26}$$

where the subscript sc indicates that this is a semiclassical definition, mimicking the form of the classical in (5.21). In the semiclassical description it is implicitly assumed that unpolarized light (i. e., the origin of the Poincaré sphere) is defined by the specific values (Karassiov 1993)

$$\langle \hat{S}_1 \rangle = \langle \hat{S}_2 \rangle = \langle \hat{S}_3 \rangle = 0 \,. \tag{5.27}$$

Sometimes the extra requirement that the Stokes parameters are temporally invariant is added to make the definition even more stringent (Barakat 1989). While this affords a very intuitive image, it has also serious flaws that give rise to strange concepts such

as that of quantum states with "hidden" polarization (Klyshko 1992). Actually, this notion leads to the paradoxical conclusion that unpolarized light has a polarization structure, which is latent when the mean intensities are measured and detectable when the noise intensities are measured (Karassiov 1995). These paradoxes can be traced back to the fact that the Stokes parameters are proportional to the second-order correlations of the field amplitudes. This may be sufficient for most classical problems, but for quantum fields higher-order correlations are crucial, as we saw in the case of visibility. Hence, we need find a generalized polarization degree.

Today, there is a wide consensus (Björk et al. 2002; Wünsche 2003) in considering unpolarized light as the only one described by quantum states that are invariant with respect to any SU(2) polarization transformation, then there is no more any "hidden" polarization. Any state satisfying this invariance condition will also fulfill the classical definition of an unpolarized state, but the converse is not true. It has been shown (Agarwal 1971; Lehner et al. 1996; Prakash and Chandra 1971) that the density operator of such quantum unpolarized states can be always written as

$$\hat{\sigma} = \bigoplus_{N=0}^{\infty} \lambda_N \hat{\mathbb{1}}_N \,, \tag{5.28}$$

where N denotes the excitation manifold in which there are exactly N photons in the field. All the coefficients λ_N are real and nonnegative and to meet the unit-trace condition of the density operator they must satisfy

$$\sum_{N=0}^{\infty} (N+1)\lambda_N = 1 \,. \tag{5.29}$$

5.3 Quantum degree of polarization as a distance

Measures of nonclassicality have been defined as the distance to an appropriate set representing classical states (Dodonov et al. 2000; Hillery 1987; Marian et al. 2002). Similarly, the minimum distance to the (convex) set of separable states has been used to introduce measures of entanglement (Vedral et al. 1997). In the same vein, we propose to quantify the degree of polarization as

$$\mathbb{P}(\hat{\varrho}) \propto \inf_{\hat{\sigma} \in \mathcal{U}} D(\hat{\varrho}, \hat{\sigma}) \,, \tag{5.30}$$

where $\mathcal{U}$ denotes the set of unpolarized states of the form (5.28) and $D(\hat{\varrho}, \hat{\sigma})$ is any measure of distance (not necessarily a metric) between the density matrices $\hat{\varrho}$ and $\hat{\sigma}$, such that $\mathbb{P}(\hat{\varrho})$ satisfies some requirements motivated by both physical and mathematical concerns. The constant of proportionality in equation (5.30) must be chosen in such a way that $\mathbb{P}$ is normalized to unity, i.e., $\sup_{\hat{\varrho}} \mathbb{P}(\hat{\varrho}) = 1$.

In (Gilchrist et al. 2005) a check list of six simple, physically-motivated criteria that should be satisfied by any good measure of distance between quantum processes can be found. For our problem, we impose the following two conditions:

(C1) $\mathbb{P}(\hat{\varrho}) = 0$ iff $\hat{\varrho}$ is unpolarized.
(C2) Energy-preserving unitary transformations $\hat{U}_E$ leave $\mathbb{P}(\hat{\varrho})$ invariant; that is, $\mathbb{P}(\hat{\varrho}) = \mathbb{P}(\hat{U}_E \hat{\varrho} \hat{U}_E^\dagger)$.

The first condition is to some extent trivial: it ensures that unpolarized and only unpolarized states have a zero degree of polarization. The second takes into account that the requirement that an unpolarized state is invariant under any SU(2) polarization transformation makes it also invariant under any energy-preserving unitary transformation (Sehat et al. 2005): these include not only the transformations generated by $\hat{\mathbf{S}}$, but also those generated by $\hat{S}_0$, which, in technical terms, corresponds to the group U(2) (Wünsche 2003).

It is clear that there are numerous nontrivial choices for $D(\hat{\varrho}, \hat{\sigma})$ (by nontrivial we mean that the choice is not a simple scale transformation of any other distance). None of them could be said to be more important *a priori* than any other, but the significance of each candidate would have to be seen through physical assumptions. To illustrate this point further, let us take an extreme example fulfilling the previous conditions (Vedral et al. 1997). Define the discrete distance

$$D_{\mathrm{dis}}(\hat{\varrho}, \hat{\sigma}) = \begin{cases} 1\,, & \hat{\varrho} \neq \hat{\sigma}\,, \\ \\ 0\,, & \hat{\varrho} = \hat{\sigma}\,. \end{cases} \tag{5.31}$$

If the degree of polarization is computed using this distance, we have

$$\mathbb{P}_{\mathrm{dis}}(\hat{\varrho}) = \begin{cases} 1\,, & \hat{\varrho} \notin \mathcal{U}\,, \\ \\ 0\,, & \hat{\varrho} \in \mathcal{U}\,. \end{cases} \tag{5.32}$$

This therefore tells us only if a given state $\hat{\varrho}$ is unpolarized or not.

There are authors demanding that $D(\hat{\varrho}, \hat{\sigma})$ is a metric (Gilchrist et al. 2005). This requires three additional properties:

1. Positiveness: $D(\hat{\varrho}, \hat{\sigma}) \geq 0$ and $D(\hat{\varrho}, \hat{\sigma}) = 0$ iff $\hat{\varrho} = \hat{\sigma}$.
2. Symmetry: $D(\hat{\varrho}, \hat{\sigma}) = D(\hat{\sigma}, \hat{\varrho})$.
3. Triangle inequality: $D(\hat{\varrho}, \hat{\tau}) \leq D(\hat{\varrho}, \hat{\sigma}) + D(\hat{\sigma}, \hat{\tau})$.

These are quite reasonable properties, since most distances used in quantum mechanics are based on an inner product and so they automatically fulfill them. However, there exist pertinent examples in which D is not a metric. For example, the quantum relative entropy (Donald 1986; Hiai and Petz 1991; Ohya 1989; Wehrl 1978)

$$S(\hat{\varrho}||\hat{\sigma}) = \mathrm{Tr}[\hat{\varrho}(\ln\hat{\varrho} - \ln\hat{\sigma})] \tag{5.33}$$

is not symmetric and does not satisfy the triangle inequality. Nevertheless, it generates a valuable measure of entanglement, and the corresponding degree of polarization satisfies both C1 and C2.

For a detailed analysis we shall consider the Hilbert-Schmidt metric

$$D_{\mathrm{HS}}(\hat{\varrho},\hat{\sigma}) = ||\hat{\varrho} - \hat{\sigma}||^2_{\mathrm{HS}} = \mathrm{Tr}[(\hat{\varrho} - \hat{\sigma})^2]\,, \tag{5.34}$$

which has been previously studied in the contexts of entanglement (Bertlmann et al. 2002; Ozawa 2000; Witte and Trucks 1999). Since $D_{\mathrm{HS}}(\hat{\varrho},\hat{\sigma})$ is a metric, condition C1 is satisfied. It follows from the unitary invariance of the Hilbert-Schmidt metric that also C2 is satisfied.

According to the general strategy outlined in (5.30), for a given state $\hat{\varrho}$ we should find the unpolarized state $\hat{\sigma}$ that minimizes the distance

$$D_{\mathrm{HS}}(\hat{\varrho},\hat{\sigma}) = \mathrm{Tr}(\hat{\varrho}^2) + \mathrm{Tr}(\hat{\sigma}^2) - 2\,\mathrm{Tr}(\hat{\varrho}\hat{\sigma})\,. \tag{5.35}$$

If we take into account that the purity of an unpolarized state is

$$\mathrm{Tr}(\hat{\sigma}^2) = \sum_{N=0}^{\infty} (N+1)\ \lambda_N^2\,, \tag{5.36}$$

we easily get

$$D_{\mathrm{HS}}(\hat{\varrho},\hat{\sigma}) = \mathrm{Tr}(\hat{\varrho}^2) + \sum_{N=0}^{\infty} [(N+1)\lambda_N^2 - 2p_N\lambda_N]\,, \tag{5.37}$$

where p_N is the probability distribution of the total number of photons

$$p_N = \sum_{k=0}^{N} \varrho_{Nk,Nk}\,, \tag{5.38}$$

and $\varrho_{Nk,N'k'} = \langle N,k|\hat{\varrho}|N',k'\rangle$. Now, it is easy to obtain the coefficients λ_N that minimize this distance. The calculation is direct and the result is

$$\lambda_N = \frac{p_N}{N+1}\,. \tag{5.39}$$

The density operator $\hat{\sigma}_{\mathrm{opt}} \in \mathcal{U}$ with these optimum coefficients λ_N satisfies the constraint (5.29) and hence minimizes the distance (5.35). Note that $\hat{\sigma}_{\mathrm{opt}}$ can be written as

$$\hat{\sigma}_{\mathrm{opt}} = \sum_{N=0}^{\infty} p_N\ \hat{\sigma}_{\mathrm{opt}}^{(N)}\,, \tag{5.40}$$

with

$$\hat{\sigma}_{\mathrm{opt}}^{(N)} = \frac{1}{N+1} \sum_{k=0}^{N} |N,k\rangle\langle N,k| \,. \tag{5.41}$$

With all this in mind, we can define the Hilbert-Schmidt degree of polarization by

$$\mathbb{P}_{\mathrm{HS}}(\hat{\varrho}) = \mathrm{Tr}(\hat{\varrho}^2) - \sum_{N=0}^{\infty} \frac{p_N^2}{N+1} \,, \tag{5.42}$$

which is determined not only by the purity $0 < \mathrm{Tr}(\hat{\varrho}^2) \leq 1$ (as it happens for other measures (Heller 1987)), but also by the distribution of the number of photons p_N. Although the maximum Hilbert-Schmidt distance between two density operators is 2, the minimum distance to an unpolarized state is normalized to unity.

Using equations (5.36) and (5.39), the Hilbert-Schmidt degree of polarization can be recast as

$$\mathbb{P}_{\mathrm{HS}}(\hat{\varrho}) = \mathrm{Tr}(\hat{\varrho}^2) - \mathrm{Tr}(\hat{\sigma}_{\mathrm{opt}}^2) \,, \tag{5.43}$$

which makes it easy to verify that it vanishes only for unpolarized states, in agreement with the condition C1.

It has been shown (Bertlmann et al. 2002; Ozawa 2000; Witte and Trucks 1999) that the Hilbert-Schmidt distance is not monotonically decreasing under every completely positive trace-preserving map (what is called the CP nonexpansive property). This has motivated that the quantum information community has identified the fidelity as a particularly important alternative approach to the definition of a distance measure for states (Nielsen and Chuang 2000).

In consequence, as our second candidate of distance we will employ the fidelity (or Uhlmann transition probability) (Uhlmann 1976)

$$F(\hat{\varrho},\hat{\sigma}) = [\mathrm{Tr}(\hat{\sigma}^{1/2}\,\hat{\varrho}\,\hat{\sigma}^{1/2})^{1/2}]^2 \,. \tag{5.44}$$

A word of caution is necessary here. There is an ambiguity in the literature: both the quantity (5.44) and its square root have been referred to as the fidelity. The reader should take this into account when comparing different sources.

The fidelity has many attractive properties. First, it is symmetric in its arguments $F(\hat{\varrho},\hat{\sigma}) = F(\hat{\sigma},\hat{\varrho})$, a fact that is not obvious from equation (5.44), but which follows from other equivalent expressions. It can also be shown that $0 \leq F(\hat{\varrho},\hat{\sigma}) \leq 1$, with equality in the second inequality iff $\hat{\varrho} = \hat{\sigma}$. This means that the fidelity is not a metric as such, but serves rather as a generalized measure of the overlap between two quantum states. A common way of turning it into a metric is through the Bures metric

$$D_{\mathrm{B}}(\hat{\varrho},\hat{\sigma}) = 2[1 - \sqrt{F(\hat{\varrho},\hat{\sigma})}] \,. \tag{5.45}$$

The origin of this distance can be seen intuitively by considering the case when $\hat{\varrho}$ and $\hat{\sigma}$ are both pure states. The Bures metric is just the Euclidean distance between the two pure states, with respect to the usual norm on the state space.

Since the larger the fidelity $F(\hat{\varrho},\hat{\sigma})$, the smaller the Bures distance $D_{\mathrm{B}}(\hat{\varrho},\hat{\sigma})$, we can define the Bures degree of polarization as

$$\mathbb{P}_{\mathrm{B}}(\hat{\varrho}) = 1 - \sup_{\hat{\sigma}\in\mathcal{U}} \sqrt{F(\hat{\varrho},\hat{\sigma})}\,. \tag{5.46}$$

An alternative definition would be $1-\sup_{\hat{\sigma}\in\mathcal{U}} F(\hat{\varrho},\hat{\sigma})$, which arises naturally in the context of quantum computation (Gilchrist et al. 2005). These definitions order the states $\hat{\varrho}$ in the same way. Unfortunately, we have not found a general expression of the unpolarized state $\hat{\sigma}$ that gives the maximum fidelity. Such a task must be performed case by case and will be illustrated with some selected examples.

5.3.1 Some examples

From equation (5.42) we infer that all pure N-photon states have the same Hilbert-Schmidt degree of polarization. For such states, we have

$$\mathbb{P}_{\mathrm{HS}}^{(N)} = \frac{N}{N+1}\,. \tag{5.47}$$

The Bures degree of polarization for these states can also be readily found:

$$\mathbb{P}_{\mathrm{B}}^{(N)} = 1 - \frac{1}{\sqrt{N+1}}\,. \tag{5.48}$$

The vacuum is the only unpolarized state, in agreement with condition C1. Note also that the expressions (5.47) and (5.48) apply, e. g., to the states $|n\rangle_H \otimes |n\rangle_V$. Since for them $\langle\hat{\mathbf{S}}\rangle = 0$, classically they would be unpolarized for every n (that is, $\mathcal{P}_{\mathrm{sc}} = 0$, even in the limit $n \gg 1$). In our distance-based approach, the degree of polarization is a function of all moments of the Stokes operators and not only of the first one, as it happens for $\mathcal{P}_{\mathrm{sc}}$, which causes this quite different behavior. We also observe that all these states lying in the $(N+1)$-dimensional invariant subspace satisfy $\mathbb{P}\to 1$ when their intensity is increased.

Next, we define the diagonal states as those that can be expressed as

$$\hat{\varrho}_{\mathrm{diag}} = \sum_{N=0}^{\infty}\sum_{k=0}^{N} p_{Nk}\,|\Psi_k^{(N)}\rangle\langle\Psi_k^{(N)}|\,, \tag{5.49}$$

where we let $p_{Nk} \geq p_{Nk+1}$, for all $k < N$, and $\{|\Psi_k^{(N)}\rangle\}_{k=0}^{N}$ is an arbitrary orthonormal basis in the excitation manifold N. It then follows from C2 that any two diagonal

states whose probability distribution $\{p_{Nk}\}_{k=0}^{N}$ coincide, must have the same degree of polarization. For any diagonal state, we have

$$\mathbb{P}_{\mathrm{HS}}(\hat{\varrho}_{\mathrm{diag}}) = \sum_{N=0}^{\infty}\sum_{k=0}^{N} p_{Nk}^2 - \sum_{N=0}^{\infty}\frac{p_N^2}{N+1} \leq \sum_{N=0}^{\infty}\frac{N p_N^2}{N+1}\,. \tag{5.50}$$

To deal with the Bures degree of polarization for this example, we first note that

$$\sqrt{F(\hat{\varrho}_{\mathrm{diag}},\hat{\sigma})} = \sum_{N=0}^{\infty}\sum_{k=0}^{N}\sqrt{\lambda_N\, p_{Nk}} = \sum_{N=0}^{\infty} s_N\,\sqrt{\lambda_N}\,, \tag{5.51}$$

where

$$s_N = \sum_{k=0}^{N}\sqrt{p_{Nk}}\,. \tag{5.52}$$

The extremal points of (5.51) are then determined by

$$\frac{s_N}{2\sqrt{\lambda_N}} - \mu(N+1) = 0\,, \tag{5.53}$$

where μ is a Lagrange multiplier that takes into account the constraint (5.29). Solving for λ_N and imposing again equation (5.29) to fix the value of μ, the optimum parameters λ_N are found to be

$$\lambda_N = \frac{s_N^2}{(N+1)^2\displaystyle\sum_{k=0}^{\infty}\frac{s_k^2}{k+1}}\,. \tag{5.54}$$

In this way, we finally arrive at

$$\mathbb{P}_{\mathrm{B}}(\hat{\varrho}_{\mathrm{diag}}) = 1 - \sqrt{\sum_{N=0}^{\infty}\frac{s_N^2}{N+1}}\,. \tag{5.55}$$

One can easily prove that

$$\sqrt{p_N} \leq s_N \leq \sqrt{(N+1)p_N}\,, \tag{5.56}$$

so we have the bound

$$\mathbb{P}_{\mathrm{B}}(\hat{\varrho}_{\mathrm{diag}}) \leq 1 - \sqrt{\sum_{N=0}^{\infty}\frac{p_N}{N+1}} < 1\,. \tag{5.57}$$

This and equation (5.42) show that $\sum_{N=0}^{\infty} p_N^2 (N+1)^{-1} \to 0$ is necessary for both $\mathbb{P}_{\rm B}(\hat{\varrho}_{\rm diag})$ and $\mathbb{P}_{\rm HS}(\hat{\varrho})$ to approach unity. The latter also requires the purity to approach unity, whereas it is clear from equation (5.57) that this is not necessary in order to have $\mathbb{P}_{\rm B}(\hat{\varrho}) \to 1$.

As another relevant example, let us consider the case in which both modes are in (quadrature) coherent states. The product of two quadrature coherent states, which we shall denote by $|\alpha_H, \alpha_V\rangle$, can be expressed as a Poissonian superposition of SU(2) coherent states (Atkins and Dobson 1971)

$$|\alpha_H, \alpha_V\rangle = \sum_{N=0}^{\infty} e^{-\bar{N}/2} \frac{\bar{N}^{N/2}}{\sqrt{N!}} \, |N, \vartheta, \varphi\rangle \,, \tag{5.58}$$

where $\bar{N} = |\alpha_H|^2 + |\alpha_V|^2$ is the average number of excitations and the SU(2) coherent states are defined as (Perelomov 1986)

$$|N, \vartheta, \varphi\rangle = \sum_{k=0}^{N} \binom{N}{k}^{1/2} \left(\sin \frac{\vartheta}{2}\right)^{N-k} \left(\cos \frac{\vartheta}{2}\right)^{k} e^{-ik\varphi} \, |N, k\rangle \,, \tag{5.59}$$

and the state parameters are connected by the relations

$$\alpha_H = e^{-i\varphi/2} \sqrt{\bar{N}} \sin \frac{\vartheta}{2} \,, \qquad \alpha_V = e^{i\varphi/2} \sqrt{\bar{N}} \cos \frac{\vartheta}{2} \,. \tag{5.60}$$

Taking into account that

$$\sum_{N=0}^{\infty} \frac{p_N^2}{N+1} = \frac{{\rm I}_1(2\bar{N})}{\bar{N}} e^{-2\bar{N}} \,, \tag{5.61}$$

where ${\rm I}_1(z)$ is the modified Bessel function, equation (5.42) reduces to

$$\mathbb{P}_{\rm HS} = 1 - \frac{{\rm I}_1(2\bar{N})}{\bar{N}} e^{-2\bar{N}} \,. \tag{5.62}$$

When $\bar{N} \gg 1$ we can retain the first term in the asymptotic expansion of ${\rm I}_1(z)$ to obtain

$$\mathbb{P}_{\rm HS} \simeq 1 - \frac{1}{2\sqrt{\pi} \bar{N}^{3/2}} \,. \tag{5.63}$$

As a last example, we consider the maximally entangled states

$$|\zeta\rangle = \hat{S}(\zeta) \, (|0\rangle_H \otimes |0\rangle_V), \tag{5.64}$$

where $\hat{S}(\zeta)$ is the two-mode squeezing operator

$$\hat{S}(\zeta) = \exp(\zeta^* \hat{a}_H \hat{a}_V - \zeta \hat{a}_H^\dagger \hat{a}_V^\dagger). \tag{5.65}$$

These states are very important because are generated in typical parametric amplification processes.

Using the disentangled form of $\hat{S}(\zeta)$, the state (5.64) can be represented as

$$|\zeta\rangle = \frac{1}{\cosh\zeta} \sum_{n=0}^{\infty} (-1)^n \tanh^n \zeta \, |n\rangle_H \otimes |n\rangle_V, \tag{5.66}$$

where we assume that ζ is real. In terms of the SU(2) invariant subspaces, (5.66) reads as

$$|\zeta\rangle = \frac{1}{\cosh\zeta} \sum_{N=0}^{\infty} (-1)^{N/2} (\tanh\zeta)^{N/2} \, |N, N/2\rangle. \tag{5.67}$$

This means that the total photon number distribution of this state is

$$p_N = \begin{cases} \dfrac{(\tanh\zeta)^N}{\cosh^2\zeta}, & \text{for } N \text{ even}, \\ \\ 0, & \text{for } N \text{ odd}. \end{cases} \tag{5.68}$$

and its average number of excitations can be expressed as $\bar{N} = 2\sinh^2\zeta$.

Using this form of p_N and after some calculations, we get

$$\sum_{N=0}^{\infty} \frac{p_N^2}{N+1} = \frac{1}{2\cosh^4\zeta \tanh^2\zeta} \ln\left(\frac{1+\tanh^2\zeta}{1-\tanh^2\zeta}\right), \tag{5.69}$$

and hence the Hilbert-Schmidt degree of polarization is

$$\mathbb{P}_{\mathrm{HS}}(|\zeta\rangle) = 1 - \frac{4\zeta}{\sinh^2(2\zeta)}. \tag{5.70}$$

When $\bar{N} \gg 1$, we can approximate (5.70) by

$$\mathbb{P}_{\mathrm{HS}}(|\zeta\rangle) \simeq 1 - 4\zeta \exp(-4\zeta), \tag{5.71}$$

so, the degree of polarization tends to unity exponentially.

If we express (5.71) in terms of $\bar{N}$ the polarization degree take the form

$$\mathbb{P}_{\mathrm{HS}}(|\zeta\rangle) \simeq 1 - \frac{2\ln(\bar{N}/2)}{\bar{N}^2}. \tag{5.72}$$

To deal with the Bures degree of polarization. We have to calculate the fidelity between the density matrix for the state (5.64) and a generic $\hat{\sigma}$. One immediately gets that

$$\hat{\sigma}^{1/2}\hat{\varrho}\hat{\sigma}^{1/2} = (1-|\eta|^2)\sum_{N,N'=0}^{\infty}\frac{\eta^{N/2}\sqrt{\lambda_N}}{\sqrt{N+1}}\frac{\eta^{*N'/2}\sqrt{\lambda_{N'}}}{\sqrt{N'+1}}|N,N/2\rangle\langle N',N'/2|. \quad (5.73)$$

with $\eta = \tanh(\zeta)$. Note that, given the particular form of these states, we have

$$(\hat{\sigma}^{1/2}\hat{\varrho}\hat{\sigma}^{1/2})^{1/2} = \sqrt{(1-|\eta|^2)}\sum_{N,N'=0}^{\infty}\frac{\eta^{N/2}\sqrt{\lambda_N}}{\sqrt{N+1}}\frac{\eta^{*N'/2}\sqrt{\lambda_{N'}}}{\sqrt{N'+1}} \times|N,N/2\rangle\langle N',N'/2|. \quad (5.74)$$

In consequence

$$\sqrt{F(\hat{\varrho},\hat{\sigma})} = \sqrt{(1-|\eta|^2)}\sum_{N=0}^{\infty}\frac{|\eta|^N\lambda_N}{N+1} \quad (5.75)$$

and the minimum of this fidelity is attained when $\lambda_N = \delta_{N,0}$. In consequence, the Bures polarization degree take the form

$$\mathbb{P}_{\mathrm{B}}(\hat{\varrho}) = 1 - \frac{1}{\cosh\zeta}, \quad (5.76)$$

which tends again exponentially to the unity.

One can ask if the Hilbert-Schmidt and Bures measures order some pairs of states differently. In the Appendix B we show that this is indeed the case, and the induced degrees of polarization are therefore fundamentally different.

5.3.2 Maximally polarized states

A number of key concepts in quantum optics can be concisely quantified in terms of distance measures. The notions of nonclassicality and entanglement to cite only a few relevant examples, have been systematically formulated within this framework. A good deal of effort has been devoted to characterize maximally nonclassical or entangled states. However, maximally polarized states have been not considered thus far, except for some trivial cases. It is precisely the purpose of this section provide a complete description of such states, as well as feasible experimental schemes for their generation.

To simplify as much as possible the discussion, we restrict our considerations here to the Hilbert-Schmidt degree of polarization and drop the corresponding subscript. It is clear from equation (5.42) that for the states living in the manifold with exactly N photons, the optimum is reached for pure states. In fact, all such pure states have the same degree of polarization (5.48) showing a typical scaling N^{-1}, when $N \gg 1$. In particular, the important SU(2) coherent states are an example of these states.

However, this scaling law N^{-1} can be easily surpassed. Perhaps the simplest example is when both modes are in (quadrature) coherent states $|\alpha_H, \alpha_V\rangle$ [described

from (5.58) to (5.60)], the polarization degree of these states (5.64) has a scaling $N^{-3/2}$. Other important example is when the state is a maximally entangled of the form (5.64). In this case, the polarization has a scaling N^{-2}.

In consequence, we are led to find optimum states for a fixed average number of photons $\bar{N}$. Obtaining the whole optimum distribution p_N in (5.42) is exceedingly difficult, since it involves optimizing over an infinite number of variables. Our strategy to attack this problem is to truncate the Hilbert space and consider only photon numbers up to some value $\mathcal{D}$, where we take the limit $\mathcal{D} \to \infty$ at the end. In this truncated space, we need to find the states that maximize (5.42) with the constraints

$$p_N \geq 0, \qquad \sum_{N=0}^{\mathcal{D}} p_N = 1, \qquad \sum_{N=0}^{\mathcal{D}} N\ p_N = \bar{N}. \tag{5.77}$$

It is clear that the optimum must be again pure states. If we introduce the notations $p^T = (p_0, p_1, \ldots, p_L)$ and $H = 2\ \mathrm{diag}[1, 1/2, \ldots, 1/(\mathcal{D}+1)]$, the task can be thus recast as

$$\begin{aligned} &\text{minimize} && \frac{1}{2} p^T H p \\ &\text{subject to} && Ap = b, \\ & && p \geq 0, \end{aligned} \tag{5.78}$$

where

$$b = \begin{pmatrix} 1 \\ \bar{N} \end{pmatrix}, \qquad A = \begin{pmatrix} 1\ 1\ 1 \cdots 1 \\ 0\ 1\ 2 \cdots \mathcal{D} \end{pmatrix}. \tag{5.79}$$

We deal then with a quadratic program that, in addition, is convex, because H is positive definite (Boyd and Vandenberghe 2004). The optimum point exists and it is unique: in fact, there are numerous algorithms that compute this optimum in a quite efficient manner. Alternatively, we may try to determine it analytically by incorporating the constraints by the method of Lagrange multipliers. The functional to be minimized is

$$\mathcal{L}(p, \lambda) = \frac{1}{2} p^T H p - \lambda^T (Ap - b)\,. \tag{5.80}$$

The first-order optimality conditions $\nabla \mathcal{L}(p, \lambda) = 0$ together with the initial equality constraint, give the system of linear equations

$$\begin{pmatrix} H & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} p \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \tag{5.81}$$

whose formal solution is

$$\lambda = (AH^{-1}A^T)^{-1} b\,, \qquad p = H^{-1} A^T \lambda\,. \tag{5.82}$$

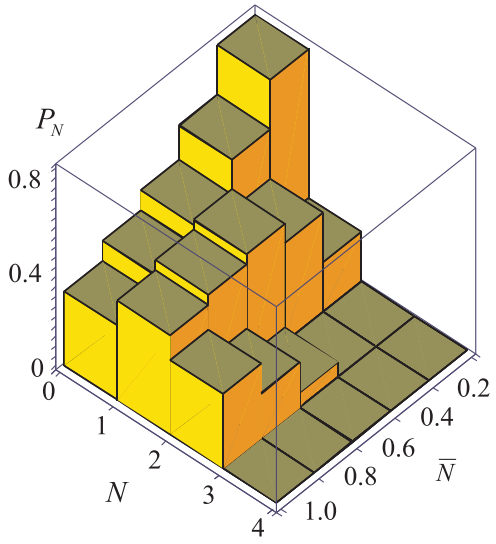

Fig. 5.2. Optimum distribution p_N, obtained by solving numerically the quadratic program (5.78), plotted as a function of the average number of photons $\bar{N}$ and N. We have taken $\bar{N}$ running from 0.2 to 1 and the dimension of the space $\mathcal{D} = 4$.

Before working out the analytical form of (5.82), in figure 5.2 we have plotted the numerical solution of the quadratic program (5.78) for some values in $0 \leq \bar{N} \leq 1$, using the MINQ code implemented in Matlab. The number of nonzero components of p_N is $[2\bar{N}+1]$, where the brackets denote integer part. The distribution presents a clear skewness and one can check that it can be well fitted to a Poisson distribution, which in physical terms means that, in this range, a quadrature coherent state $|\alpha_H, \alpha_V\rangle$ can be considered as optimum. To better assess this behavior, we have calculated the associated Mandel Q parameter (Mandel and Wolf 1995)

$$Q = \frac{\langle(\Delta\hat{N})^2\rangle}{\langle\hat{N}\rangle} - 1\,, \tag{5.83}$$

where $\langle(\Delta\hat{N})^2\rangle$ is the variance, which is a standard measure of the deviation from the Poisson statistics. In figure 5.3 we have represented Q in terms if $\bar{N}$. As we can see, Q increases linearly with $\bar{N}$ and is zero only near $\bar{N} \simeq 3$.

In figure 5.4 we have plotted the optimum distribution p_N for different integer values of $\bar{N}$ running from 1 to 9. The truncation value has been chosen to be 25 in all the cases, although it is sufficient to ensure, for each value of $\bar{N}$, that $p_N \simeq 0$ for $N > \mathcal{D}$. Three distinctive features can be immediately discerned: the solutions are

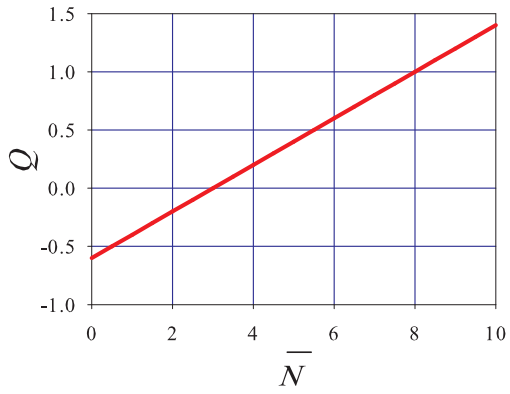

Fig. 5.3. Plot of the Mandel Q parameter for the optimum distribution p_N obtained numerically from (5.78) in terms of the average number of photons $\bar{N}$ of the state.

symmetric around $\bar{N}$, they are parabolic, and extend in a range from 0 to $2\bar{N}$. The two first facts are in agreement with the symmetry properties of the original problem (5.78). The third one means a variance that scales as $\bar{N}^2$, at difference of what happens for standard coherent optical processes presenting a variance linear with $\bar{N}$ (as for, e.g., in Poissonian or Gaussian statistics). In other words, the optimum states are extremely noisy and fluctuating. When $\bar{N}$ is not integer (or semi-integer), one can appreciate a small asymmetry that is less and less noticeable as $\bar{N}$ increases.

We conclude that we can take the dimension $\mathcal{D}$ to be $2\bar{N}$ without serious error. Given the very simple form of H and A, we can express the final solution (5.82) in a closed analytic form:

$$p_N = 3\frac{(\bar{N}+1)^2-(N-\bar{N})^2}{(2\bar{N}+1)(\bar{N}+1)(2\bar{N}+3)} \simeq \frac{3}{2\bar{N}}\left(\frac{N}{\bar{N}}-\frac{1}{2}\frac{N^2}{\bar{N}^2}\right), \tag{5.84}$$

which is properly normalized and shows all the aforementioned characteristics, with a maximum value of $p_{\bar{N}} \simeq 3/(4\bar{N})$. If we use $x = N/(2\bar{N})$, which can be taken as a quasicontinuous variable $0 \le x \le 1$, we can convert (5.84) in equation $p(x) = (3/\bar{N})\, x(1-x)$, which is the Beta distribution of parameters $(2,2)$ (Evans et al. 2000). For the solution (5.84), the corresponding degree of polarization is

$$\mathbb{P}_{\text{opt}} = 1 - \frac{3}{(2\bar{N}+1)(2\bar{N}+3)} \sim 1 - \frac{3}{4\bar{N}^2}\,. \tag{5.85}$$

This provides a full characterization of the optimum states we were looking for. However, their physical implementation stands as a serious problem. The crucial issue for

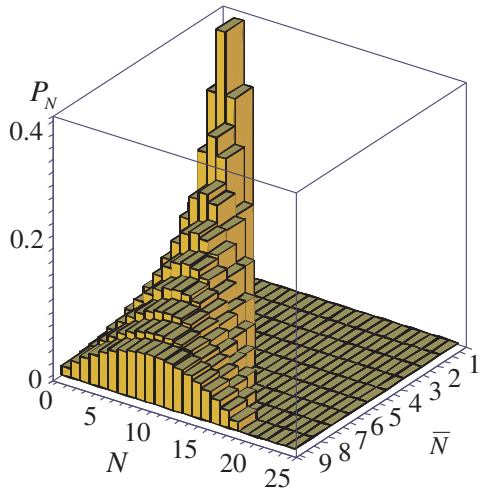

Fig. 5.4. Optimum distribution p_N plotted as a function of the average number of photons $\bar{N}$ and N. We have taken $\bar{N}$ to be a integer running from 1 to 9 and the dimension of the space $\mathcal{D} = 25$.

the scaling in (5.85) is the fact that distribution variance is proportional to $\bar{N}^2$. It turns out that, for the discrete uniparametric distributions usually encountered in physics, this is distinctive of the thermal (or geometric) distribution

$$p_N = \frac{1}{\bar{N}+1}\left(\frac{\bar{N}}{\bar{N}+1}\right)^N . \tag{5.86}$$

But this is the photon statistics associated with the states (5.66) which are precisely the twin beams generated in an optical parametric amplifier with a vacuum-state input. The distribution (5.86) presents a skewness absent in the exact solution (5.84), but a calculation of the state degree of polarization gives

$$\mathbb{P} \simeq 1 - \frac{2\ln(\bar{N}/2)}{\bar{N}^2} . \tag{5.87}$$

Apparently, this is different from (5.85), but as soon as $\bar{N} \gg 1$ they both approach unity in essentially the same way, which means that the (maximally entangled) squeezed vacuum (5.66) is very close to optimum when $\bar{N} \gg 1$.

Before ending, two important remarks seem in order. First, we observe that in classical optics fully polarized fields have a perfectly defined relative phase between H- and V-polarized modes (Brosseau 1998). Such a relation does not necessarily hold

in the quantum domain: while the quadrature coherent states (5.59) have a sharp relative phase, the twin-photon beams (5.66) have an almost random relative phase. Second, the maximally polarized states we have found have a highly nonclassical behavior, even in the limit $\bar{N} \gg 1$, which makes the classical limit of these polarized states a touchy business.

6 Conclusions

In this work, we have studied discrete quantum systems. In the following, we summarize the most important results that have been obtained:

- We have analysed the basic properties of two- and three-dimensional quantum systems. We have shown their simmetries and how one can describe these systems. To this end, we have studied and compared several possible descriptions of the phase based in a polar decomposition, in a POVM or as the complementary variable to amplitude.
- We have investigated an appropriate operator for the description of the relative phase in the interaction of two-level and three-level atoms with quantum fields. We have resorted to a proper polar decomposition of the corresponding amplitudes, which has been justified on physical grounds as well as using the theory of polynomial deformations of su(2) and su(3), respectively. The finite dimension of the invariant subspaces implies that the spectrum of the relative phase is discrete, which is very surprising from a physical point of view. From the phase states obtained in this procedure, we have defined a probability distribution function for the relative phase and studied its time evolution, showing how the formalism could be applied to understanding more involved phenomena.
- Inspired by the behavior of linear models with simmetry su(2), we have introduced small "rotations" generated by polinomial deformations. When a controlable parameter (normally the inverse of a detuning) becomes small, our method allows us to describe the original model in terms of an effective Hamiltonian.
- We have shown a criterion to define a two-mode state with well-defined relative phase. The relative-phase eigenstates satisfy this criterion in spite of displaying less than unity visibility. This has led us to define a generalization of a visibility measure, and subsequently to derive a criterion for when the generalized visibility can be equal to unity. We have shown that all states with a well-defined relative phase can display unit generalized visibility, whereas the converse is not true.
- Quantum optics entails polarization states that cannot be suitably described by the (semi)classical formalism based on Stokes parameters. We have advocated the use of a degree of polarization based on an appropriate distance to the set of

unpolarized states. Such a definition is closely related to other recent proposals in different areas of quantum optics and is well behaved even when the classical formalism fails. Finally, taking into account this, we have provided a complete description of maximally polarized states, as well as feasible experimental schemes for their generation.

A Positive operator-valued measures

In quantum mechanics it is fundamental to understand the form in which we extract the information from the physics systems. This form is very important because the taking of measures entails a projection of the system over one of the eigenstates of the quantum variable that this being measured. The successive measurements on equivalent systems provide us the statistic of the observable for each one of their possible results.

Given a system, the state at the initial moment (this is before making some measurement) can be represented, in general, like a linear superposition of the eigenstates of an observable which we desire to measure

To be more precise, it turns out suitable to observe that, given an obsevable $\hat{M}$, the state of the system $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \sum_a \alpha_a |u_a\rangle, \tag{A.1}$$

where $|u_a\rangle$ are the ortogonal eigenstates of the operator $\hat{M}$

$$\hat{M} = \sum_a |u_a\rangle M_a \langle u_a|, \tag{A.2}$$

and M_a are the eigenvalues of $\hat{M}$.

We will detect a value M_a for the observable $\hat{M}$ with probability $|\alpha_a|^2$, in in that case we will have prepared the system in an eigenstate of $\hat{M}$ of $|u_a\rangle$. One can conclude, that the initial state $|\psi\rangle$ or quantum system, is projected on $|u_a\rangle$ with probability $|\langle u_a|\psi\rangle|^2$. This forms the model of orthogonal measurement of Von Neumann.

A.1 Generalized measurements

If one try to generalize the measurement concept beyond these orthogonal measurements considered by Von Neummann, one way to arrive at that idea is supose that the sistem A is extended from the Hilbert state $\mathcal{H}_A$ to a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$,

and that we perform orthogonal measurements in the tensor product, which will not necessarily be orthogonal measurements in A.

We assume that the Hilbert space $\mathcal{H}_A$ is a part of a larger space that has structure of a *direct sum*

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B. \tag{A.3}$$

The observers living in $\mathcal{H}_A$ have acces only to observables with support in $\mathcal{H}_A$, observables $\hat{M}_A$ such that

$$\hat{M}_A|\psi^\perp\rangle = 0 = \langle\psi^\perp|\hat{M}_A, \tag{A.4}$$

for any $|\psi^\perp\rangle \in \mathcal{H}_A^\perp$. Anyway, when we perform orthogonal measurements in $\mathcal{H}$, preparing one of a set of mutually orthogonal states (eigenstates of a observer M), the observer will know only about the component of that state in his space $\mathcal{H}_A$. Since these components are not necessarily orthogonal in $\mathcal{H}_A$, he will conclude that the measurement prepares one of a set or non-orthogonal states.

Suppose that the initial density matrix $\hat{\varrho}_A$ has support in $\mathcal{H}_A$, (this is reasonable because the observer will build the eigenstate in his enviroment) and that we perform an orthogonal measurement in $\mathcal{H}$. We will consider the case in which each E_a, is one-dimensional projector, which will be general for our purposes. Thus , $\hat{E}_a = |u_a\rangle\langle u_a|$, where $|u_a\rangle$ is a normalized vector in $\mathcal{H}$. This vector has a unique orthogonal decomposition

$$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^\perp\rangle, \tag{A.5}$$

where $|\tilde{\psi}_a\rangle$ and $|\tilde{\psi}_a^\perp\rangle$ are (unnormalized) vectors in $\mathcal{H}_A$ and $\mathcal{H}_A^\perp$ respectively. After the measurement, the new density matrix will be $|u_a\rangle\langle u_a|$ with probability $\langle u_a|\hat{\varrho}_A|u_a\rangle = \langle\psi_a|\hat{\varrho}_A|\psi_a\rangle$ since $\hat{\varrho}_A$ has no support in $\mathcal{H}_A^\perp$.

But to the observer who knows nothing of $\mathcal{H}_A$, there is no physical distinction between $|u_a\rangle$ and $|\tilde{\psi}_a\rangle$ (aside from normalization). If we write $|\tilde{\psi}_a\rangle = \sqrt{\lambda_a}|\psi_a\rangle$, where $|\psi_a\rangle$ is a normalized state, then for the observer limited to observations in $\mathcal{H}_A$, we might as well say that the outcome of the measurement is $|\psi_a\rangle\langle\psi_a|$ with probability $\langle\psi_a|\hat{\varrho}_A|\psi_a\rangle$.

Let us define an operator

$$\hat{\Delta}_a = \hat{E}_A\hat{E}_a\hat{E}_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a|\psi_a\rangle\langle\psi_a|, \tag{A.6}$$

where $\hat{E}_A$ is the orthogonal projection taking $\mathcal{H}$ to $\mathcal{H}_A$. Then we may say that the outcome a has a probability $P(a) = \mathrm{Tr}(\hat{\Delta}_a\varrho)$. It is evident that each $\hat{\Delta}_a$ is hermitian and nonnegative, but these are not projections unless $\lambda_a = 1$. Furthermore, $\hat{\Delta}_a$ fullfills

$$\sum_a \hat{\Delta}_a = \hat{E}_A \sum_a (\hat{E}_a)\hat{E}_A = \hat{E}_A = \hat{\mathbb{1}}_A, \tag{A.7}$$

that is, the sum of $\hat{\Delta}_a$ gives the identity in A.

A partition of unity by nonnegative operators $\hat{\Delta}_a$ is called a positive operator-valued measure (POVM). In summary, the set operators $\hat{\Delta}_a$ must fulfill

$$\hat{\Delta}_a^\dagger = \hat{\Delta}_a, \qquad \hat{\Delta}_a \geq 0, \qquad \sum_a (\hat{\Delta}_a) = \mathbb{1}, \tag{A.8}$$

In general, each outcome has a probability that can be expressed as

$$P(a) = \mathrm{Tr}(\hat{\Delta}_a \varrho), \tag{A.9}$$

hence, the second and third conditions of (A.8) guarantee that the probabilities are nonnegative and with sum the unity respectively.

In the analized situation, we have imposed that the operators are one-dimensional. The generalation of this is trivial. The main advantaje of we use POVMs is that we do not need a description of the observables in terms of hermitian operators to get a complete description of these observables. Thus, we can obtain the probability distribution imposing additional restrictions to the restrictions imposed in (A.8), based on fundamental properties of the observables that we want to describe.

A.2 Example: POVMs for angular variables

Se trata de obtener, a través del formalismo de POVM explicado anteriormente, las POVMs que describen variables ángulo de manera general. Para ello, analizaremos las restricciones adicionales que son necesarias para la contrucción de dicho formalismo.

When dealing with generic angle-action variables, one imposes that the complex exponential of the angle (denoted by $\hat{E}$) and the action variable (denoted by $\hat{L}$) satisfy

$$[\hat{E}, \hat{L}] = \hat{E}. \tag{A.10}$$

If $\hat{E}$ were unitary, its action on the basis of eigenstates of $\hat{L}$ (denoted by $|m\rangle$) will be as a ladder operator

$$\hat{E}|m\rangle = |m-1\rangle. \tag{A.11}$$

The eigenstates of $\hat{E}$ (denoted by $|\theta\rangle$) provide then an adequate description of the quantum angle (Luis and Sánchez-Soto 1998). To ensure that $\hat{\Delta}(\theta)$ provides a meaningful description of the angle as a canonically conjugate variable with respect L we require

$$e^{i\theta' \hat{L}}\ \hat{\Delta}(\theta)\ e^{-i\theta' \hat{L}} = \hat{\Delta}(\theta + \theta'), \tag{A.12}$$

which reflects nothing but the basic feature that an angle shifter is an angle-distribution shifter. This condition restricts the form of the POVM to

$$\hat{\Delta}(\theta) = \frac{1}{2\pi} \sum_{n,m=0}^{\infty} b_{n,m}\ e^{i(m-n)\theta}\ |m\rangle\langle n|. \tag{A.13}$$

We must take also into account that a shift in $\hat{L}$ should not change the angle distribution. A shift in $\hat{L}$ is expressed by the operator $\hat{E}$, since it shifts the distribution of $\hat{L}$ by one step. Therefore, we require as well

$$\hat{E}\ \hat{\Delta}(\theta)\ \hat{E}^{\dagger} = \hat{\Delta}(\theta), \tag{A.14}$$

which, loosely speaking, is the physical translation of the fact that angle should be complementary to the action variable. This implies the invariance

$$b_{n,m} = b_{n-m}, \tag{A.15}$$

that allows us to recast Eq. (A.13) as

$$\hat{\Delta}(\theta) = \frac{1}{2\pi} \sum_{\nu}^{\infty} b_{-\nu} e^{-i\nu\theta} \hat{E}^{\nu}, \tag{A.16}$$

while conditions (A.8) read now as now

$$|b_{\nu}| \leq 1, \quad b_{\nu}^{*} = b_{\nu}. \tag{A.17}$$

Expressing $\hat{E}$ in terms of its eigenvectors $|\theta\rangle$, we finally arrive at the general form of a POVM describing the angle variable and fulfilling the natural requirements (A.12) and (A.14):

$$\hat{\Delta}(\theta) = \int_{2\pi} d\theta'\ B(\theta')\ |\theta + \theta'\rangle\langle\theta + \theta'|, \tag{A.18}$$

where

$$B(\theta) = \frac{1}{2\pi} \sum_{\nu=0}^{\infty} b_{\nu} e^{i\nu\theta}. \tag{A.19}$$

This convolution shows that this effectively represents a noisy measurement, the function $B(\theta)$ giving the resolution provided by this POVM (Luis and Sánchez-Soto 1998).

B Hilbert-Schmidt and Bures degrees of polarization

In this appendix, we will show that the Hilbert-Schmidt and the Bures distances induce fundamentally different degrees of polarization. To this end, we consider the states

$$\hat{\rho}_{N_1N_2} = \sum_{j,k=1}^{2} \rho_{jk} |\Psi^{(N_j)}\rangle\langle\Psi^{(N_k)}| \,, \tag{B.1}$$

where $|\Psi^{(N_1)}\rangle$ and $|\Psi^{(N_2)}\rangle$ are orthogonal pure states with N_1 and N_2 photons, respectively. We here assume that $N_1 \neq N_2$, and note that $\hat{\rho}_{NN}$ is a diagonal state of the form (5.49). To simplify calculations, we shall use the notation

$$p = \rho_{11} \,, \qquad 1-p = \rho_{22} \,, \qquad q = \rho_{12} = \rho_{21}^* \,. \tag{B.2}$$

The states $|\Psi^{(N_1)}\rangle$ and $|\Psi^{(N_2)}$ then correspond to $p = 0$ and $p = 1$, respectively, and the purity becomes

$$\mathrm{Tr}(\hat{\rho}_{N_1N_2}^2) = p^2 + (1-p)^2 + 2|q|^2 \,. \tag{B.3}$$

We note in passing that $1 - 2p(1-p) \leq \mathrm{Tr}(\hat{\rho}_{N_1N_2}^2) \leq 1$ and $0 \leq |q|^2 \leq p(1-p)$.

In the basis $(|\Psi^{(N_1)}\rangle, |\Psi^{(N_2)}\rangle)$, we can write

$$\hat{\sigma}^{1/2}\hat{\rho}\hat{\sigma}^{1/2} = \begin{pmatrix} \lambda_{N_1}p & \sqrt{\lambda_{N_1}\lambda_{N_2}}q \\ \sqrt{\lambda_{N_1}\lambda_{N_2}}q^* & \lambda_{N_2}(1-p) \end{pmatrix} . \tag{B.4}$$

Since the eigenvalues of this matrix are

$$\chi_\pm = \frac{1}{2}\left\{\lambda_{N_1}p + \lambda_{N_2}(1-p) \pm \sqrt{[\lambda_{N_1}p - \lambda_{N_2}(1-p)]^2 + 4\lambda_{N_1}\lambda_{N_2}|q|^2}\right\} , \tag{B.5}$$

the fidelity can be expressed as

$$F(\hat{\rho},\hat{\sigma}) = \chi_+ + 2\sqrt{\chi_+\chi_-} + \chi_- = \lambda_{N_1}p + \lambda_{N_2}(1-p)$$

$$+ 2\sqrt{\lambda_{N_1}\lambda_{N_2}[p(1-p) - |q|^2]} \,. \tag{B.6}$$

For any fixed λ_{N_1}, λ_{N_2}, and p, the fidelity decreases as $|q|^2$ increases. This could be expected, since the unpolarized states do not have any off-diagonal elements.

The restriction (5.29) implies for this problem that

$$\lambda_{N_2} = \frac{1-(N_1+1)\lambda_{N_1}}{N_2+1}\,. \tag{B.7}$$

In consequence, the coefficients that optimize the fidelity are determined by

$$\begin{aligned}\frac{\partial F}{\partial \lambda_{N_1}} = 0 = p - \frac{(1+N_1)(1-p)}{1+N_2} \\ + [1-2\lambda_{N_1}(1+N_1)]\sqrt{\frac{p(1-p)-|q|^2}{\lambda_{N_1}[1-\lambda_{N_1}(1+N_1)](1+N_2)}}\,.\end{aligned} \tag{B.8}$$

We first consider pure states, for which $|q|^2 = p(1-p)$. Choosing λ_{N_1} according to

$$\begin{aligned}&\lambda_{N_1} = 0\,, && p < \frac{1+N_1}{2+N_1+N_2}\,,\\ &0 \le \lambda_{N_1} \le \frac{1}{1+N_1}\,, && p = \frac{1+N_1}{2+N_1+N_2},\\ &\lambda_{N_1} = \frac{1}{1+N_1}\,, && p > \frac{1+N_1}{2+N_1+N_2}\,,\end{aligned} \tag{B.9}$$

then maximizes the fidelity:

$$\sup_{\hat\sigma\in\mathcal{U}} F(\hat\rho,\hat\sigma) = \begin{cases} \dfrac{1-p}{1+N_2}\,, & p \le \dfrac{1+N_1}{2+N_1+N_2}\,,\\[2ex] \dfrac{p}{1+N_1}\,, & p \ge \dfrac{1+N_1}{2+N_1+N_2}\,. \end{cases} \tag{B.10}$$

On the other hand, when $|q|^2 \neq p(1-p)$ (i.e., when $0 < p < 1$), the solution of equation (B.8) is

$$\lambda_{N_1} = \frac{1}{2(N_1+1)}\left[1 - \frac{(1+N_1)(1-p)-(1+N_2)p}{\sqrt{[1+N_1(1-p)+N_2p]^2 - 4(1+N_1)(1+N_2)|q|^2}}\right]. \tag{B.11}$$

Depending on the parameters, this solution can take any value in the interval $0 < \lambda_{N_1} < 1/(1+N_1)$. In fact, one can check that the choice (B.11) gives the closest unpolarized state. Combining equations (B.6), (B.7), and (B.11), thus allows one to obtain the fidelity and hence $\mathbb{P}_{\mathrm{B}}$.

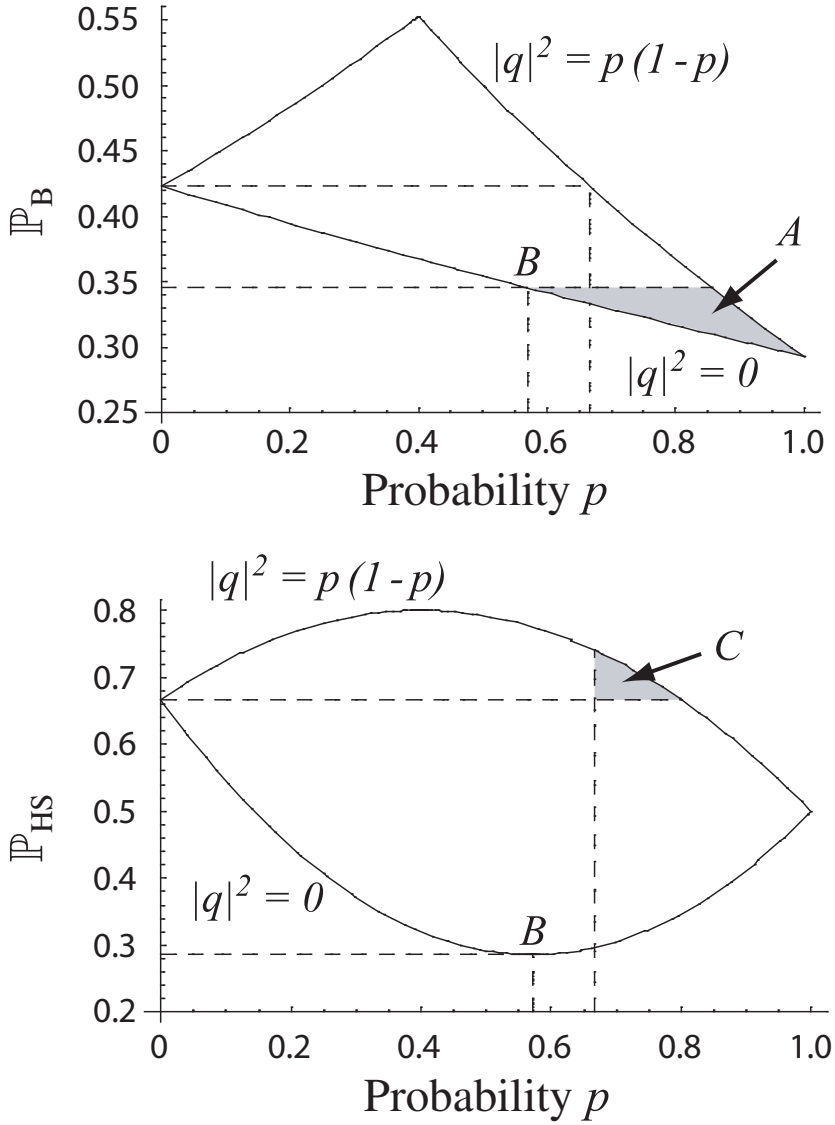

Fig. B.1. Polarization degrees for two-dimensional states with $N_1 = 1$ and $N_2 = 2$. For any given p, the maximum and minimum fidelities are given by $|q|^2 = p(1-p)$ and $|q|^2 = 0$, respectively. Region A corresponds to states satisfying $p > 4/7$ and $\mathbb{P}_{\rm B} < 1 - \sqrt{3/7}$. For any state $\hat{\rho}_A$ in this region,we have $\mathbb{P}_{\rm B}(\hat{\rho}_A) < \mathbb{P}_{\rm B}(\hat{\rho}_B)$ while $\mathbb{P}_{\rm HS}(\hat{\rho}_A) > \mathbb{P}_{\rm HS}(\hat{\rho}_B)$, where the state $\hat{\rho}_B$ is characterized by $p = 4/7$ and $|q|^2 = 0$. In region C, the states satisfy $p > 2/3$ and $\mathbb{P}_{\rm HS} > 2/3$. For any such state $\hat{\rho}_C$, we have $\mathbb{P}_{\rm HS}(\hat{\rho}_C) > \mathbb{P}_{\rm HS}(|\Psi^{(2)}\rangle)$ and $\mathbb{P}_{\rm HS}(\hat{\rho}_C) < \mathbb{P}_{\rm HS}(|\Psi^{(2)}\rangle)$, where $|\Psi^{(2)}\rangle$ is the (arbitrary) pure two-photon state corresponding to $p = 0$.

In the figure B.1, we have plotted the Hilbert-Schmidt and Bures degree of polarization for some two-dimensional states. From the explanation in the caption, we see that the two measures order some pairs of states differently. The Hilbert-Schmidt and Bures distances thus induce two fundamentally different degrees of polarization.

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