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CONJUNTOS INVARIANTES E INTEGRALES PRIMERAS DE SISTEMAS DINÁMICOS

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CONJUNTOS INVARIANTES E INTEGRALES PRIMERAS DE SISTEMAS DINÁMICOS

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Capítulo 1

Introducción

Sea X un campo de vectores (suave o analítico) en $\mathbb{R}^n$. La función $I : \mathbb{R}^n \to \mathbb{R}$, que se asume suficientemente suave, es una integral primera de X si $X(I) = 0$. El significado geométrico de esta definición es el siguiente: las órbitas de X son tangentes a los conjuntos de nivel de I en sus puntos regulares ($\nabla I \neq 0$). Cada hoja $I = c$, $c \in \mathbb{R}$, es un conjunto invariante de X.

La noción de conjunto invariante es más general que la de integral primera. El conjunto $\Sigma \subset \mathbb{R}^n$ es invariante bajo X si $\phi_t(p) \in \Sigma$ para todo $(p, t) \in \Sigma \times \mathbb{R}$ (el campo se asume completo sin pérdida de generalidad). Recordemos que ϕ_t es el flujo uniparamétrico inducido por X.

En este trabajo estudiamos las integrales primeras y los conjuntos invariantes de X desde diferentes puntos de vista. La literatura sobre estos objetos es extensa, en cada bloque haremos referencia al estado actual de cada cuestión así como a las novedades introducidas.

La estructura de la tesis es la siguiente:

- En el capítulo 2 estudiamos la relación entre integrales primeras y estabilidad (tanto de puntos críticos como frente a perturbaciones).

- En el capítulo 3 analizamos la conexión entre integrales primeras y simetrías.

- En el capítulo 4 se presentan algunas aplicaciones de las integrales primeras a modelos físicos específicos, por ejemplo campos magnéticos creados por configuraciones de hilos.

- Finalmente en el capítulo 5 se estudian dos tipos de conjuntos invariantes no asociados a integrales primeras: los conjuntos invariantes de las ecuaciones Newtonianas y los atractores.

Al final se incluye una sección de conclusiones así como una lista de otros artículos, no directamente relacionados con el tema de tesis, en los que se ha visto envuelto el solicitante. Todos los trabajos que aqui se detallan corresponden al período 1999-2005. Me gustaría expresar mi agradecimiento al Profesor Francisco González Gascón por haber confiado hace muchos años en mí como colaborador científico.

Capítulo 2

Integrales primeras: estabilidad

Sea el campo de vectores X con (al menos) una integral primera I, suave o analítica. Las trayectorias del campo yacen sobre los conjuntos de nivel de I, reduciendo así la dimensionalidad del sistema. Esta reducción tiene consecuencias topológicas, por ejemplo en la estabilidad de los puntos críticos, en la acotación y no acotación de órbitas o en los ω–límite de las curvas integrales de X. En este capítulo describimos algunos de estos fenómenos.

2.1. Instability of vector fields induced by first integrals: J. Math. Phys. 40 (1999) 3099

Consideremos en $\mathbb{R}^n$ un campo de vectores X con un cero aislado en el origen. Si el punto crítico es hiperbólico entonces su estabilidad se puede averiguar a partir de la linealización del campo [1]. En el caso degenerado no existen criterios universales. El uso de funciones de Liapunov es una herramienta extendida [2], nosotros sin embargo tomaremos otro camino: probar la inestabilidad del punto crítico cuando se conocen integrales primeras analíticas que verifican ciertas condiciones.

Si X es un sistema Hamiltoniano analítico con 1 o 2 grados de libertad entonces la inestabilidad del cero se sigue si el origen no es un mínimo del potencial [3]. En dimensión más alta este teorema no ha sido probado en general, aunque sí imponiendo más condiciones [4].

En el artículo obtenemos un resultado análogo al caso Hamiltoniano, pero para campos no Hamiltonianos en $\mathbb{R}^3$ (y su posible extensión a $\mathbb{R}^n$) con una integral primera analítica que verifica que el origen es un punto regular o una silla. La demostración depende fuertemente de la dimensionalidad del sistema (teorema de Bendixon-Poincaré [1]) y de la analiticidad de la integral primera (teorema de estructura de Lojasiewicz [5]).

2.2. A separation bound for non-Hamiltonian differential equations with proper first integrals: J. Math. Phys. 41 (2000) 2922

Las integrales primeras juegan un papel relevante no sólo en la estabilidad de puntos críticos, sino en la estabilidad de las órbitas del campo cuando éste es perturbado. Uno de los resultados más importantes en esta línea es el teorema KAM [6], que bajo ciertas condiciones garantiza la persistencia de toros invariantes cuando se perturban sistemas Hamiltonianos integrables. Esto implica que con 1 ó 2 grados de libertad las órbitas permanecen confinadas en regiones acotadas del espacio de fases, pudiendo escapar cuando el número de grados de libertad es mayor (difusión de Arnold [7]).

Las teorías del promedio y de los invariantes adiabáticas [6] también permiten estudiar la estabilidad de las órbitas y obtener cotas para la separación de las variables acción en el caso Hamiltoniano. El principal problema de estas técnicas es que generalmente exigen que las variables acción-ángulo estén definidas globalmente [8] o que las trayectorias del campo sean periódicas.

En este artículo se estudia la perturbación de campos de vectores en R^n con integrales primeras propias. Cuando la perturbación verifica ciertas condiciones, así como las integrales primeras, se prueba que la separación entre las órbitas del campo X y del campo $X + X_p$ es polinómica (en tiempo finito), en contraste con los resultados clásicos de separación exponencial. La demostración depende fuertemente de ciertas propiedades algebraicas de la integral primera y de que el módulo del gradiente esté acotado superiormente.

2.3. Unbounded trajectories of dynamical systems: Appl. Math. Lett. 17 (2004) 253

La idea de estudiar las propiedades topológicas de las hojas de las integrales primeras y su posible conexión con las propiedades de las órbitas del campo es debida a Smale [9], que explotó este enfoque para sistemas Hamiltonianos, en particular potenciales centrales y el problema de N cuerpos en $\mathbb{R}^2$. La mayor parte de la literatura sobre la relación entre topología de conjuntos invariantes y acotación de órbitas se centra en la existencia de órbitas periódicas [10].

En este artículo se muestra cómo la existencia de integrales primeras cuyas hojas tienen cierta topología implica la existencia de órbitas no acotadas si el campo inducido no tiene puntos críticos y es de divergencia nula. La demostración depende fuertemente de la dimensionalidad del sistema (dimensión 2). Es importante señalar que se exige que la divergencia del campo inducido sea cero con respecto a alguna forma de volumen, no necesariamente la forma de volumen inducida. En los ejemplos esta propiedad se pone de manifiesto ya que la forma que se considera es la de Godbillon [11], no la heredada del espacio ambiente.

2.4. Bagpipes configurations in Mechanics and Electromagnetism: Math. Comput. Modelling 42 (2005) 921-930

Las herramientas más extendidas para estudiar la estabilidad de puntos críticos de campos de vectores son el teorema de LaSalle y las funciones de Liapunov [2]. Las hipótesis fundamentales son la existencia de una función que decrezca con el flujo y la acotación de las órbitas del campo. En este caso se puede garantizar que el conjunto límite, estable, está contenido en el conjunto estacionario de la función de Liapunov. Si bien un punto crítico estable generalmente posee función de Liapunov [12] el cálculo práctico de ésta es inviable, con lo cual el criterio no se puede aplicar normalmente para averiguar la estabilidad.

En este artículo enfocamos el problema desde una perspectiva distinta, usando la existencia de integrales primeras que poseen cierta estructura topológica. El criterio puede verse como una generalización del sencillo resultado que afirma que si las hojas de una integral primera alrededor del punto crítico son esferas topológicas entonces el punto crítico es estable. Las integrales primeras que nosotros consideramos tienen forma de gaita, además se asume que el campo es asintoticamente estable en el esqueleto de la gaita. La demostración depende fuertemente de estas dos hipótesis. Esto permite definir una región de trampa que fuerza a la estabilidad del campo.

La forma de gaita de los conjuntos de nivel de la integral primera surge al estudiar el campo magnetico creado por N hilos rectilíneos que se cortan en el origen. Esto permite demostrar que las líneas magnéticas son círculos topológicos cerca de los hilos.

Capítulo 3

Integrales primeras: simetrías

La existencia de cantidades conservadas suele asociarse a simetrías del sistema, posiblemente ocultas. No es claro que una integral primera I necesariamente proceda, en cierto sentido, de una simetría S del campo X, al igual que una simetría no genera necesariamente una integral primera. En este capítulo estudiamos integrales primeras y conjuntos invariantes que son consecuencia de la existencia de simetrías u otras estructuras algebraicas relacionadas con el campo de vectores.

3.1. Symmetries and first integrals of divergence-free $\mathbb{R}^3$ vector fields: Int. J. Nonlinear Mech. 35 (2000) 589

La relación entre la existencia de simetrías de campos de vectores y la existencia de integrales primeras, foliaciones invariantes o conjuntos invariantes es un tema clásico en la literatura [13]. Bajo ciertas condiciones se puede ver que las simetrías permiten integrar el campo local o globalmente, y que ciertas estructuras complejas de las órbitas, por ejemplo atractores extraños o ergodicidad en abiertos, no pueden darse [14]. Estos resultados son revisados en el artículo, también se obtienen ciertos conjuntos invariantes a partir de la existencia de simetrías.

Un caso particularmente interesante que también se estudia es el de campos de vectores de divergencia nula (como el campo magnético o el campo de velocidades de un fluído). Se obtienen integrales primeras y conjuntos invariantes de estos campos asumiendo la existencia de simetrías que verifican ciertas propiedades.

La existencia de integrales primeras, obtenida con los anteriores procedimientos, se aplica al análisis de la estabilidad o inestabilidad de los puntos críticos. Los resultados que se obtienen en esta línea están contenidos en los artículos del capítulo anterior.

3.2. Dynamical Systems embedded into Lie algebras: J. Math. Phys. 42 (2001) 5741

Este artículo sigue la línea del anterior, obtener conjuntos invariantes, integrales primeras u otras propiedades cualitatívas de las órbitas a partir de la existencia de simetrías. En este caso se asume que los campos de vectores cierran a la Lie con constantes, dando lugar a un álgebra de Lie. Este contexto es nuevo en la literatura ya que los campos de vectores que cierran no necesariamente lo hacen como simetrías de X. Se analiza con detalle los casos de dos campos de vectores ($A_{2,2}$) y tres campos de vectores ($A_{3,3}$).

La hipótesis de algebra de Lie no es suficiente para obtener conjuntos invariantes o integrales primeras. En el artículo se imponen condiciones extra, como ciertas relaciones en los coeficientes que definen el álgebra, existencia de integrales primeras de los campos que cierran con X o existencia de formas diferenciales invariantes bajo X o los otros campos. Todos estos resultados se aplican a conocidas ecuaciones que surgen en Física, como por ejemplo el sistema de Lorenz.

Capítulo 4

Integrales primeras: aplicaciones

En este capítulo se consideran sistemas dinámicos concretos que surgen en diferentes contextos físicos: Mecánica de Fluidos, Electrodinámica, ... La existencia de integrales primeras para estos campos es relevante a la hora de estudiar su complejidad orbital o la presencia de caos y turbulencia. Analizaremos algunos sistemas físicos que poseen integrales primeras y por tanto presentan comportamientos ordenados.

4.1. On the first integrals of Lotka-Volterra systems: Phys. Lett. A 266 (2000) 336

El sistema de Lotka–Volterra en $\mathbb{R}^3$ aparece en contextos tan diferentes como biología matemática [15], física de fluidos [16] y cinética química [17]. Se trata de un conjunto de 3 ecuaciones diferenciales ordinarias que dependen de 9 parámetros. El problema esencial consiste en entender la estructura de las órbitas del campo de vectores asociado en función de los parámetros. En este artículo se obtienen 9 nuevos casos para los que Lotka–Volterra tiene una integral primera, generalmente local, complementando así otros estudios en la literatura [18].

La técnica de la demostración se basa en encontrar 2 simetrías generalizadas (independientes en casi todo $\mathbb{R}^3$) del campo de vectores de Lotka-Volterra. Una de ellas es bien conocida, la simetría de dilatación, la otra se busca para diferentes valores de los parámetros mediante cálculos con el ordenador. Una vez se tienen estas simetrías el cálculo de las integrales primeras es inmediato aplicando los algoritmos clásicos de integración.

4.2. Ordered behavior in force-free magnetic fields: Phys. Lett. A 292 (2001) 75

En este trabajo se estudia un tipo particular de campos magnéticos de especial relevancia en magnetohidrodinámica, los campos "force-free" [19]. La ecuación que se verifica es $\mathrm{rot}B = \lambda B$, donde λ es una función. El caso $\lambda =$ constante es el más interesante desde el punto de vista de la complejidad de las órbitas, ya que es fácil ver que λ es una integral primera de B.

Los campos "force-free" pueden ser muy complejos (e.g. ergodicos en abiertos de $\mathbb{R}^3$, como el célebre campo ABC [20]), nosotros nos restringimos sin embargo al caso en el que existen integrales primeras. Por primera vez en la literatura se obtienen obstrucciones a la geometría de las integrales primeras de estos campos. También se estudian integrales primeras con simetría euclídea e integrales primeras inducidas por la existencia de simetrías euclideas de B. Las técnicas que se emplean son nuevamente los algoritmos de integración cuando se conocen simetrías.

4.3. Motion of a charge in the magnetic field created by wires, impossibility of reaching the wires: Phys. Lett. A 333 (2004) 72

Dada una configuración de hilos que genera un campo magnético es un problemas clásico, así como difícil, el estudiar la relación entre la estructura del campo y el movimiento de las cargas sometidas a dicho campo [21]. En este artículo se prueba que cuando el campo posee dos integrales primeras que verifican ciertas propiedades entonces las ecuaciones del movimiento heredan una integral primera diferente de la energía. Esto permite probar que para ciertas configuraciones, i.e. hilos rectilíneos paralelos e hilos planos circulares coaxiales, las partículas nunca pueden alcanzar los cables, encontrándose, por tanto, apantallados. Al final del artículo se prueba que las hipótesis del criterio implican la existencia de una simetría euclídea de B. Esta propiedad de inaccesibilidad de los hilos magnéticos es la primera vez que se encuentra en la literatura.

Capítulo 5

Otros conjuntos invariantes

Los conjuntos invariantes de sistemas dinámicos no están necesariamente asociados a una integral primera. Este capítulo estudia dos ejemplos relevantes en los que sucede precisamente esto, a saber, los conjuntos invariantes de las ecuaciones de Newton y los atractores. Ambos casos tienen interés físico, el primero en Mecánica Clásica, el segundo en ecuaciones de evolución (como por ejemplo Navier-Stokes) donde se sabe de la existencia de conjuntos atractores de dimensión finita.

5.1. Invariant sets of second order differential equations: Phys. Lett. A 325 (2004) 340

Las ecuaciones de Newton, tanto en Mecánica Clásica como Relativista, son EDOs de segundo orden, que definen un campo de vectores en el espacio de fases [6]. Un tipo concreto de conjunto invariante de estas ecuaciones, particularmente interesante para las aplicaciones, es el dado por conjuntos invariantes en el espacio de fases cuya proyeccion en el espacio de configuracion es también invariante. Aparte de ejemplos concretos (e.g. potenciales centrales), este tipo de conjuntos está poco estudiado en la literatura, y el caso relativista nunca es considerado [22].

En este trabajo las ecuaciones diferenciales no lineales que definen estos conjuntos son obtenidas. Como resultado se prueba que si la fuerza no depende de la velocidad entonces los conjuntos invariantes son siempre planos. El caso de fuerzas cuadráticas en la velocidad es también analizado, concluyéndose la posibilidad de conjuntos invariantes curvos así como su no existencia. En el caso relativista se obtiene el interesante resultado de que los conjuntos invariantes curvos están prohibidos, lo mismo sucede en las ecuaciones de la óptica geométrica. Las técnicas que se usan en las demostraciones son esencialmente de tipo analítico. Esta clase de conjuntos invariantes incluye las subvariedades totalmente geodésicas de una variedad Riemanniana.

5.2. Note on a paper of J. Llibre and G. Rodríguez concerning algebraic limit cycles: J. Diff. Eqs. 217 (2005) 249

El estudio del número y distribución de los ciclos límite de campos polinómicos es un problemas clásico que se remonta a Hilbert [23]. El problema inverso de averiguar si cualquier configuración de ciclos en $\mathbb{R}^2$ puede realizarse, salvo homeomorfismo, por un campo polinómico fue resuelto por Llibre y Rodríguez [24] usando la teoría de integrabilidad de Darboux. En este trabajo se prueba el mismo resultado de forma más sencilla usando una construcción clásica. Un resultado análogo se demuestra en $\mathbb{R}^n$ ($n > 2$), siendo particularmente importante el caso $n = 3$, debido a la existencia de nudos y links. Esto responde una pregunta de Ronald Sverdlove formulada en 1981 [25]. La técnica de la demostración envuelve el teorema de Nash-Tognoli, el teorema de Liapunov y una construcción explícita. Además se muestra la estabilidad estructural de los ciclos límite y la no existencia de ceros del campo.

Capítulo 6

Conclusiones

En esta tesis se han obtenido diversos resultados sobre integrales primeras y conjuntos invariantes de campos de vectores, generalmente analíticos, en $\mathbb{R}^n$. Las propiedades que se han estudiado son, básicamente, la estabilidad de puntos críticos y de soluciones cuando se conocen integrales primeras, la relación entre simetrías, integrales primeras y conjuntos invariantes, y la existencia de conjuntos invariantes atractores. Estos resultados son de interes fundamentalmente matemático. La tesis también ha aportado aplicaciones a diferentes contextos físicos, que incluyen las ecuaciones de la mecánica de Newton, campos magnéticos creados por hilos y campos de Lotka-Volterra.

Capítulo 7

Otros artículos

El solicitante ha participado en otros artículos, no directamente relacionados con el tema de la tesis, los cuales se detallan a continuación:

- F.G. Gascón, D. Peralta-Salas y J.M. Vegas-Montaner: Limit velocity of charged particles in a constant electromagnetic field under friction. Phys. Lett. A 251 (1999) 39

- F.G. Gascón y D. Peralta-Salas: Escape to infinity in a Newtonian potential. J. Phys. A: Math. Gen. 33 (2000) 5361

- F.G. Gascón y D. Peralta-Salas: Escape to infinity under the action of a potential and a constant electromagnetic field. J. Phys. A: Math. Gen. 36 (2003) 6441

- F.G. Gascón y D. Peralta-Salas: On the construction of global coordinate systems in Euclidean spaces. Nonlinear Anal. 57 (2004) 723

- J. Almeida, D. Peralta-Salas y M. Romera: Can two chaotic systems give rise to order?. Phys. D 200 (2005) 124

- F. Mañosas y D. Peralta-Salas: Note on the Markus–Yamabe conjecture for gradient dynamical systems. J. Math. Anal. Appl., aceptado para su publicación

- D. Peralta-Salas: A geometric approach to the classification of the equilibrium shapes of self–gravitating fluids. Comm. Math. Phys., aceptado para su publicación

- A. Enciso y D. Peralta-Salas: On the classical and quantum integrability of Hamiltonians without scattering states. Theor. Math. Phys., aceptado para su publicación

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Instability of vector fields induced by first integrals

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It is shown that when a first integral of a $\mathbb{R}^3$ vector field $\mathbf{X}$ is known, instabilities are induced on the equilibrium points of $\mathbf{X}$. © *1999 American Institute of Physics.* [S0022-2488(99)03106-0]

I. INTRODUCTION

Let $\mathbf{X}$ be an analytic (C^w) $\mathbb{R}^n$ vector field (v.f.) with an isolated singularity at the origin, i.e., $\mathbf{X}(\mathbf{0})=\mathbf{0}$. We are concerned here with establishing criteria for the instability of $\mathbf{X}$ at $\mathbf{0}$ (the origin).

It is a classical result that when the linear part $\mathbf{X}_L$ of $\mathbf{X}$ at $\mathbf{0}$ has an eigenvalue of positive real part then $\mathbf{0}$ is an unstable equilibrium point of $\mathbf{X}$ (Ref. 1). This criterion gives no information concerning instability when there are not eigenvalues of $\mathbf{X}_L$ to the right of the imaginary axis.

When $\mathbf{X}_\mathrm{H}$ is a Hamiltonian v.f. and H is an analytic function of the form

$$H=\sum_{i,j=1}^{m} p_i p_j a_{ij}(\mathbf{q})+V(\mathbf{q}) \quad (\mathbf{q}\in\mathbb{R}^m, \quad n=2m), \tag{1}$$

and (i) $a_{ij}(\mathbf{q})$ is definite positive for any $\mathbf{q}$, (ii) $\mathbf{0}$ is a critical point of V, (iii) $\mathbf{0}$ is not a strict minimum of V, and (iv) $m=1,2$. Then $\mathbf{0}$ is an unstable equilibrium point of $\mathbf{X}_\mathrm{H}$ (Ref. 2).

When $m>2$, the instability of $\mathbf{X}_\mathrm{H}$ at $\mathbf{0}$, under the above assumptions, is an unproved conjecture. Nevertheless, the unstable behavior of $\mathbf{X}_\mathrm{H}$ at $\mathbf{0}$ has been obtained under additional requirements on $V(\mathbf{q})$ (Ref. 3).

The stability of periodic solutions of Hamiltonian v.f. when first integrals are known has also been recently investigated (Ref. 4).

The technique proposed in this paper is valid for $\mathbb{R}^3$ v.f. with an isolated singularity (equilibrium point) at $\mathbf{0}$ and with a known C^w first integral I. The technique is illustrated with examples that show that the method is valid, even in the case of trivial center (that is, when all the eigenvalues of $\mathbf{X}_L$ lie on the imaginary axis).

The method proposed here is based on the well-known fact that the w limit of a bounded trajectory of a planar vector field must include either a singularity or a closed trajectory (Bendixon–Poincaré theorem).

The possibilities of extending the new technique to $\mathbb{R}^n$ v.f. ($n>3$) are also discussed.

II. INSTABILITY INDUCED BY FIRST INTEGRALS

Let $\mathbf{X}$ be a $\mathbb{R}^3$ dynamical system with an isolated singularity at $\mathbf{0}$ and I a C^w first integral of $\mathbf{X}$. Assume that either

$$\text{(i)} \quad \nabla I_{|\mathbf{0}}\neq\mathbf{0},$$

or (2)

$$\text{(ii)} \quad \nabla I(\mathbf{P})=\mathbf{0},\ \ \mathbf{P}\in N_0\Rightarrow\mathbf{P}=(0,0,0),$$

and I has a saddle at the origin.

Then $\mathbf{0}$ is an unstable equilibrium point of $\mathbf{X}$.

Remember that ∇I stands for the gradient of I. On the other hand, I by definition has a saddle at $\mathbf{0}$ if $\nabla I_{|\mathbf{0}}=\mathbf{0}$ and there are points P and Q arbitrarily near $\mathbf{0}$ on which I takes values of opposite signs. Remember that we assume in this paper that the first integral I has the value 0 at $\mathbf{0}$.

Proof: Assume that $\nabla I_{|\mathbf{0}}\neq\mathbf{0}$. In this case local coordinates (u_1,u_2,u_3) can be introduced on N_0 (a neighborhood of $\mathbf{0}$) on which I takes the canonical form $I=u_1$. Therefore, if $\mathbf{X}$ is assumed stable at $\mathbf{0}$ its trajectories will lie in N_0 and on the local planes $u_l=k_1$. The w limits of these trajectories must be, on account of the Poincaré–Bendixon theorem,[5] singular points of $\mathbf{X}$, polygonals whose vertices are singular points of $\mathbf{X}$ or closed trajectories.

In any of these three cases, singular points of $\mathbf{X}$, lying on the planes $u_1=k_1$ and arbitrarily near $\mathbf{0}$, are obtained. But since $\mathbf{0}$ was assumed to be an isolated singularity of $\mathbf{X}$, we get a contradiction. Therefore $\mathbf{X}$ cannot be stable at $\mathbf{0}$.

Assume now that ∇I vanishes on N_0 just at $\mathbf{0}$ and that I has a saddle at $\mathbf{0}$.

These assumptions imply (as we now explain) that on a certain domain $Z_0\subset N_0$ the level sets of I resemble locally topological planes, to which the above reasoning can be applied, getting again a contradiction if $\mathbf{0}$ is assumed to be a stable singularity of $\mathbf{X}$. Therefore $\mathbf{0}$ must be an unstable singularity of $\mathbf{X}$, as we desired to prove.

We now show that if I is an $\mathbb{R}^3$ analytic function with a saddle at $\mathbf{0}$ and $\nabla I_{|N_0}$ vanishes just at $\mathbf{0}$, then a domain $Z_0\subset N_0$ exists on which the sets $I^{-1}(c)\cap Z_0$ are local planes (disks).

In fact, the analiticity of I implies that $I^{-1}(0)\cap N_0$ is the finite union of the surfaces C_i, $i\in J$, through $\mathbf{0}$. Condition (ii) of Eq. (2) implies that the surfaces C_i do not intersect each other on $N_0-\{\mathbf{0}\}$. The surfaces C_i divide N_0 into solid zones Z_j, whose boundary is made up of one or several of the surfaces C_i.

By topological reasons it is not too difficult to show that one at least (say Z_0) of the zones Z_j is diffeomorphic to $\mathbb{R}^3$. This is due to the fact that C_i is, inside N_0, either a topological plane (if C_i has a tangent at $\mathbf{0}$) or a topological cone (if C_i has not a tangent plane at $\mathbf{0}$); in any case, each C_i separates N_0 into zones, one of which is clearly diffeomorphic to $\mathbb{R}^3$. This property, valid for any of the surfaces C_i, is the geometric reason underlying the existence of the zone Z_0.

For example, consider the functions $I_1=(x^2+y^2-z^2)z$, $I_2=(x^2+y^2-z^2)(x^2+y^2-4z^2)$. I_1 and I_2 have clearly a saddle at $\mathbf{0}$, and it is easy to check that ∇I_i $(i=1,2)$ vanishes just at $\mathbf{0}$. The set Z_0 diffeomorphic to $\mathbb{R}^3$ can be chosen to be

$$Z_0=\{(x,y,z)|x^2+y^2<z^2,\quad z>0\}. \tag{3}$$

Consider now the C^w curves φ_α, defined either by

$$\varphi_\alpha=I^{-1}(c)\cap Z_0\cap\pi_\alpha, \tag{4}$$

π_α standing for a family of planes through $\mathbf{0}$, intersecting Z_0, or by

$$I_{|\pi_\alpha\cap Z_0}=c. \tag{5}$$

Calling $I_{|\pi_\alpha}$ by I_α^*, we have the following.

(1) (0,0) is a saddle of I_α^*. This is a consequence of the fact that the sign of I changes on the surfaces C_i, since otherwise $\nabla I=\mathbf{0}$ on points of $N_0-\{\mathbf{0}\}$.

(2) ∇I_α^* has an isolated zero at (0,0). In fact, if $\nabla I^*_{\alpha|\varphi}=\mathbf{0}$, where φ is a curve through (0,0) we would get $I_{|\varphi}=0$, in contradiction with the fact that $I\neq 0$ inside Z_0. A similar contradiction is obtained if ∇I_α^* vanishes on a succession of points tending to (0,0).

Summarizing, the curves φ_α are the zeros of plane C^w functions with a saddle at (0,0) and an isolated critical point at (0,0). Therefore (Ref. 6), φ_α is just an open segment. The union of these segments, when the plane π_α varies is, given the topology of Z_0, a local plane (a disk).

Therefore $I^{-1}(c)\cap Z_0$ is locally a plane.

The reasoning above is sketchy and probably can be improved.

We have not been able to improve it by consultations with professional mathematicians. We now give some examples of $\mathbb{R}^3$ v.f. whose instability at $\mathbf{0}$ can be detected with the above-mentioned techniques. To our knowledge they cannot be integrated via quadratures and they are interesting since most of them have a vanishing linear part.

A. Consider the $\mathbb{R}^3$ v.f.

Here

$$\mathbf{X}=(y(1+z^2)-x-z)\partial_x+(-y-x(1+z^2))\partial_y+((x^2+y^2+xz)(1+z^2))\partial_z. \tag{6}$$

It is easy to check that this v.f. has an isolated zero at (0,0,0) and the eigenvalues of $\mathbf{X}_L$ at $\mathbf{0}$ are 0 and $-1\pm i$. Therefore the eigenvalues cannot decide between stability and instability at $\mathbf{0}$.

This v.f. has the first integral $I=\frac{1}{2}(x^2+y^2)-\arctan(z)$. Note that $\boldsymbol{\nabla} I_{|\mathbf{0}}\neq\mathbf{0}$. Therefore by (i) of Eq. (2), $\mathbf{X}$ is unstable at $\mathbf{0}$.

B. Consider the v.f.

Here

$$\mathbf{X}=(x^4(y^2+z^2)+x^2+y^2)\partial_x-(2x^3(y^2+z^2)(1+y)+x^2+y^2+z^2)\partial_y+(2x(x^2+y^2)(1+y)$$
$$-x^2(x^2+y^2+z^2))\partial_z. \tag{7}$$

It is easy to check that (i) $\mathbf{0}$ is an isolated zero of $\mathbf{X}$ and that $\mathbf{X}_L$ (the linear part of $\mathbf{X}$ at $\mathbf{0}$) is identically zero; (ii) $I=x^2(1+y)-z$ is a first integral of $\mathbf{X}$. (iii) $\nabla I_{|\mathbf{0}}\neq\mathbf{0}$.

Therefore, according to (i) of Eq. (2), $\mathbf{0}$ is an unstable singular point of $\mathbf{X}$.

C. Let X be the v.f.

Here

$$\mathbf{X}=(2x(y-z)(x^2+y^2+z^2))\partial_x-((3x^2+y^2+z^2)(x^2+y^2+z^2)2x^2yz)\partial_y$$
$$+((3x^2+y^2+z^2)(x^2+y^2+z^2)+2x^2y^2)\partial_z. \tag{8}$$

It is easy to check that (i) $\mathbf{0}$ is an isolated zero of $\mathbf{X}$ and $\mathbf{X}_L=0$; (ii) $I=x(x^2+y^2+z^2)$ is a first integral of $\mathbf{X}$. The first integral has a saddle at $\mathbf{0}$ and its gradient vanishes just at $\mathbf{0}$.

According to (ii) of Eq. (2), $\mathbf{0}$ is an unstable singularity of $\mathbf{X}$.

D. Consider the $\mathbb{R}^3$ v.f.

Here

$$\mathbf{X}=-2(y^2+zx^2+zxy^2+xz^3)\partial_x+(-(z^2+3x^2)y+2xyz(x^2+y^2+z^2))\partial_y$$
$$+((x^2+y^2+z^2)(3x^2+2y^2+z^2))\partial_z. \tag{9}$$

It is easy to check that (i) $\mathbf{0}$ is an isolated zero of $\mathbf{X}$ and $\mathbf{X}_L=\mathbf{0}$; (ii) $I=z^2x+x^3-y^2$ is a first integral of $\mathbf{X}$. In addition, I has a saddle at $\mathbf{0}$ and $\boldsymbol{\nabla} I$ vanishes just at $\mathbf{0}$.
Therefore, by applying (ii) of Eq. (2), we can conclude that $\mathbf{X}$ is unstable at $\mathbf{0}$.

We conclude by noting that our instability criterion can be applied to $\mathbb{R}^n$ v.f. $(n>3)$ when $(n-2)$ first integrals of $\mathbf{X}$ are known and $\operatorname{rank}(\boldsymbol{\nabla} I_1,\ldots,\boldsymbol{\nabla} I_{n-2})_{|\mathbf{0}}=n-2$. This can be seen by introducing local coordinates $(u_1,\ldots,u_n)$ in N_0 on which the first integrals take the local form $I_1=u_1,\ldots,I_{n-2}=u_{n-2}$.

Therefore the local level sets of $I_1,\ldots,I_{n-2}$ will be local planes (two-dimensional disks). By applying to them the considerations used to demonstrate of Eq. (2), we get instability of $\mathbf{X}$ at $\mathbf{0}$.

When $\operatorname{rank}(\nabla I_1,\ldots,\nabla I_{n-2})_{|\mathbf{0}}<n-2$, a general criterion for instability seems difficult to get. We list now several partial results in this direction.

(i) Let $\mathbf{X}$ be a $\mathbb{R}^4$ v.f. with an isolated zero at $\mathbf{0}$ and the two first integrals:

$$\begin{aligned} I_1 &= (1+x^2)y^2+(1+x^4)z^2-(1+e^x)u^4, \\ I_2 &= x. \end{aligned} \tag{10}$$

Note that $\operatorname{rank}(\nabla I_1,\nabla I_2)_{|\mathbf{0}}=1$.

On the level set $I_2=0$, I_1 and $\mathbf{X}$ become

$$\begin{aligned} I_1^* &= y^2+z^2-2u^4, \\ \mathbf{X}^* &= a\partial_y+b\partial_z+c\partial_u . \end{aligned} \tag{11}$$

It is clear that ∇I_1^* vanishes just at (0,0,0), that $\mathbf{X}^*$ has an isolated zero at (0,0,0), and that I_1^* has a saddle at $\mathbf{0}$. Therefore the couple $(\mathbf{X}^*,I_1^*)$ satisfies the assumptions of (ii) of Eq. (2), and we conclude that $\mathbf{X}^*$, and therefore $\mathbf{X}$, is unstable at (0,0,0,0).

Examples of this type are not only academic, since they appear in the study of systems of the type

$$\begin{aligned} \ddot{x} &= V_{,x}(x,y), \\ \ddot{y} &= V_{,y}(x,y), \end{aligned} \tag{12}$$

whenever a pair of first integrals of a $\mathbb{R}^4$ v.f. are known and the gradient of one of them does not vanish at $\mathbf{0}$ (Ref. 7). The second first integral is, usually, linear in the components of the velocity.

In fact, via a local change of variables this first integral can be reduced to a canonical form similar to the function I_2 of (10). This fact gives generality to the couple of first integrals chosen in (10).

(ii) Let I_1 and I_2 be defined by

$$\begin{aligned} I_1 &= u^n-P(x,y,z), \\ I_2 &= x^m-Q(y,z), \end{aligned} \tag{13}$$

where n and m are positive integers $(n,m>1)$, P and Q non-negative polynomials and $\operatorname{rank}(\nabla I_1,\nabla I_2)_{|\mathbf{0}}=0$.

It is immediate to check that the level sets

$$\begin{aligned} I_1 &= C_1, \\ I_2 &= C_2, \end{aligned} \tag{14}$$

are planes when $C_1,C_2>0$ (one has just to get u and x as global functions of y and z). Therefore, by using similar arguments to those given in the proof of (2), any $\mathbb{R}^4$ v.f. with an isolated zero at $\mathbf{0}$ and these first integrals is unstable at $\mathbf{0}$.

(iii) Let I_1 and I_2 be defined by

$$\begin{aligned} I_1 &= y^2-f(x), \\ I_2 &= xu-zy, \end{aligned} \tag{15}$$

$f(x)$ being a non-negative function and $f'(0)=0$.

Note that I_2 has the form of an angular momentum and that $\operatorname{rank}(\nabla I_1,\nabla I_2)_{|\mathbf{0}}=0$. On the other hand, the level sets,

$$y^2 - f(x) = C,$$

$$xu - zy = D, \tag{16}$$

$$C > 0,$$

can be globally parametrized in the form

$$\left(x, \pm\sqrt{C+f(x)}, \frac{xu-D}{\pm\sqrt{C+f(x)}}, u \right), \tag{17}$$

and they are a couple of two-dimensional planes (note that the parameters x and u are free). Therefore any v.f. with an isolated zero at **0** and the two first integrals (15) is unstable at **0**.

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A separation bound for non-Hamiltonian differential equations with proper first integrals

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It is shown that when a dynamical system $\mathbf{X_0}$ with a proper set of global first integrals is perturbed, the phase space region accessible to the orbits of the perturbed vector field $\mathbf{X_0}+\mathbf{X_p}$ is bounded (we are assuming here that the time variable runs over a *finite* interval). A polynomial new bound is obtained for the separation between the solutions of $\mathbf{X_0}$ and $\mathbf{X_0}+\mathbf{X_p}$. Perturbations near an equilibrium point of $\mathbf{X_0}$ are also considered.
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I. INTRODUCTION

The role played by first integrals of vector fields (v.f.'s) in the integration of them via quadratures and other reduction mechanisms is well known.[1] Remember that a smooth function I is called first integral of the v.f. $\mathbf{Y}$ when $\mathcal{L}_{\mathbf{Y}}(I)=0$, $\mathcal{L}_{\mathbf{Y}}$ standing for the Lie derivative of I along the streamlines of $\mathbf{Y}$. Most of the first integrals considered in this paper are *proper* first integrals: a function I is *proper* when $I^{-1}(K)$ is a compact set whenever K is compact. The reader will have no difficulty in proving that when $\lim_\infty I(\mathbf{x})=\infty$, then I is a proper function (of $\mathbb{R}^n$ in $\mathbb{R}$). More information and some examples of proper functions can be found in Appendix B.

First integrals have also been used in other contexts: to estimate limiting possibilities of optimal control systems,[2] in averaging techniques of perturbed Hamiltonian v.f.'s,[1,3] and in the obtention of bounds for the number of periodic orbits surviving when a completely degenerate, linear, Hamiltonian system is perturbed.[4] We now show that they also play an interesting role in relation to (i) the wideness of the phase space region accessible to the perturbed orbits and (ii) the obtention of bounds for the separation of perturbed and unperturbed solutions. Since the only way of studying the perturbed v.f. is, in general, numerical, these phase space domains and bounds could be useful in order to control the errors of the numerical computations.

Let us now compare our method with other perturbations methods. Consider, for example, KAM theory. In this theory $n/2$ first integrals, in involution, of an unperturbed v.f. $\mathbf{X_0}$ are used. $\mathbf{X_0}$ is Hamiltonian and n is the phase space dimension. On the phase space domain where the level sets of the first integrals meet in tori and where the Kolmogorov condition holds[1,5] most of the nonresonant tori survive the perturbation and do not disappear, but are slightly deformed (the perturbing term $\mathbf{X_p}$ is assumed conveniently small).

For $n=2,4$ KAM implies the boundedness of the perturbed solutions. But when $n>4$, unbounded orbits can appear (Arnold diffusion). The theory is not applicable if the first integrals are not in involution or if they are but the geometry where its level sets meet is not toruslike ($\mathbf{X_0}$ might vanish on one of these compact intersections). The same applies if $\mathbf{X_0}$ (the unperturbed v.f.) is not Hamiltonian.

Concerning the relation between KAM and the work developed here, we note the following: (i) The bounds obtained in this paper are valid for any v.f. $\mathbf{X_0}$, integrable or not, Hamiltonian or not, in so long as its first integrals form a proper set of first integrals. This implies that the sets where the level sets meet are compact. The theory is also applicable when *one* of the connected components of the set where the level sets meet is a compact set. On the other hand, the geometry of these compact intersections of level sets is not necessarily toruslike: any compact differential manifold is equally valid to us. (ii) Our bounds are valid for finite intervals of time, while KAM's are valid for infinite intervals of time. (iii) Local coordinates around the compact sets defined by the first integrals are never used in this paper. Therefore we do not get problems when trying to globalize them.[6]

Now, we will give some information concerning our method and the techniques of averaging and adiabatic invariants.[1] Most of these methods are designed in order to study the perturbations of Hamiltonian v.f.'s, with $n/2$ first integrals in involution and with global action-angle variables[6] on a certain compact set filled with tori, or for v.f.'s with parameters drifting ''slowly with time.'' We have already mentioned that global action-angle variables do not always exist, because of topological obstructions. On the other hand, in order to define the term ''slowly with time,'' used in the theory of adiabatic invariants, some authors are compelled to consider only Hamiltonian v.f.'s $\mathbf{X_0}$ of degenerate type; that is, v.f.'s, all of whose orbits are of type S^1 (topological circles), at least on a certain phase space domain. The periods of the S^1 orbits can be used as a scale of time in order to give a certain meaning to the term ''slowly with time.'' We have to say that in our approach the v.f. $\mathbf{X_0}$ is not constrained to be degenerate.

We explain now why our treatment has little in common with the so-called ''averaging methods.''[1] In these methods bounds for the separation between the evolution of the *action* variables (slow variables) in the v.f. $\mathbf{X_0}+\mathbf{X_p}$ and $A(\mathbf{X_0}+\mathbf{X_p})$ are obtained, $A(\mathbf{Y})$ standing for the average of $p(\mathbf{Y})$ over the angle variables [$p(\mathbf{Y})$ is the projection of $\mathbf{Y}$ over the action variables space].

On the contrary, we get bounds for the separation $\|\mathbf{x}(t)-\mathbf{y}(t)\|$ between the position vectors of the solutions of $\mathbf{X_0}$ and $\mathbf{X_0}+\mathbf{X_p}$ at time t. We show that for large values of t this separation cannot grow faster than a polynomial function of t. This result improves previous exponential bounds in the literature.

The plan of the paper is the following: the bounding regions are introduced in Sec. II. Bounds for the separation between the unperturbed and perturbed solutions, with the same initial conditions, are given in Sec. III, and these bounds are compared, in Sec. IV, with other bounds in the literature. An application to the perturbations near a stable equilibrium point is given in Sec. V.

II. THE BOUNDING REGIONS

We prove in this section the following Proposition:

Assume that (i) I is a uniformly bounded and smooth first integral [see formula (6)] of $\mathbf{X_0}$; (ii) $\mathbf{y_0}$ is a common initial condition of the v.f. $\mathbf{X_0}$ and $\mathbf{X_0}+\mathbf{X_p}$ with corresponding solutions $\mathbf{x}(t)$ and $\mathbf{y}(t)$ satisfying $\mathbf{x}(0)=\mathbf{y_0}$, $\mathbf{y}(0)=\mathbf{y_0}$; and (iii) $\mathbf{X_p}$ satisfies Eq. (7).

Under these assumptions $\mathbf{y}(t)$ must lie inside the phase space domain defined by Eq. (8). Certain consequences of Eq. (8) are also discussed at the end of this section.

In fact, consider the differential equations associated with the v.f. $\mathbf{X_0}$ and $\mathbf{X_0}+\mathbf{X_p}$,

$$\dot{\mathbf{x}}=\mathbf{X_0}(\mathbf{x}), \tag{1}$$

$$\dot{\mathbf{y}}=\mathbf{X_0}(\mathbf{y})+\mathbf{X_p}(t,\mathbf{y}), \tag{2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are vectors in R^n and $\mathbf{X_p}$ is the perturbing term. The rate of change of I along the solutions of $\mathbf{X_0}+\mathbf{X_p}$ is

$$\dot{I}=\nabla I\cdot(\mathbf{X_0}+\mathbf{X_p})=\nabla I\cdot\mathbf{X_p}, \tag{3}$$

where ∇ stands for the gradient operator. Note that the identity $\nabla I\cdot\mathbf{X_0}=0$ has been used in (3), since I is a first integral of $\mathbf{X_0}$. We immediately obtain from (3)

$$-|\nabla I\cdot \mathbf{X_p}|\leqslant \dot{I}\leqslant |\nabla I\cdot \mathbf{X_p}| \tag{4a}$$

and

$$-\|\nabla I\|\cdot\|\mathbf{X_p}\|\leqslant \dot{I}\leqslant \|\nabla I\|\cdot\|\mathbf{X_p}\|, \tag{4b}$$

$\|\ \|$ standing for the Euclidean norm of $\mathbb{R}^n$. Integrating (4b) between $t=0$ and $t=T$, $T>0$, we obtain

$$-\int_0^T \|\nabla I\|\cdot\|\mathbf{X_p}\|\,dt\leqslant I-I_0\leqslant \int_0^T \|\nabla I\|\cdot\|\mathbf{X_p}\|dt, \tag{5}$$

with $I_0=I(\mathbf{y_0})$.

We discuss now some consequences of Eq. (5). Assume that ∇I satisfies the uniform boundedness condition:

$$\|\nabla I\|\leqslant K, \quad \forall x\in \mathbb{R}^n \tag{6}$$

(see Appendix A for a study concerning this point) and that $\mathbf{X_p}$ factorizes in the form

$$\mathbf{X_p}=f(t)\cdot \hat{\mathbf{X}}_\mathbf{p}(\mathbf{y}),$$

$$\|\hat{X}_\mathbf{p}\|\geqslant K', \quad \forall \mathbf{y}\in \mathbb{R}^n \tag{7}$$

$$f(t)\in C^0.$$

Under these requirements we get from (5)

$$-K\cdot K'\int_0^T |f(t)|\,dt\leqslant I(\mathbf{y})-I_0\leqslant K\cdot K'\int_0^T \quad |f(t)|\,dt. \tag{8}$$

A similar equation holds when $\mathbf{X_p}$ is a linear combination of terms of type (7).

Let us discuss now some consequences of Eq. (8).

(i) If we assume, in addition, that $\int_0^{+\infty}|f(t)|\,dt$ is bounded and that I is proper (see Appendix B), Eq. (8) defines a bounded domain of R^n where $\mathbf{y}(t)$ lies when t runs over the interval $[0,T]$ for any value of T. Therefore $\mathbf{y}(t)$ cannot blow up to infinite in a finite time. These conclusions hold as well if we assume that $I_1,\ldots,I_s$ is a proper set of first integrals of $\mathbf{X_0}$ such that $\|\nabla I\|<K$ $(I\equiv {I_1}^2+\cdots+{I_s}^2)$. In fact, it is easy to show that I is a proper function.

(ii) Assume now that I is a first integral of $\mathbf{X_0}$ not necessarily proper and that the connected component of the level set $I^{-1}(I_0)$ through $\mathbf{y_0}$ is compact. On the other hand, we do not assume the validity of Eq. (6) on the whole of $\mathbb{R}^n$, as it is obviously verified on any compact set C containing the compact component of $I^{-1}(I_0)$. Under these assumptions the perturbed solution $\mathbf{y}(t)$ remains in C when $\underline{t}$ $(t>0)$ is sufficiently small. We get in this way, through Eq. (8), a restriction on the phase space domain (contained in C) accessible to the perturbed solution $\mathbf{y}(t)$.

(iii) Let us clarify the meaning of this section and the last paragraph with an example. Assume that $\mathbf{X_0}$ is the electromagnetic induction $\mathbf{B_0}(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^3$, and I is not necessarily a proper first integral of $\mathbf{B_0}$, whose level sets, inside a certain compact set C, are tori. Let us compare the orbits of $\mathbf{B_0}$ and $\mathbf{B_0}+\mathbf{B_p}$, with the same initial conditions, while $\mathbf{y}(t)$ lies in C. Note that the bound K of Eq. (6) can be made arbitrarily small if C is chosen near the central line φ of the tori, since ∇I vanishes on φ. This fact implies that the term $K\cdot K'\cdot\int_0^T|f(t)|\,dt$ in Eq. (8) can be made small, and small will also be the domain accessible to $\mathbf{y}(t)$ defined by Eq. (8).

We obtain, in this way, a convenient finite time confinement of the orbits of $\mathbf{X_0}+\mathbf{X_p}$. This confinement has been induced by the presence of the first integral I and its compact level sets on C.

(iv) Note that a first integral of $\mathbf{B_0}$ with toruslike level sets can be obtained if $\mathbf{B_0}$ possesses a transverse symmetry vector $\mathbf{S}$ of zero divergence.

That is, when $\mathbf{S}$ commutes with $\mathbf{B_0}$ and $\mathrm{Div}\,\mathbf{S}=0$.

This is what happens, for instance, when $\mathbf{B_0}$ is symmetric under rotations around the z axis. In this case $\mathbf{S}=\partial_\varphi$, $\mathrm{Div}\,(\partial_\varphi)=0$, and I is given by

$$\mathbf{B_0}\rfloor\,\partial_\varphi\rfloor\,(dx\wedge dy\wedge dz)=dI, \tag{9}$$

$\rfloor$ standing for the contraction operator between v.f. and differential forms.

Under these conditions it is easy to see that the compact components of the level sets of I are tori. One just has to remember that the function I defined in Eq. (9) is also a first integral of ∂_φ.

III. A NEW BOUND OF $\|\mathbf{x}(t)-\mathbf{y}(t)\|$

We prove in this section the following Proposition:

Assume that (i) I is a proper, or locally proper, polynomial first integral of Eq. (1), (ii) $|f(t)|<K'',\forall t$, and (iii) the assumptions used to obtain Eq. (8).
Under these requirements a polynomial upper bound for $\|\mathbf{x}(t)-\mathbf{y}(t)\|$ is obtained [see Eq. (12)].

In fact, under the above requirements, Eq. (8) implies

$$-K'''\cdot T\leqslant I-I_0\leqslant K'''\cdot T, \tag{10}$$

K''' being the product of the bounds of $\|\nabla I\|$, $\|\mathbf{X_p}\|$, and $|f(t)|$ on C. Remember that C is a compact set containing $\mathbf{y}(t)$ for $t\in[0,T]$.

We see in (10) that I cannot increase faster than linearly along the solutions of $\mathbf{X_0}+\mathbf{X_p}$. We can now obtain, out of Eq. (10), a bound for the maximal separation $\|\mathbf{x}(t)-\mathbf{y}(t)\|$. In fact, we can write

$$\|\mathbf{x}(t)-\mathbf{y}(t)\|\leqslant D(T)\leqslant 2R(T), \tag{11}$$

$D(T)$ being the diameter of the bounded set $I^{-1}[I_0-K'''\cdot T, I_0+K'''\cdot T]$, and $R(T)$ being the maximum distance from the points of this set to any fixed arbitrary point of $\mathbb{R}^n$.

Now, it is shown in Appendix C that when I is a polynomial $R(T)$ cannot increase, for large values of T, faster than $T^m (m\in\mathbb{N})$. Therefore, we obtain from Eq. (11)

$$\|\mathbf{x}(t)-\mathbf{y}(t)\|\leqslant aT^m, \quad m\in\mathbb{N}, \tag{12}$$

a standing for a positive real number.

Let us compare next the bound (12) with other bounds in literature.

IV. COMPARING THE POLYNOMIAL BOUND WITH OTHER BOUNDS

We compare now the polynomial bound of Sec. III with some classical bounds.

(i) First of all, consider the well-known expression[7]

$$\|\mathbf{x}(t)-\mathbf{y}(t)\|\leqslant K'\cdot \mathrm{L}^{-1}\cdot[\exp(\mathrm{L}\cdot T)-1], \quad t\in[0,T], \quad T>0, \tag{13}$$

L being a Lyschitz constant of $\mathbf{X_0}$ and K' a bound of $\|\mathbf{X_p}\|$. Remember that since $\mathbf{X_0}$ is analytic, L is just a bound of the matrix $D\mathbf{X_0}$ (D is the differential operator).

We see in (13) that $\|\mathbf{x}(t)-\mathbf{y}(t)\|$ increases exponentially with T. Therefore, the bound (13) is worse (when T is large) than the polynomial bound in T obtained in Sec. III. Our improvement is to be ascribed to the presence of the proper uniformly bounded first integrals.

(ii) Assume now that $\mathbf{X_0}=A_0\cdot\mathbf{x}$, where A_0 is a constant $n\times n$ matrix and $\mathbf{X_p}=\mathbf{f}(t)$ with $\mathbf{f}(t)$ satisfying $\|\mathbf{f}(t)\|\leqslant K'$ for any t. In this case writing the explicit expressions for $\mathbf{x}(t)$ and $\mathbf{y}(t)$ we obtain

$$\|\mathbf{x}(t)-\mathbf{y}(t)\|\leqslant\|\exp(A_0\cdot t)\|\cdot\left\|\int_0^t\exp(-A_0\cdot s)\cdot\mathbf{f}(s)\,ds\right\|. \tag{14}$$

Using now the inequality[8]

$$\|\exp A\|\leqslant(n-1)+\exp\|A\|, \tag{15}$$

where A is again a (n,n) matrix, we get from Eqs. (14) and (15)

$$\|\mathbf{x}(t)-\mathbf{y}(t)\|\leqslant K'\cdot[(n-1)+\exp(\|A_0\|\cdot T)]^2\cdot T\qquad t\in[0,T],\quad T>0. \tag{16}$$

Equation (16) is a new bound of $\|\mathbf{x}(t)-\mathbf{y}(t)\|$, of exponential type, and valid when $\mathbf{X_0}$ is a linear v.f. This bound is, therefore, worse than the polynomial bound of Sec. III.

In particular cases the bound (16) becomes linear in T. Assume, for example, that the eigenvalues of A_0 are purely imaginary and simple. Then it is easy to see that $\exp(\|A_0\|\cdot T)$ is bounded for any T. Let k be a bound of $\exp(\|A_0\|\cdot T)$.

In this case we can write (16) in the form

$$\|\mathbf{x}(t)-\mathbf{y}(t)\|\leqslant K'\cdot[(n-1)+k]^2\cdot T, \tag{17}$$

which is a bound of $\|\mathbf{x}(t)-\mathbf{y}(t)\|$ linear in T.

It is easy to see that this improvement is due to the presence of a proper set of first integrals. Indeed, under the hypothesis considered on the eigenvalues of A_0, $\mathbf{X_0}$ has a set of proper, and quadratic, first integrals.[9]

What we learn from this example is that it is again the presence of proper first integrals that induces improvements of the bounds of $\|\mathbf{x}(t)-\mathbf{y}(t)\|$.

V. PERTURBATIONS AROUND STABLE EQUILIBRIUM POINTS

We show now that the existence of bounding regions and the separation bound of Eq. (12) are sufficient to explain the stability of systems of linear oscillators under nonlinear perturbations. Assume that $\mathbf{0}$ is an equilibrium point of $\mathbf{X_0}$ and $\mathbf{X}_{\boldsymbol{\theta}}+\mathbf{X_p}$ and that I is a proper first integral of $\mathbf{X_0}$, with $I(\mathbf{0})=0$, $\nabla I(\mathbf{0})=\mathbf{0}$. These assumptions imply that the level sets of I near $\mathbf{0}$ are topological spheres. We also assume that I and the v.f. $\mathbf{X_0}$ and $\mathbf{X_p}$ are analytic. We can, therefore, write

$$\dot{I}=\nabla I\cdot\mathbf{X}_p=\sum_{i=n_0}^{\infty}A_i(\theta)\cdot\rho^i,\qquad n_0\geqslant 2, \tag{18}$$

$(\rho,\ \theta)$ standing for the generalized spherical coordinates in $\mathbb{R}^n$ around $\mathbf{0}$.

For convenient values of $\mathbf{y_0}$, $\mathbf{y}(t)$ lies inside an arbitrary ball B_r of radius r centered at $\mathbf{0}$, and we obtain from Eq. (18)

$$-\left(\sum_{i=n_0}^{\infty}\hat{A}_i\cdot r^i\right)\cdot T\leqslant I-I_0\leqslant\left(\sum_{i=n_0}^{\infty}\hat{A}_i\cdot r^i\right)\cdot T, \tag{19}$$

$\hat{A}_i$ being the maximum of $A_i(\theta)$ on the unit sphere $\|\mathbf{x}\|=1$. When r is small the series $\Sigma_{i=n_0}^{\infty}\hat{A}_i\cdot r^i$ behaves like its leading term $\hat{A}_{n_0}\cdot r^{n_0}$ and we can write

$$\Delta I \approx \hat{A}_{n_0} \cdot r^{n_0} \cdot T. \tag{20}$$

We see in (20) that ΔI becomes quite small, even if T is large, when B_r is sufficiently small. This implies that $\|\mathbf{x}(t)-\mathbf{y}(t)\|$ becomes small, since (see Sec. III) $\|\mathbf{x}(t)-\mathbf{y}(t)\|$ is proportional to ΔI. This fact makes the solutions $\mathbf{x}(t)$ and $\mathbf{y}(t)$ practically indistinguishable.

In some physical problems (motion of a spherical pendulum near the equilibrium position, and related problems, normal modes of vibration of molecular systems[10]) $\mathbf{X_0}$ is a linear v.f. with a proper and quadratic first integral (the total energy). The perturbed v.f. $\mathbf{X_0}+\mathbf{X_p}$ has a first integral $\hat{I}$. Here $\hat{I}$ has the structure $I+I_p$, where I_p is a perturbation of I near $\mathbf{0}$.

In the above physical problems $\hat{I}$ is also proper. Therefore, it is possible to get $\mathbf{x}(t)$ and $\mathbf{y}(t)$ inside B_r *for any t*. One has just to choose the initial condition $\mathbf{y_0}$ sufficiently near $\mathbf{0}$. Under these conditions Eq. (20) can be applied, but now T is an *unrestricted* positive number, since $\mathbf{x}(t)$ and $\mathbf{y}(t)$ never get out of B_r.

The key to this stability of linear systems with proper integrals is

(i) the presence in (20) of the factor $r^{n_0}\cdot T$, which can be made small even if T is large, and
(ii) the existence of proper integrals of $\mathbf{X_0}$ and $\mathbf{X_0}+\mathbf{X_p}$.

The smallness of ΔI and $\|\mathbf{x}(t)-\mathbf{y}(t)\|$ explains why the theory of linear oscillations is useful, since the separation between the small amplitude solutions of $\mathbf{X_0}$ and those of $\mathbf{X_0}+\mathbf{X_p}$ is so small that its detection is practically impossible.

VI. FINAL REMARKS

The effect of proper first integrals on the separation $\|\mathbf{x}(t)-\mathbf{y}(t)\|$ between the solutions of the perturbed and the unperturbed systems has been studied. It has been shown that under certain conditions this separation cannot become, when T is large, larger than a polynomial function of T, while in the absence of proper first integrals the separation is exponential in T.

The influence of proper integrals on the stability of linear systems has also been considered. Open problems in this field are the following.

(i) To get bounds of $\|\mathbf{x}(t)-\mathbf{y}(t)\|$, improving the exponential bound of Equation (13), when the first integrals do not form a proper set or when they are not polynomials. Note that for molecular systems and for the motion of a point on the surface $z=f(x,y)$, where f is a polynomial and $\lim_{\infty} f=+\infty$, there are proper and polynomial first integrals.
(ii) To improve the bounds of this paper when *more than one* polynomial, proper first integrals of $\mathbf{X_0}$ are known.
(iii) To obtain relations between I and the integer m of Eq. (12).

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APPENDIX A: UNIFORM BOUNDESNESS OF PROPER FIRST INTEGRALS

Assume that I is a first integral of $\mathbf{X}_0$. We show that if I is proper, an increasing function, f, can be obtained such that

$$\|\nabla f(I)\| \leqslant 1. \tag{A1}$$

Note that $f(I)$ is proper whenever I is proper. In fact, let $I(\mathbf{x})=C$ be the compact level sets of I. Define

$$M(C)=\mathrm{Max}\|\nabla I\| \quad \text{on} \quad I(\mathbf{x})=C. \tag{A2}$$

Note that in general $M(C)$ is continuous but not differentiable.

We define f by

$$f(z)=\int^{z} M(C)^{-1}\, dC. \tag{A3}$$

Note that f is a C^1 function [its first derivatives are continuous, but is not in general a smooth (C^∞) function].

Let us see that $f(I)$ satisfies Eq. (A1):

$$\|\nabla f(I)\|=\left\|\frac{df}{dI}\cdot\nabla I\right\|=M(C)^{-1}\cdot\|\nabla I\|\leqslant 1, \tag{A4}$$

as we wanted to prove.

Note that if $\{I_i\}$, $i=1,...,s$, is a proper set of first integrals, then $I=I_1{}^2+\cdots+I_s{}^2$ is a proper function, to which the above construction can be applied.

Remark that Eq. (A3) is of difficult handling, since the analytical expression of $M(C)$ can rarely be obtained and, on the other hand, the integrand in (A3) becomes singular at those values of C corresponding to the singular level sets of I [manifolds degenerating into points, curves,..., manifolds of dimension $(n-2)$]. Because of these problems it is preferable to use Eq. (A3) in order to get suggestions on the form of possible functions $\hat{f}$ for which $\hat{f}(I)$ satisfies Eq. (A1).

Let us now give some examples. In all of them $\hat{f}$ has been suggested by the form of $M(C)$. This form can be obtained using the Lagrange multipliers rule to get the extrema of $\|\nabla I\|$ on $I=C$.

Example 1: Let I be given by the following polynomial,

$$I(\mathbf{x})=\sum_{i=1}^{n} a_i\cdot x_i^{2p_i},$$

where a_i are positive real numbers and p_i are natural numbers. In this case $\hat{f}$ is of the form $z^{1/p}$ with $p=\mathrm{Greater}(2p_1,\ldots,2p_n)+1$. It is easy to check that $\hat{f}(I)=k\cdot I^{1/p}$ is proper and satisfies (A1) for a suitable value of k. Note that $\hat{f}(I)$ is C^∞ on $\mathbb{R}-\{0\}$.

Example 2:

$$I(\mathbf{x})=P_m(\mathbf{x})+p_{m-1}(\mathbf{x}),$$

where P_m is a homogeneous polynomial of even degree (m), $\lim_{\infty} P_m=+\infty$, and P_{m-1} is a polynomial of degree $m-1$. In this case the computations with $M(C)$ suggest that $\hat{f}(I)$ is of the form $k\cdot\ln I$. In fact, $k\cdot\ln I$ is proper and satisfies Eq. (A1) for a suitable value of k.

Example 3:

$$I(\mathbf{x})=\sum_{i=1}^{n} e^{x_i^2}.$$

In this case $\hat{f}(I)=k\cdot\ln(\ln I)$.

Note that in these two last examples $\hat{f}(I)$ is C^∞ in $\mathbb{R}^+-\{0\}$. This local C^∞ behavior is sufficient in order to be able to apply the techniques of Appendix C. On the other hand, the local first integral $\hat{f}(I)$ can be of interest in so far as the interval $[I_0-K'''\cdot T,\, I_0+K'''\cdot T]$ lies inside the region where $\hat{f}(I)$ is smooth (C^∞).

APPENDIX B: A SUMMARY ON PROPER FUNCTIONS

A summary of certain useful properties concerning *proper* functions is given here.

A function I is called proper if $I^{-1}(K)$ is compact for every compact set K, $K \subset \mathbb{R}$. The function (total energy) $\dot{\mathbf{z}}^2 + V(\mathbf{z})$ is proper when $\lim_{\infty} V = +\infty$.

A function I is called locally proper if $I^{-1}(K)$ is compact for any compact set $K, K \subset \mathcal{D} \subset \mathbb{R}$ ($\mathcal{D}$ is a fixed subset of $\mathbb{R}$). The total energy of a pendulum is a locally proper function of the angular displacement (θ) and $\dot{\theta}$. When $\dot{\theta}$ is small, the level sets of E are bounded.

The set of functions $\{I_i\}$, $i=1,...,s$ is proper if $\bigcap_{i=1}^{s} I_i^{-1}(K)$ is a compact set for any $K \subset \mathbb{R}^s$. If $\{I_i\}$ is a proper set, then $I_1{}^2 + \cdots + I_s{}^2$ is a proper function, since its level sets are formed by the compact union of the (compact) level sets of I_i.

The functions

$$
\begin{aligned}
I_1 &= x_1{}^2 + x_2{}^2 + \mathrm{sen}(x_3), \\
I_2 &= x_2{}^2 + x_3{}^2 + \exp(-x_1{}^2),
\end{aligned} \tag{B1}
$$

are not proper, but they form a proper pair. Therefore $I_1{}^2 + I_2{}^2$ is a proper function, as the reader can check directly.

A set of functions I_i $i=1,...,s$, is locally proper on $\mathcal{D}$ if for any compact set contained in $\mathcal{D} \subset \mathbb{R}^s$ the set $\bigcap_{i=1}^{s} I_i^{-1}(K)$ is compact. The energy and the angular momentum form a proper local set of integrals of Kepler's problem. In this case $\mathcal{D}$ is any $\mathbb{R}^2$ domain on which $E(\text{energy}) < 0$, $L(\text{angular momentum}) \neq 0$.

Note that when $\{I_i\}$, $i=1,...,s$, is a locally proper set of functions on $\mathcal{D}$ the function $I_1{}^2 + \cdots + I_s{}^2$ is not always locally proper on $\mathcal{D}$. For instance, the energy and the angular momentum of Kepler's problem do not satisfy this requirement. In fact, the level sets of $E^2 + L^2$ are always unbounded.

APPENDIX C: A USEFUL BOUND FOR $R(T)$

We now get a bound for $R(T)$, the maximum distance from $\mathbf{0} \in \mathbb{R}^n$ to the set S defined by

$$
S = \{\mathbf{x} | I_0 - K''' \cdot T \leqslant I(\mathbf{x}) \leqslant I_0 + K''' \cdot T\}. \tag{C1}
$$

The following evaluations shall be made by computing the maximum distance from $\mathbf{0}$ to the part of the Boundary(S) given by the compact set $I^{-1}(I_0 + K''' \cdot T)$. By Sard's theorem[11] we may assume that I^{-1} $(I_0 + K''' \cdot T)$ is a differential manifold. Note that the same evaluations apply in computing the maximum distance from $\mathbf{0}$ to the compact set I^{-1} $(I_0 - K''' \cdot T)$.

Consider the projections pr_i (M_T) of the compact set $M_T = I^{-1}$ $(I_0 + K''' \cdot T)$ on the coordinate axis x_i. That is,

$$
pr_i(x_1, \ldots, x_n) = x_i. \tag{C2}
$$

We can also write

$$
\begin{aligned}
&pr_i(M_T) \subset [a_i(T), b_i(T)], \\
&a_i(T) = \mathrm{Min}\, pr_i(M_T), \\
&b_i(T) = \mathrm{Max}\, pr_i(M_T).
\end{aligned} \tag{C3}
$$

Now, since we assume that I is a polynomial it can be shown using the Tarski theorem[12] that $a_i(T)$ and $b_i(T)$ are semialgebraic in T (Ref. 12) and when $T\rightarrow+\infty$ they are bounded by an integer power of T, T^{mi}, $m_i\in\mathbb{N}$.

It follows that $R(T)=D_{\max}(\mathbf{0},M_T)$ shall be bounded by

$$\{(T^{m_1})^2+\cdots+(T^{m_n})^2\}^{1/2}\approx T^m, \tag{C4}$$

m standing for the maximum of the natural numbers $(m_1,\ldots,m_n)$. Therefore $R(T)$ cannot increase, for large values of T, faster than T^m, as we desired to prove.

When I is a nonpolynomial first integral we can use the Stone–Weierstrass theorem[13] to approximate I, and a finite number of its derivatives, near M_T by a polynomial $P_{m(\varepsilon,T)}(\mathbf{x})$ of degree $m(\varepsilon,T)$. Moreover, by the Thom isotopy lemma,[14] the sets defined by

$$i(\mathbf{x})=I_0+K'''\cdot T, \tag{C5}$$

$$P_{m(e,T)}(\mathbf{x})=I_0+K'''\cdot T, \tag{C6}$$

are diffeomorphic and the set defined by (C5) lies in a neighborhood of the set defined by (C6).

Fixing now the value of ε (say $\varepsilon=1$) and *assuming* that the coefficients of $P_{m(1,T)}(\mathbf{x})$ depend algebraically on T (Ref. 12), we define the projections $pr_i(M_T)$ and obtain again, via Tarski theorem,[12] the polynomial bound $R(T)\approx T^m$. It must be said that when the dependence on T of the coefficients of $P_{m(1,T)}(\mathbf{x})$ is *not* semialgebraic, the problem of obtaining a bound of $R(T)$ is a very difficult one and no general solution of it is known to us.

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Unbounded Trajectories of Dynamical Systems

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Abstract—It is shown that when a divergence-free vector field without zeros $\mathbf{X}$ is defined on a two-dimensional, noncompact manifold, which is not a cylinder, then $\mathbf{X}$ must possess an unbounded orbit.

Keywords—Divergence-free vector fields, Unbounded orbits, Topological techniques, Foliations.

1. INTRODUCTION

In recent papers [1,2], the blow-up of R^n vector fields (v.f.) has been studied by means of local series around movable singularities (Painlevé analysis) [3–7].

In this paper, we study a related problem: the existence of unbounded orbits of differential equations. They shall be called escape orbits, and they play an important role in Newtonian gravitation, in which unbounded orbits of equations of type

$$\begin{aligned} &m\ddot{\mathbf{x}} = -\nabla V, \\ &V = -Gm \sum \frac{m_i}{\|\mathbf{x}-\mathbf{x_i}\|}, \\ &\mathbf{x_i} = \text{position of the attracting masses}, \\ &G = \text{gravitational constant} \end{aligned} \tag{1}$$

appear [8].

The following are additional examples of forces admitting escape solutions.

(i) The magnetic force $\dot{\mathbf{x}} \wedge \mathbf{B}(\mathbf{x})$, where $\mathbf{B}(\mathbf{x})$ is parallel to a fixed direction (say the z-axis) and $\mathbf{B}(\mathbf{x})$ is constant.

(ii) A constant gravitational force $\mathbf{g}(\mathbf{x})$ parallel to the z-axis.

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The reader can easily check that for any initial conditions $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0 \neq 0)$ the solutions to $\ddot{\mathbf{x}} = \dot{\mathbf{x}} \wedge \mathbf{B}(\mathbf{x})$ and $\ddot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$ verify

$$|z(t)| \to +\infty, \qquad \text{when } t \to +\infty.$$

The time taken by the particle in reaching $\|\mathbf{x}\| = +\infty$ can be finite or infinite, but we are not interested here in this issue.

A first formulation of our problem is: can we choose the initial conditions $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$ in such a way that the corresponding solution $\mathbf{x}(t)$ of equation (1) is unbounded in $\mathbf{x}(t)$, in $\dot{\mathbf{x}}(t)$, or in both $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$?

Note that when $V(\mathbf{x})$ is a central potential this question has an easy reply since in this case equation (1) is integrable. However, when several attracting masses are present, equation (1) is no longer integrable and escape to infinity must be analyzed in other ways.

Escape to infinity in the presence of non-Newtonian [9–19] or Newtonian potentials [8,20–22] has been analyzed via analytical techniques. On the contrary, we shall study escape to infinity using *topological* means.

Topological means were suggested by Smale [23] in order to get properties of the orbits of equation (1) when topological invariants of a certain number of first integrals I of equation (1) are known (Betti numbers, homotopy, or homology groups of the level sets of $I, \dots$).

This paper follows exactly this line and its main result is as follows.

THEOREM.

Let $\mathbf{X}$ be the v.f. representing the dynamics. Let $\mathbf{X}_{|V_2}$ be the restriction of $\mathbf{X}$ to an invariant unbounded differential manifold V_2 of dimension two, where V_2 is not a cylinder. Assume, finally, that

(i) *$\mathbf{X}_{|V_2}$ is divergence free, and*
(ii) *$\mathbf{X}$ is free from zeros.*

Then, there is an unbounded orbit of $\mathbf{X}_{|V_2}$ on V_2.

The proof of this theorem appears in Section 2.

In ending this introduction, we must say that the study of escape orbits of differential equations of type

$$\begin{aligned} m\ddot{\mathbf{x}} &= \mathbf{F}(t, \mathbf{x}), \\ \mathbf{x} &\in R^n \end{aligned} \tag{2}$$

was initiated by Kneser [24]. Hartman and Wintner [25] extended Kneser theory to include velocity dependent forces, when $n = 1$, and finally (see [26]; see also [27], where systems of linear repulsive forces are considered) for arbitrary values of n.

The techniques used by all these authors are analytical.

2. PROOF OF THE THEOREM

We shall prove the theorem in Section 1 by contradiction. That is, if we assume that *all* the orbits of $\mathbf{X}$ are bounded, we get a contradiction. Remember that $\mathbf{X}$ is a divergence-free v.f. without zeros defined on a two-dimensional unbounded manifold V_2 (a surface).

In fact, if $\mathbf{X}$ is free from zeros and is divergence free then the w-limit set of any bounded orbit ϑ of $\mathbf{X}$ is an S^1 orbit [28,29]. Therefore, if *all* the orbits of $\mathbf{X}$ were bounded, V_2 would be foliated by disjoint circles.

Now, it is immediate to prove (see below) that the only unbounded surface V_2 covered with topological circles is the topological cylinder. Therefore, if V_2 is *not* a topological cylinder, at least one of the orbits of $\mathbf{X}$ must be unbounded.

We now briefly sketch the proof that an unbounded V_2 orientable manifold covered with circles is a topological cylinder. This fact is an easy consequence [30] of the fact that we can form in a neighbourhood of every S^1 orbit in V_2 a local tubular neighbourhood N which is covered with S^1 orbits. Under prolongation of N repeatedly one gets either a cylinder or a torus-like (compact) surface. As we are working with unbounded manifolds V_2 the last case is excluded.

3. EXAMPLES

Illustrative examples in R^3 and R^4 are now given. The common features underlying them are the following:

(i) the construction of divergence-free v.f. $\mathbf{X}$ in R^3 or R^4,

(ii) the appearance of one or two first integrals I_i of $\mathbf{X}$,

(iii) the manifold V_2 of Sections 1 and 2 is obtained as a common level set of the first integrals;

(iv) the v.f. $\mathbf{X}_{|V_2}$ (the restriction of $\mathbf{X}$ to V_2 is divergence free [31], by choosing conveniently a volume form w_2 on V_2.

For R^3 v.f., w_2 has the form

$$\begin{gathered}\frac{i_{\nabla I_1}\Omega_3}{\|\nabla I_1\|^2},\\ \Omega_3 = dx\wedge dy\wedge dz,\\ i = \text{contraction operator.}\end{gathered}\tag{3}$$

For R^4 v.f. w_2 has the form

$$\begin{gathered}\frac{i_{\nabla I_1} i_{\nabla I_2}\Omega_4}{\begin{vmatrix}(\nabla I_1)^2 & \nabla I_1\nabla I_2\\ \nabla I_1\nabla I_2 & (\nabla I_2)^2\end{vmatrix}}\\ \Omega_4 = dx\wedge dy\wedge dz\wedge dt\end{gathered}\tag{4}$$

Note that (3) and (4) are valid on the level sets of I_1 (or I_1 and I_2) when $\operatorname{rank}(\nabla I_1) = 1$ (or $\operatorname{rank}(\nabla I_1, \nabla I_2) = 2$) on them.

EXAMPLE 3(i). Consider the divergence-free v.f. in R^3

$$\begin{gathered}\mathbf{X} = \left(-y\Pi, x\Pi, y - x + z\left(y\Pi_{,x} - x\Pi_{,y}\right)\right),\\ \Pi \equiv \left(x^2+y^2\right)\left((x-1)^2+y^2\right).\end{gathered}\tag{5}$$

One can check that the function I_1, defined as

$$I_1 = z\Pi + x + y,\tag{6}$$

is a first integral of $\mathbf{X}$ and that $\mathbf{X}$ is free from zeros on the level sets

$$\begin{gathered}z\Pi + x + y = c,\\ c\neq 0,\qquad c\neq 1.\end{gathered}\tag{7}$$

On the other hand, ∇I_1 does not vanish on the level sets defined in (7); the topology of these level sets is that of a plane with two points deleted, as the reader will easily check out. Therefore, they are not cylinders.

We can, therefore, apply to these level sets the results of Section 1 and conclude that the v.f. $\mathbf{X}$ of equation (5) possesses an unbounded orbit on *each* of the manifolds defined in equation (7).

EXAMPLE 3(ii). Consider the complex polynomial

$$\begin{gathered}P(z_1, z_2) = \frac{z_2^2}{2} - z_1\left(z_1^2 - 1\right),\\ z_1, z_2 \in C,\end{gathered}$$

and the Hamiltonian differential equations [32–34]

$$\dot{z}_1 = \frac{\partial P}{\partial z_2}, \qquad \dot{z}_2 = -\frac{\partial P}{\partial z_1}, \tag{9}$$

that is,

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = 3z_1^2 - 1. \tag{10}$$

Writing $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $x_i, y_i \in R$, in (10), one gets the R^4 v.f.

$$\mathbf{X} = x_2\partial_{x_1} + y_2\partial_{y_1} + \left(3x_1^2 - 3y_1^2 - 1\right)\partial_{x_2} + 6x_1y_1\partial_{y_2}, \tag{11}$$

with the first integrals

$$\begin{aligned} I_1 &= \frac{x_2^2 - y_2^2}{2} - x_1^3 + 3x_1y_1^2 + x_1, \\ I_2 &= x_2y_2 - 3x_1^2y_1 + y_1^3 + y_1. \end{aligned} \tag{12}$$

It is immediate to check that $\mathbf{X}$, I_1, and I_2 satisfy all the conditions of our theorem on the level sets

$$\begin{gathered} I_1 = c_1, \\ I_2 = c_2, \\ c_1 \neq \pm\frac{2}{3\sqrt{3}}, \qquad c_2 \neq 0. \end{gathered} \tag{13}$$

Since these sets [33] have the topology of a torus with a point deleted they are not cylinders. Accordingly, in each of them lies an unbounded trajectory of $\mathbf{X}$. Nevertheless, we can predict in this case the existence of unbounded orbits in another way. In fact, from equation (10), we get

$$\ddot{z}_1 = 3z_1^2 - 1, \tag{14}$$

and therefore,

$$\begin{aligned} \ddot{x}_1 &= 3x_1^2 - 3y_1^2 - 1, \\ \ddot{y}_1 &= 6x_1y_1. \end{aligned} \tag{15}$$

A particular solution of equation (15) is $y_1(t) = 0$ and $x_1(t)$ any solution $\tilde{x}_1(t)$ of the second-order differential equation

$$\ddot{x}_1 = 3x_1^2 - 1. \tag{16}$$

This last equation is integrable and trivially possesses unbounded solutions. From this, the existence of the unbounded solutions of $\mathbf{X}$ immediately follows

$$\begin{aligned} x_1 &= \tilde{x}_1(t), \\ x_2 &= \dot{\tilde{x}}_1(t), \\ y_1 &= 0, \\ y_2 &= 0. \end{aligned} \tag{17}$$

EXAMPLE 3(iii). Consider now the complex polynomial

$$P(z_1, z_2) = \frac{z_2^2}{2} + iz_1 + \frac{z_1^3}{3}, \tag{18}$$

and the associated Hamiltonian differential equations

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -i - z_1^2, \end{aligned} \tag{19}$$

whose real form is

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{y}_1 &= y_2,\\ \dot{x}_2 &= -x_1^2 + y_1^2,\\ \dot{y}_2 &= -1 - 2x_1y_1.\end{aligned} \tag{20}$$

The reader can check that all the assumptions of our theorem are met on the level sets

$$\begin{aligned}I_1 &= c_1,\\ I_2 &= c_2,\\ c_1 + ic_2 &\neq \pm\frac{2}{3}i - i,\end{aligned} \tag{21}$$

I_1, I_2 defined by

$$\begin{aligned}I_1 &= \frac{x_2^2 - y_2^2}{2} - y_1 + \frac{x_1^3 - 3x_1y_1^2}{3},\\ I_2 &= x_2y_2 + x_1 + \frac{3x_1^2y_1 - y_1^3}{3}.\end{aligned} \tag{22}$$

These level sets are again [33,34] of type torus with a point deleted, and therefore, unbounded orbits in R^4 must appear on the level sets defined by equation (21).

4. TWO EXAMPLES RELATED TO ELECTROMAGNETIC FIELDS

EXAMPLE 4(i). Let $\mathbf{E}(x, y)$ be the R^2-vector field created by the $N \geq 2$ electric charges $(q_i, \mathbf{x}_i)$, $\mathbf{x}_i$ standing for the position of the charge q_i ($\mathbf{x}_i \in R^2$). We assume that $\mathbf{E}(x, y)$ is given by

$$\mathbf{E}(x, y) = \sum_{i=1}^{N} \frac{q_i(\mathbf{x} - \mathbf{x}_i)}{(x - x_i)^2 + (y - y_i)^2}. \tag{23}$$

The reader can check that this vector field is divergence-free ($\operatorname{div}(\mathbf{E}) \stackrel{\text{def}}{=} \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}$) and has a finite number of zeros; this last property can be immediately shown by eliminating the denominators appearing in $\mathbf{E} = \mathbf{0}$ and introducing the complex variable $z = x + iy$. We get in this way an expression whose zeros are just the zeros of a complex polynomial of degree $N - 1$.

Therefore, we can apply to $\mathbf{E}(x, y)$ the results of Sections 1 and 2 on the manifold V_2 defined by

$$V_2 = R^2 - Z - S, \tag{24}$$

Z being the set of zeros of $\mathbf{E}$ and S the singular points of $\mathbf{E}$ (these last points being, of course, the positions $\mathbf{x}_i$ of the charges q_i).

Since V_2 is not certainly a cylinder (the cardinality of the set $Z \cup S$ is greater than one for $N \geq 2$), we conclude that there is, at least, an unbounded orbit of $\mathbf{E}$ on V_2.

EXAMPLE 4(ii). Consider now the magnetic field $\mathbf{B}$ created by a planar circular wire on which a electric current of intensity I flows. It is well known [35,36] that if the planar wire W lies on the xy-plane and the origin of coordinates coincides with the center of W, then any plane π containing the z-axis is invariant under $\mathbf{B}$. On the other hand, $\mathbf{B}$ is singular on π at the two points given by

$$W \cap \pi, \tag{25}$$

and $\mathbf{B}$ is free from zeros on π [35,36].

Since $\mathbf{B}$ is divergence free [35,36], that is,

$$\operatorname{div}(\mathbf{B}) = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0, \tag{26}$$

it is easy to check that the vector field $\mathbf{B}_{|\pi}$ (the restriction of $\mathbf{B}$ to π) is also divergence free.

The definition domain of $\mathbf{B}_{|\pi}$ is

$$V_2 = R^2 - (W \cap \pi), \tag{27}$$

which is *not* a topological cylinder (remember that V_2 is just R^2 with two points deleted).

Therefore, as $\mathbf{B}_{|\pi}$ is free from zeros on V_2, we can apply to this example the results of Sections 1 and 2 and conclude that $\mathbf{B}_{|\pi}$ has at least an unbounded orbit on V_2. In fact, it can be shown [35,36] that the z-axis is the unique unbounded orbit of $\mathbf{B}_{|\pi}$.

5. FINAL COMMENTS

A criterion in order to get unbounded solutions of differential equations by topological means has been obtained. Several examples are given in Section 3. Examples related to electromagnetism are developed in Section 4. A similar criterion for bounded solutions (for example, for periodic solutions) would be interesting.

A related problem is that of knowing whether or not a divergence-free zero free v.f. $\mathbf{X}$, on any unbounded manifold V_2, exists such that V_2 is foliated by orbits of type R. Note that when the v.f. $\mathbf{X}$ is *not* necessarily divergence free, it is known [37] that V_2 can indeed be foliated by the type R orbits of $\mathbf{X}$.

Another problem related to this one is that of knowing if orbits of type R (unbounded) and S^1 (periodic) can coexist on two-dimensional unbounded manifolds V_2 when $\operatorname{div}\mathbf{X} = 0$ and $\mathbf{X}$ is free from zeros. The coexistence is impossible for $V_2 = R^2$ since a periodic orbit of $\mathbf{X}$ implies the appearance of a zero of $\mathbf{X}$. When $V_2 = R^2$ the condition $\operatorname{div}\mathbf{X} = 0$ plays no role, but we suspect it does when $V_2 \neq R^2$.

Finally, in order to get a generalization of the results of this paper to unbounded manifolds V_n ($n \geq 3$) a previous study and classification of the manifolds that can be foliated by circles would be necessary [38]. The main difficulty of this study is that the w-limit sets of orbits, of three-dimensional vector fields (in contrast with what happens in V_2), are, up to now, not topologically classified.

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Bagpipes Configurations in Mechanics and Electromagnetism

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Abstract—It is shown that when a first integral I of a vector field (v.f. in what follows) $\mathbf{X}$ is known, the level sets of I resemble bagpipes, and $\mathbf{X}$ is asymptotically stable (a.s. in what follows) on the skeleton of I (the set where ∇I vanishes), then the v.f. is stable at $\mathbf{0}$ (a singular, not necessarily isolated, zero of $\mathbf{X}$). A similar bagpipes configuration is shown to appear concerning the orbits of the magnetic field created by a set of concurrent straight line wires.

1. INTRODUCTION

It is well known that if $\mathbf{0}$ is an isolated critical point of $\mathbf{X}$ (a smooth dynamical system) and I is a first integral of $\mathbf{X}$ with an isolated (or strict) minimum (or maximum) at $\mathbf{0}$, then $\mathbf{0}$ is a stable equilibrium point of $\mathbf{X}$ [1–6]. The essence of this result is that the level sets of I near zero ($I^{-1}(\varepsilon)$, $|\varepsilon|$ small) are bounded sets; in fact, they are topological (or deformed) spheres. Note that we have assumed that the value of I at $\mathbf{0}$ is zero. This can *always* be achieved by the addition to I of a trivial constant.

When $\mathbf{0}$ is a critical point of $\mathbf{X}$ but it is *not* an isolated minimum of I the above criterion fails as the reader can check by considering the differential equation of a nonrelativistic moving charge ($q = m = 1$) in a constant magnetic field $\mathbf{B}_0$ ($\mathbf{B_0} \neq \mathbf{0},\ \mathbf{B_0} \in R^3$),

$$\ddot{\mathbf{x}} = \dot{\mathbf{x}} \wedge \mathbf{B_0}, \qquad \mathbf{x} \in R^3, \tag{1}$$

and the classical first integral I of equation (1),

$$I = \dot{\mathbf{x}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2. \tag{2}$$

Note that ∇I vanishes on the three-dimensional manifold M_3 given by

$$M_3 = \{(\mathbf{x}, \dot{\mathbf{x}}),\ \text{such that } \dot{\mathbf{x}} = \mathbf{0}\} = (x, y, z, 0, 0, 0). \tag{3}$$

Moreover, I and ∇I vanish on the manifold M_3 of equation (3).

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Note that I possesses an absolute minimum (equal to zero) on M_3.

Note also that the v.f. $\mathbf{X} = (\dot{\mathbf{x}}, \dot{\mathbf{x}} \wedge \mathbf{B_0})$ associated to equation (1) vanishes on M_3; that is the zeros of this v.f. are never isolated.

On the other hand, the zeros of $\mathbf{X}$ are *not* stable. Indeed, consider the following solution of equation (1),

$$\mathbf{x} = \varepsilon \mathbf{B_0} t, \quad \dot{\mathbf{x}} = \varepsilon \mathbf{B_0}, \qquad \varepsilon \in R, \tag{4}$$

corresponding to the initial conditions,

$$\mathbf{x}(t=0) = \mathbf{0}, \qquad \dot{\mathbf{x}}(t=0) = \varepsilon \mathbf{B_0}. \tag{5}$$

The initial conditions (5) are, for small values of ε, arbitrarily near $P_0 = (\mathbf{0}, \mathbf{0}) \in M_3$. The solution of equation (1) corresponding to $\mathbf{x_0} = \mathbf{0}$, $\dot{\mathbf{x}}_\mathbf{0} = \mathbf{0}$ is

$$\mathbf{x}(t) = \mathbf{0}, \qquad \dot{\mathbf{x}}(t) = \mathbf{0}. \tag{6}$$

A look at equations (4) and (6) shows that the initial point P_0 is unstable. Moreover, the corresponding solution given in equation (4) is *unbounded*.

In the following, we shall study stability around zeros $\mathbf{x}_0$ of v.f. Under certain conditions that we will specify in Section 2, a new stability criterion is given. The criterion is valid for critical points (not necessarily isolated) of R^n v.f. (possessing a nonnegative first integral I). The zero ($\mathbf{x}_0$) of $\mathbf{X}$ must be on the *critical* level set of I, that is, $\mathbf{x}_0$ must satisfy the equation $\nabla I|_{\mathbf{x_0}} = \mathbf{0}$.

Along this paper the only allowed critical sets C of I (the set where $\nabla I = \mathbf{0}$) are a finite number of straight lines. Namely, $C = \bigcup_{i=1}^N L_i$, L_i being straight lines meeting at $\mathbf{x}_0$.

The reader can check that the criterion in section II holds when L_i is substituted by $L_i^\top$ (a closed topological curve diffeomorphic with R).

Note that assuming $I(\mathbf{x}_0) = 0$ implies (since $\nabla I = \mathbf{0}$ on C) $I(L_i) = 0$, $\forall\, i$; therefore,

$$I\left(\bigcup_{i=1}^N L_i\right) = 0,$$

that is, I reaches an absolute minimum on $\bigcup_{i=1}^N L_i$.

To the above hypotheses we must add a final requirement: the v.f. $\mathbf{X}|_C$, induced by $\mathbf{X}$ on C (note that C lies on the level set I=0) must be a.s. at $\mathbf{x}_0$.

Under these conditions it is shown that $\mathbf{X}$ is stable at $\mathbf{x}_0$. The proof of this criterion is given in Section 2 and applications of it are discussed in Section 3.

In Section 4, it is shown that the orbits of the magnetic field $\mathbf{B}$ created by N straight line wires w_j ($j = 1, \dots, N$) intersecting at $\mathbf{x}_0$, are (near the wires) topological circles. Therefore, a bagpipes structure appears concerning the magnetic induction v.f. created by a set of current carriers.

In ending this introduction, we must say that our criterion is similar, but has *not* very much to do with LaSalle invariance principle [7,8] (connecting limit sets of orbits with the set of zeros of a function $V(\mathbf{x})$ satisfying $\dot{V} = \nabla V \cdot \mathbf{X} \leq 0$) for the following reasons.

- The functions $V(\mathbf{x})$ considered in LaSalle invariance principle satisfy the condition $\nabla V \cdot \mathbf{X} \leq 0$, while our first integral I satisfies the condition $\nabla I \cdot \mathbf{X} = 0$ on all R^n.
- On the other hand, LaSalle invariance principle deals with the limit sets $\omega^+(\mathbf{x}(t))$ of the *bounded* solutions of $\mathbf{X}$ (for $t > 0$) and asserts that for *bounded* $\mathbf{x}(t)$, we get

$$\omega^+(\mathbf{x}(t)) \subset A, \tag{7}$$

A being the maximal invariant set of $\mathbf{X}$ lying in the set B defined by the equation,

$$B = \{\mathbf{x} \;:\; \nabla V \cdot \mathbf{X} = 0\}. \tag{8}$$

In our case, equation (8) holds on all R^n (since it holds $\forall\, \mathbf{x} \in R^n$ when V is a first integral of $\mathbf{X}$) and the *maximal* invariant set (of $\mathbf{X}$) contained in R^n is again R^n. Therefore, LaSalle's theorem is useless concerning the stability question of this paper.

2. A NEW STABILITY CRITERION

In this section, the new stability criterion is established (Section 2.1). The proof is given in Section 2.2. For simplicity, the proof is given in R^3, but the criterion holds in R^n, $n \geq 3$.

2.1. Criterion

Assume that $\mathbf{0}$ is a critical point of $\mathbf{X}$, an R^n v.f., ($\mathbf{X}(\mathbf{0}) = \mathbf{0}$), that I is a nonnegative first integral of $\mathbf{X}$, such that $I(\mathbf{0})$=0. Assume that $I^{-1}(0)$ is the union of a finite number of straight lines L_i ($i = 1, \ldots, N$) through $\mathbf{0}$ and that $I^{-1}(c)$, $c \neq$0, are bagpipes with $2N$ pipes (see Figure 1). Assume, finally that $\mathbf{X}|_{L_i}$ is, for every L_i, a.s. at $\mathbf{0}$ (the reader will easily check that the straight lines L_i are invariant sets under the v.f. $\mathbf{X}$). Then, it follows that $\mathbf{X}$ is stable at $\mathbf{0}$.

Note that the above conditions imply that $\nabla I|_{I^{-1}(0)} = \mathbf{0}$, as $\mathbf{0}$ is the absolute minimum of I.

An R^n hypersurface S is called a bagpipes with $2N$ pipes if S is homeomorphic to the surface of a sphere S^{n-1} with $2N$ points deleted. When $N = 1$, S is just a cylinder. See Figure 1 for $N = 2$.

Geometrically, the lines L_i ($i = 1, \ldots, N$) act as the skeleton of the bagpipes. A cylinder can be thought of as the surface obtained by blowing-up this skeleton (the single line L_1). Analogously,

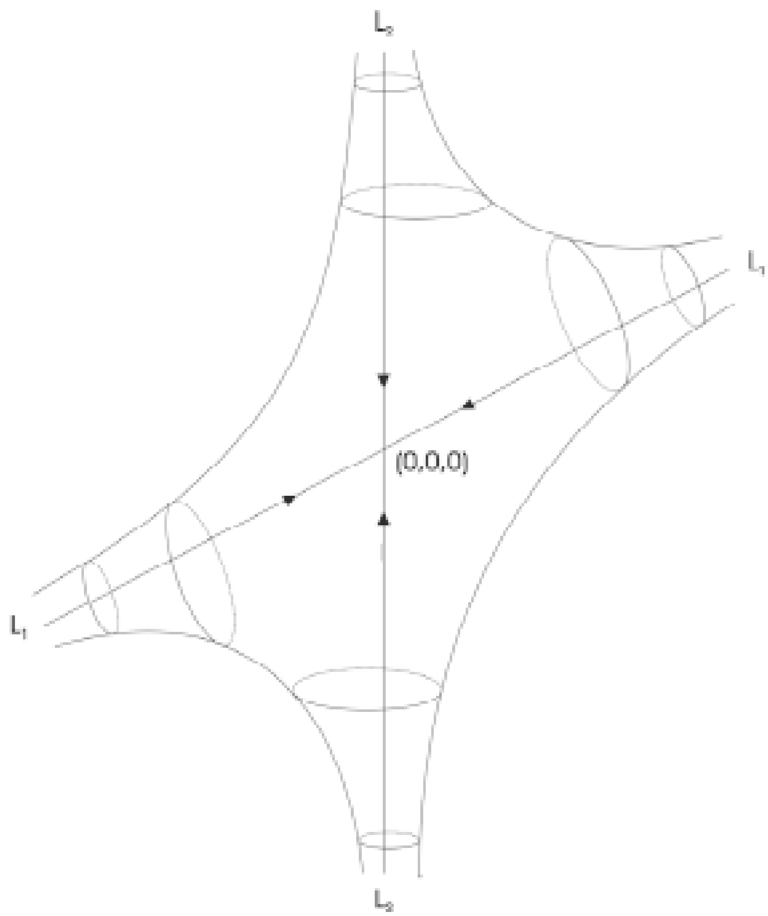

Figure 1. A typical bagpipes with four pipes ($N = 2$). See Example 3.(iii).

a bagpipes can be thought of as the surface obtained by blowing-up simultaneously the lines, $L_1, \dots, L_N$.

Note that the intersections of an R^3 bagpipes with planes π orthogonal to the lines L_i ($\mathbf{0} \notin \pi$) are topological circles (S^1). For an R^n bagpipes these intersections are (n-2)-dimensional spheres (S^{n-2}).

Note also that the straight lines L_i can be replaced by other more general curves $L_i^\top$ through $\mathbf{0}$ and stability at $\mathbf{0}$ would also follow.

Finally, some comments on the relation of this criterion with the *Liapunov's stability theorem* are in order.

- It is well known [9,10] that, under very general conditions, a Liapunov's function $F(\mathbf{x})$ exists around any stable equilibrium point of a v.f. $\mathbf{X}$. Nevertheless, except when $\mathbf{X}$ is a linear v.f., the practical computation of $F(\mathbf{x})$ is a very difficult problem (see reference [10]). In general, the construction of $F(\mathbf{x})$ depends on the prior knowledge of the general solution of the v.f. $\mathbf{X}$. Moreover, in some cases, F is necessarily time dependent [10].
- Our criterion is similar to Liapunov's in that $I \geq 0$ and $\dot{I} = 0$. Nevertheless the level sets of $I(\mathbf{x})$ are unbounded, while the level sets of $F(\mathbf{x})$, near the critical point, are compact (topological spheres). Moreover, our criterion assumes that $\mathbf{X}|_{L_i}$ is a.s. at $\mathbf{0}$. This last assumption is not made in the usual Liapunov's stability theorem.

In Section 3, we show some examples for which the stability of $\mathbf{0}$ is proved by applying our criterion. It is not clear how to construct Liapunov's functions in all these cases and therefore, Liapunov's stability theorem seems to be useless. A detailed discussion of this fact is made in example 3.(i), where no obvious Liapunov's functions, e.g., trivial modifications of the first integral $I(\mathbf{x})$, are found.

2.2. Proof of the Criterion

Let us prove stability of $\mathbf{X}$ at $\mathbf{0}$, when $N = 1$.

Let B_ε be the closed ball of radius ε centered at $\mathbf{0}$ and P_ε any of the two points where L_1 intersects ∂B_ε. Assume that L_1 lies on the z-axis. Let us find a $\delta(\varepsilon) > 0$ such that for the points $\mathbf{x_0} \in B_{\delta(\varepsilon)}$ it follows $\|\mathbf{x}(t, \mathbf{x_0})\| < \varepsilon$ ($\mathbf{x_0}$ being the initial condition underlying $\mathbf{x}(t, \mathbf{x_0})$).

Indeed, for ε sufficiently small the vector $\mathbf{X}(\mathrm{P}_\varepsilon)$ points towards $\mathbf{0}$ (remind that $\mathbf{X}$ is a.s., on L_1, at $\mathbf{0}$) and by continuity reasons $\mathbf{X} \cdot \mathbf{n} < 0$ on a region R_ε of $\partial(B_\varepsilon)$ containing P_ε ($\partial(B_\varepsilon)$ is the spherical surface $\|\mathbf{x}\|^2 = \varepsilon^2$ and $\mathbf{n}$ the outer normal to $\partial(B_\varepsilon)$). Under these conditions R_ε is a repellor for the orbits of $\mathbf{X}$ arriving to R_ε from the interior of B_ε.

Let $I^{-1}(c_\varepsilon)$ be a level set of I, such that

$$\left(I^{-1}(c_\varepsilon) \cap \partial B_\varepsilon\right) \subset R_\varepsilon, \tag{9}$$

and consider the set C_ε formed by the points $P \in R^3$ satisfying

$$\begin{aligned} P &\in I^{-1}(c_\varepsilon), \\ \text{distance } (P, L_1) &< \varepsilon, \\ |z_P| &< \varepsilon. \end{aligned} \tag{10}$$

These hypotheses can be accomplished since the set of points $(0,0,z)$, $|z| \leq \varepsilon$ is compact and $I^{-1}(c_\varepsilon) \cap (z = k)$ are topological circles of radius $r_{\varepsilon,k}$ vanishing with ε ($|k| \leq \varepsilon$).

Consider, finally, the ball B_δ (centered at $\mathbf{0}$) of radius δ defined by

$$\delta = \text{distance } \left(\mathbf{0}, C_\varepsilon\right). \tag{11}$$

It is easy to see that if $\mathbf{x_0}$ is an initial condition lying on B_δ the corresponding solution $\mathbf{x}(t, \mathbf{x_0})$ will always remain inside B_ε.

Therefore, $\mathbf{X}$ is stable at $\mathbf{0}$.

The reader will have noticed that the proof holds also when the number of pipes is greater than one.

Some applications of the criterion are given in Section 3.

Note that stability at (0,0,0) can also be obtained if the level sets of I induce a *local* bagpipes structure in R^3 and these level sets are bounded near (0,0,0) in the region $|\mathbf{x}\cdot\mathbf{u}_i| < k_i$, $k_i \in R$ $\forall\, i = 1,\ldots,N$, $\mathbf{u}_i$ being a unitary vector along L_i (see Section 3, Example (iii), for the definition of local bagpipes structure and an example).

In ending this section some comments are in order.

(i) When C is near $\mathbf{x}_0$ a curve or the local union of several curves branching at $\mathbf{x}_0$, it is easy to show that the proof of this section holds, and therefore, $\mathbf{x}_0$ inherits stability in R^3 from the assumed a.s., of $\mathbf{X}|_C$ at $\mathbf{x}_0$.

(ii) The same thing happens when C is a circle where f vanishes and the sets $f > 0$ are tori blowing up from the circle C. Therefore, stability of $\mathbf{X}$ at $\mathbf{x}_0 \in C$ holds when asymptotic stability of $\mathbf{X}|_C$ is assumed (see Example 3.(iv)).

3. EXAMPLES

Five examples are now given for which the criterion of Section 2 can be applied.

3.1. Example (i)

Let $\mathbf{X}$ be the family of v.f.,

$$\mathbf{X} = (-ya(x,y,z))\partial_x + (xa(x,y,z))\partial_y + (-z^n + xb(x,y,z) + yc(x,y,z))\partial_z, \tag{12}$$

a standing for a positive function of (x,y,z), b, and c for arbitrary functions of $(x,\ y,\ z)$ and n for an odd positive integer.

This v.f. vanishes at (0,0,0) and has the first integral $I = x^2+y^2$. It is clear that $I^{-1}(0) = z$-axis and $I^{-1}(c)$ $(c>0)$ are cylinders (bagpipes with two pipes).

On the other hand,

$$\mathbf{X}|_{z-\text{axis}} = -z^n\partial_z. \tag{13}$$

Therefore, the family of v.f. $\mathbf{X}$ meets all the conditions required in the criterion of Section 2 and we can say that (0,0,0) is a stable point of $\mathbf{X}$.

Note that Liapunov's theorem cannot be applied in this case because there is no obvious candidate to be a Liapunov's function $F(x,y,z)$. Indeed, since the level sets of F must be compact (topological spheres) around the critical point $\mathbf{0}$, we can verify whether the typical Liapunov's function for a critical point, $F(x,y,z) = x^2+y^2+z^2$, works in this example. It is straight forward to see that $\dot{F} = 2z(xb(x,y,z) + yc(x,y,z) - z^n)$, which in general does not satisfy Liapunov's condition $\dot{F} \le 0$ around the origin. We have not found easy modifications of this $F(x,y,z)$ which work, and therefore the task of finding a suitable Liapunov's function for this family is so difficult that it does not seem clear how to ascertain stability of $\mathbf{0}$ without invoking our criterion.

3.2. Example (ii)

In this example the level sets of I (the first integral) are also topological cylinders (bagpipes with two pipes).

Let $\mathbf{X}$ be the v.f.,

$$\mathbf{X} = \left(2z^2x + x^2y\right)\partial_x + \left(2z^2y - x^3\right)\partial_y - z\left(3x^2 + 3y^2 + z^2\right)\partial_z, \tag{14}$$

This v.f. has a nonhyperbolic zero at (0,0,0), in fact $\mathbf{X}$ vanishes on the y-axis. Therefore, its stability character cannot be ascertained via the computation of the linear approximation $\mathbf{X}_L$ of $\mathbf{X}$ at (0,0,0).

The function $I = (x^2+y^2)(x^2+y^2+z^2)^2$ is a first integral of $\mathbf{X}$.

It is immediate to check (introduce cylindrical coordinates) that the level sets $I^{-1}(c)$ (c¿0) are cylinders (bagpipes with two pipes) and $I^{-1}(0)$ is the z-axis. On the other hand the v.f. induced by $\mathbf{X}$ on the z-axis is

$$\mathbf{X}|_{z-\text{axis}} = -z^3\partial_z. \tag{15}$$

Since the v.f. in equation (15) is a.s., at $z=0$, we can apply the criterion of Section 2 and conclude that $\mathbf{X}$ is stable at (0,0,0).

3.3. Example (iii)

In this example the level sets of I resemble bagpipes with four pipes near the z and y axis, as it is explained immediately.

Let $\mathbf{X}$ be the v.f.,

$$\mathbf{X} = \left(x^2+y^2+z^2\right)\left(M\partial_x + N\partial_y + P\partial_z\right), \tag{16}$$

M, N, P being defined by

$$\begin{aligned} M &= -\frac{Ny\left(Ax^2+B\right)+Pz\left(A+Bx^2\right)}{x\left[\left(1+z^2\right)B+\left(1+y^2\right)A\right]}, \\ N &= x - y\left(\frac{y^2}{1+x^2+z^2}-z^2\right), \\ P &= z\left(\frac{y^2}{1+x^2+z^2}-z^2\right), \\ A &= x^2\left(1+z^2\right)+y^2, \\ B &= x^2\left(1+y^2\right)+z^2. \end{aligned} \tag{17}$$

Note that the term x in the denominator of $\mathbf{X}$ gets cancelled by the same factor x appearing in the numerator of $\mathbf{X}$. We have preferred keeping x at the denominator of $\mathbf{X}$, instead of simplifying it in order not to complicate the formulas.

The reader can check that the v.f. $\mathbf{X}$ defined by (16) and (17) is C^1 (on R^3) and has a nonhyperbolic zero at (0,0,0).

The reader can also check that $I = [x^2(1+z^2)+y^2][x^2(1+y^2)+z^2]$ is a first integral of $\mathbf{X}$. Its level sets $I^{-1}(c)$ are as follows.

(i) The z-axis and the y-axis when $c=0$.

(ii) When $c>0$ and c is small the intersection of the surface $I(x,y,z)=c$ with the planes $z=k$ ($k\neq 0$) are topological circles (deformed circles) near the z-axis, as follows from the fact that I has a strict minimum on $z=k$ at the point (0,0,k). The same thing happens with the intersection of $I(x,y,z)=c$ with the planes $y=k'$ ($k'\neq 0$) when $c>0$ is small.

By definition, when (i) and (ii) hold, we say that the first integral I induces a local bagpipes structure in a neighbourhood of the z and y axis.

The z-axis and the y-axis are invariant sets under $\mathbf{X}$ and the v.f. induced by $\mathbf{X}$ on them are

$$\begin{aligned} \mathbf{X}|_{y-\text{axis}} &= -y^5\partial_y, \\ \mathbf{X}|_{z-\text{axis}} &= -z^5\partial_z. \end{aligned} \tag{18}$$

On the other hand, the points of the level set $I=c$ ($c>0$) for which $|z|<k_1$, $|y|<k_2$ form a bounded set in R^3, since we get for them

$$c = \left[x^2(1+z^2)+y^2\right]\left[x^2(1+y^2)+z^2\right] \geq x^4, \tag{19}$$

and therefore, $|x| \leq c^{1/4}$.

Since the v.f. of (18) are a.s. at y=0 and z=0, the reader will check that the stability criterion of Section 2 remains valid for local bagpipes structures whose level sets are bounded for $|\mathbf{x} \cdot \mathbf{u_i}| < k_i$, $k_i \in R$, $\forall\, i = 1, \ldots, N$, $\mathbf{u}_i$ being a unitary vector along L_i. Therefore, $\mathbf{X}$ is stable at (0,0,0).

3.4. Example (iv)

In this example the critical level set of the first integral I, I=0, is a circle (φ) and the level sets of I, $I = c$ ($c > 0$, c small), are topological tori around φ.

Let $\mathbf{X}$ be the family of v.f. in R^3

$$\begin{aligned} \mathbf{X} &= X_1\partial_x + X_2\partial_y + X_3\partial_z, \\ X_1 &= 4xy(y-1)\cdot a(x,y,z) - 2yz\cdot b(x,y,z), \\ X_2 &= 2yz\cdot c(x,y,z) - 4x^2y\cdot a(x,y,z), \\ X_3 &= 4y(x^2+(y-1)^2-1)(x\cdot b(x,y,z) - (y-1)\cdot c(x,y,z)), \end{aligned} \tag{20}$$

a standing for a positive function of (x,y,z) and b, c for arbitrary functions of (x,y,z).

This v.f. vanishes at (0,0,0) (a nonisolated zero of $\mathbf{X}$) and has the first integral,

$$I = \left(x^2 + (y-1)^2 - 1\right)^2 + z^2.$$

It is clear that $I^{-1}(0) = \varphi = \{(x,y,0) : x^2 + (y-1)^2 = 1\}$ (a circle on the $z = 0$ plane) and $I^{-1}(c)$ ($c > 0$, c small) are topological tori around φ.

On the other hand the v.f. induced by $\mathbf{X}$ on the circle φ is

$$\mathbf{X}|_\varphi = \left[4xy\cdot a\,(x,y,0)\,((y-1)\,\partial_x - x\partial_y)\right]|_{x^2+(y-1)^2=1}.$$

The point ($x = 0$, $y = 0$) is an isolated zero of the v.f. defined in (21). The reader can also check that it is a.s. at ($x = 0$, $y = 0$) (remember that $a(x,y,z)$ is a positive function). Therefore, we can apply the criterion of Section 2 and conclude that $\mathbf{X}$ is stable at (0,0,0).

3.5. Example (v)

This is an example in R^4.

Let $\mathbf{X}$ be the v.f.,

$$\mathbf{X} = xu^4\partial_x + x^2u^4\partial_y + x^2u^4\partial_z + u^3\left(-1 - u^2 - y - z\right)\partial_u.$$

The v.f. $\mathbf{X}$ vanishes at $\mathbf{0}$ (a nonhyperbolic singular point of $\mathbf{X}$).

It is immediate to check that $I = (1+u^2)x^2 + y^2 + z^2$ is a first integral of $\mathbf{X}$. Its level sets $I^{-1}(c)$ are cylinders of type S^2xR ($c > 0$) or the u-axis ($c = 0$). The v.f. induced by $\mathbf{X}$ on this axis is

$$\mathbf{X}|_{u-\text{axis}} = -u^3 - u^5. \tag{23}$$

Since the v.f. in equation (23) is a.s. at $u = 0$, we can again conclude that $\mathbf{X}$ is stable at $\mathbf{0}$.

4. THE MAGNETIC FIELD CREATED BY N CONCURRENT WIRES: A PHYSICAL EXAMPLE OF A LOCAL BAGPIPES STRUCTURE

A similar bagpipes configuration arises concerning the level sets of a first integral (see I_T in equation (31) of the magnetic field created by N straight-line wires (W_j, i_j) ($j = 1, \ldots, N$) concurrent at (0,0,0); i_j stands for the intensity of the current flowing through the W_j wire. Note that i_j can be a positive or a negative real number (depending on the j index).

The reader should note that no stability claiming is made in this section, in contrast with the contents of Section 2. Note also that the first integrals of Sections 2 and 3 are global while the first integrals of the magnetic field $\mathbf{B}$ in this section are just local.

Let $\mathbf{B}_j(x,y,z)$ be the magnetic field created by just the wire $(W_j,\, i_j)$ at the point (x,y,z). It is immediate [11] that

$$I = x^2 + y^2 + z^2, \tag{24}$$

is a first integral of $\mathbf{B}_j$ and $\mathbf{B}_{\text{Total}} = \sum_{j=1}^{N} \mathbf{B_j}$.

Let us now show that the orbits of $\mathbf{B}_{\text{Total}}$ on

$$S_c = \left\{ x^2 + y^2 + z^2 = c^2,\ c > 0 \right\}, \tag{25}$$

near the singular points $S_c \cap W_j$ are topological circles (that is, deformed circles).

In fact, from a general result [12] in the theory of R^3 divergence-free v.f. ($\mathbf{B}_j$ in our case), and the presence of the first integral $x^2 + y^2 + z^2$, whose level sets S_c $(c > 0)$ are of trivial first homotopy group, one gets a first integral I_j of $\mathbf{B}_j$ on S_c provided that

$$\int_{\varphi} i_{\mathbf{B_j}} \Omega_2 = 0, \tag{26}$$

Ω_2 standing for

$$\begin{aligned} \Omega_2 &\equiv \frac{i_{\nabla I}\Omega_3}{\|\nabla I\|^2}, \\ \Omega_3 &= dx \wedge dy \wedge dz, \end{aligned} \tag{27}$$

φ being a closed curve around each one of the singular points $S_c \cap W_j$ of $\mathbf{B}_j$ on S_c and i being the contraction operator of v.f. and differential forms [12].

Now, in a spherical coordinate system around W_j as polar line, we can write $\Omega_3 = \rho^2 \sin\theta\, d\rho \wedge d\theta \wedge d\phi$ and $\nabla I = 2\rho\partial_\rho$. Therefore, $i_{\nabla I}\Omega_3 = 2\rho^3 \sin\theta\, d\theta \wedge d\phi$ and we get $\Omega_2 = (1/2)\rho \sin\theta\, d\theta \wedge d\phi$ and $\int i_{\mathbf{B_j}}\Omega_2$ trivially vanishes (remember that the orbits of $\mathbf{B}_j$ on S_c are the lines $\theta =$ constant). Accordingly equation (26) holds.

Therefore, the first integral I_j exists and is defined by

$$i_{\mathbf{B_j}}\Omega_2 = dI_j. \tag{28}$$

Remember that although S_c is simply connected, $S_c - (S_c \cap W_j)$ is not. Therefore, equation (26) is necessary in order that equation (28) defines a function I_j globally defined on $S_c - (S_c \cap W_j)$.

It is straightforward to show that I_j is given by

$$I_j = i_j \int \frac{d\theta_j}{\sin\theta_j}, \tag{29}$$

θ_j being the angle formed by the vector $(x,y,z) \in S_c$ and the line W_j.

Moreover, from equation (28) we get (by adding on j)

$$i_{\mathbf{B}_{\text{Total}}}\Omega_2 = dI_T. \tag{30}$$

Therefore I_T is given by

$$I_T = \sum_{j=1}^{N} i_j \int \frac{d\theta_j}{\sin\theta_j}. \tag{31}$$

Note that this formula for I_T is valid for any value of the intensities i_j ($i_j \neq 0$). Note also that near the points (x,y,z) of S_c defined by θ_j=0 or $\theta_j = \pi$, we get either

$$\begin{aligned} &\lim I_T = +\infty, \\ &\text{for } i_j > 0, \qquad j = 1, \dots, N, \end{aligned} \tag{32}$$

or

$$\lim I_T = -\infty, \\ \text{for } i_j < 0, \qquad j = 1, \ldots, N. \tag{33}$$

Equations (31)–(33) and the fact that I_T is a continuous function for any $P \in S_c - (S_c \cap W_j)$, imply that the level sets of I_T near $S_c \cap W_j$ are topological circles and therefore, the orbits of $\mathbf{B}_{\text{Total}}$ near the singular points $S_c \cap W_j$ are topological circles as well. It follows that the level sets of the function I_T defined by equation (31) are bagpipes in a neighbourhood of the wires W_j.

Note that although the function I_T is only defined in $S_c - (S_c \cap W_j)$ the structure of I_T makes it clear that I_T extends to a global function on $R^3 - \bigcup_{j=1}^{N} W_j$. Therefore, I_T defines a local bagpipes structure in $R^3 - \bigcup_{j=1}^{N} W_j$, as we desired to prove.

5. FINAL REMARKS

It would be interesting to know under what circumstances stability at $\mathbf{0}$ can be obtained when the level sets of I are not bagpipes. Can additional topological conditions on the level sets of I near C (the critical set of I) be found in order to guarantee stability at $\mathbf{0}$?

Concerning the presence of bagpipes in the first integrals of the magnetic field $\mathbf{B}$ created by wires the problem remains of studying the possible influence of these structures on the motion of charged particles (m,q) subjected to a purely magnetic electromagnetic field $\mathbf{B}$ (that is, the electric field $\mathbf{E}$ vanishing everywhere) with bagpipes structures. A magnetic field $\mathbf{B}$ is said to possess a bagpipes structure when $\mathbf{B}$ has a first integral whose level sets are bagpipes. For instance, can a first integral of a purely magnetic field $\mathbf{B}$, with a bagpipes structure, prevents a material particle (m,q), subjected to $\mathbf{B}$, from approaching the wires creating $\mathbf{B}$ indefinitely?

Another interesting problem is that of studying the possible relation between bagpipes structures in the orbits of the magnetic field $\mathbf{B}$ and the existence of solutions of the Lorentz equation,

$$m\ddot{\mathbf{x}} = q\dot{\mathbf{x}} \wedge \mathbf{B}, \tag{34}$$

confined to remain inside a certain domain of the configuration space.

We can easily check, by using cylindrical coordinates, that the motion of a particle subjected to the magnetic field created by a straight-line wire is confined to an annular domain of the configuration space around the wire, the axis of the annular region being the straight-line wire (remember that a cylinder is just a bagpipes with two pipes, that is, N=1). In fact, as we now show, this domain is defined for each solution of equation (34) by an interval $[r_1, r_2]$, $r_1 > 0$, r being the radial cylindrical coordinate.

Indeed, in cylindrical coordinates around the wire the differential equations of motion of a unit-mass, unit-charge particle under the action of the magnetic field $\mathbf{B}$ created by the straight-line wire are (recall that $\|\mathbf{B}\|$ is proportional to $1/r$, $r = \sqrt{x^2 + y^2}$):

$$\begin{aligned} \ddot{r} - r\dot{\phi}^2 &= -\frac{\dot{z}}{r}, \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0, \\ \ddot{z} &= \frac{\dot{r}}{r}. \end{aligned} \tag{35}$$

From equations (35), we get

$$\begin{aligned} \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 &= E, \\ r^2\dot{\phi} &= L, \\ \dot{z} - \mathrm{Ln}(r) &= A. \end{aligned} \tag{36}$$

The last of equations (36) implies that $r(t)$ cannot reach the z-axis since otherwise we would have that $\mathrm{Ln}(r) \to -\infty$ which is in contradiction with the first of equations (36).

On the other hand by eliminating $\dot{\phi}$ and $\dot{z}$ in equations (36) we get:

$$\dot{r}^2 + \frac{L^2}{r^2} + (A + \mathrm{Ln}(r))^2 = E, \tag{37}$$

and therefore, r is limited by the equation

$$\frac{L^2}{r^2} + (A + \mathrm{Ln}(r))^2 \leq E, \tag{38}$$

which defines an annulus.

Therefore, the solutions of equations (35) satisfy

$$r_1 \leq r(t) \leq r_2, \tag{39}$$

r_1, r_2 being the roots of the equation,

$$\frac{L^2}{r^2} + (A + \mathrm{Ln}(r))^2 = E. \tag{40}$$

Generalizing to N¿1, and considering the magnetic field created by N straight-line wires intersecting in (0,0,0), does a *bagpipes type domain* (of the configuration space) with $2N$ pipes exist (depending on the initial conditions $\mathbf{x_0}$, $\dot{\mathbf{x}}_\mathbf{0}$) in which the particle *remains forever*?

Concerning section IV an interesting question is to generalize the physical sources of the magnetic field by assuming that wires are not concurrent and/or they are no longer straight lines. Under these circumstances, can we assert that a bagpipes configuration holds?

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Symmetries and first integrals of divergence-free $\mathbb{R}^3$ vector fields

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Abstract

First integrals and invariant sets of divergence-free vector fields with symmetries are obtained; the results are applied to the solution of certain stability questions.

1. Introduction

The orbit structure of $\mathbb{R}^3$ vector fields (v.f.) $\mathbf{X}$ is quite tangled and phenomena like the presence of strange attractors [1–4], ergodicity [5], chaos [6–8], turbulence [9], etc. complicate it enormously.

On the other hand, important v.f. in physics, like the magnetic induction $\mathbf{B}$ [10,11] and the field of velocities of a fluid [12] are $\mathbb{R}^3$ v.f.. These v.f. have been recently studied by adapting to them Hamiltonian-like structures [13–16]. $\mathbb{R}^3$ v.f. arise as well in connection with the study of magnetic force-free v.f. ($\operatorname{rot}\mathbf{B} = \lambda(x)\mathbf{B}$, $x \in \mathbb{R}^3$) [17–19] appearing in the study of solar flares, superconductors and plasma confinement [20].

It would be expected that the orbit structure of divergence-free $\mathbb{R}^3$ v.f. ($\operatorname{Div}\mathbf{X} = 0$) be simpler. But this is not the case; in fact [21–23] ergodicity and chaos seem to be compatible with the restriction $\operatorname{Div}\mathbf{X} = 0$.

Divergence-free vector fields (v.f.) with symmetries have recently been studied concerning their integrability [24–31]. Nevertheless, most of the results obtained in these papers are *local* and unsuitable (see Section 2 for the explanation) in order to study the behaviour of the orbits of a v.f. $\mathbf{X}$ near an equilibrium point of $\mathbf{X}$ (that is, near a point P where $\mathbf{X}$ vanishes). This study is essential, for example, if the stability or instability of $\mathbf{X}$ at P is required. Certain of these difficulties are overcome in this paper. This has been achieved via the obtention of global first integrals and invariant sets out of the symmetry vectors.

Our results are useful in order to ascertain the stability of $\mathbf{X}$ around a point (we shall call it $\mathbf{0}$) where $\mathbf{X}$ vanishes. These results are valid for $\mathbb{R}^n$ v.f. but in order not to complicate the notation most of the v.f. considered in this paper are $\mathbb{R}^3$ v.f..

A brief summary of other methods used in the literature in order to ascertain stability is given below.

When no symmetries are known but $\mathbf{0}$ is an hyperbolic equilibrium point of $\mathbf{X}$, the Hartman–Grobman theorem [32] can be applied and we can assert that $\mathbf{X}$ is locally topologically similar to its linear part $\mathbf{X}_L$. Remember that $\mathbf{X}$ is hyperbolic at $\mathbf{0}$ if $\mathbf{X}_L$ is free from eigenvalues of zero real part.

When $\mathbf{X}$ is a $\mathbb{R}^n$ v.f. non-hyperbolic at $\mathbf{0}$ little is known of the local structure of its orbits near $\mathbf{0}$.

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Leaving aside the trivial case $n = 1$, when $n = 2$ and $\mathbf{X}$ is analytic the organization of the orbits of $\mathbf{X}$ near $\mathbf{0}$ is known [33]; blowing-up techniques are often used ([34], see also Ref. [1]) in order to resolve the singularity of $\mathbf{X}$ at $\mathbf{0}$ (in the sense that by using a finite number of singular changes of variables the organization of the orbits near $\mathbf{0}$ can be ascertained).

In contrast with the situation in $\mathbb{R}^2$, little advance in the direction of knowing the organization of the orbits near $\mathbf{0}$ (when $n > 2$) has been made.

Concerning this point we quote the work of-Samardzija [35], who studied polynomial v.f. in $\mathbb{R}^3$, Golubitsky, Schaefer and Stewart who studied the bifurcations of $\mathbb{R}^3$ v.f. symmetric under continuous or discrete *linear* groups [36–38], and Dumortier, Roussarie, Sotomayor, Zoladek and Bonckaert [39–44], who considered the bifurcations of three-parameter families of v.f.. Many of the results obtained in these papers are based on the use of normal-forms and blowing-up techniques.

Our techniques complement the above ones in the sense that they have a geometrical base: the use of symmetry vectors. On the other hand, our techniques hold when the symmetry vectors are *non-linear*, in contrast with the results in Refs. [13–16].

The paper is organized as follows: In Section 2 the classical local integration algorithm and its difficulties near equilibrium points of $\mathbf{X}$ are reviewed. Section 3 deals with the obtention of invariant sets and its consequences concerning stability matters. Global first integrals of divergence-free v.f. are obtained in Section 4. In Section 5 we apply the methods of Section 4 to a higher-dimensional example. Finally, an instability criterion for $\mathbb{R}^3$ v.f. with a first integral is given in Section 6.

2. The classical integration algorithm

We now summarize the classical local integration algorithm of the $\mathbb{R}^3$ v.f. $\mathbf{X}$ when two symmetry vectors of it are known [24–28]. We shall see that the algorithm fails at the equilibrium points of $\mathbf{X}$.

Let $\mathbf{X}$ be an analytic v.f. vanishing at $\mathbf{0}$ and $\mathbf{S}_1$, $\mathbf{S}_2$ a pair of independent symmetry vectors. By independent we mean that the function Δ defined by

$$\Delta = \mathrm{Det}(\mathbf{X}, \mathbf{S}_1, \mathbf{S}_2) \tag{1}$$

is not identically zero.

Recall that $\mathbf{S}$ is a symmetry of $\mathbf{X}$ if

$$\mathscr{L}_{\mathrm{S}}\mathbf{X} = \lambda(x)\cdot\mathbf{X}, \tag{2}$$

$\mathscr{L}_{\mathrm{S}}$ standing for the Lie derivative along the streamlines of $\mathbf{S}$ and $\lambda(x)$ being an arbitrary function of x. Many times along this paper we consider symmetries for which the function $\lambda(x)$ is the zero function.

Let us now see how $\mathbf{X}$ can be locally integrated when two symmetries of it are known.

Consider the 1-form w_1 defined by

$$w_1 = (i_{\mathrm{X}} i_{\mathrm{S}_1}\cdot\Omega_3)\cdot\Delta^{-1}, \tag{3}$$

where i_{Y} stands for the operator of contraction between v.f. and differential forms [45], $\Omega_3 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ is the standard volume form of $\mathbb{R}^3$, and Δ is the function defined by (1).

It is easy to verify that w_1 is closed ($\mathrm{d}w_1 = 0$). Therefore, we can locally write

$$w_1 = \mathrm{d}I, \tag{4}$$

the function I satisfying

$$i_{\mathrm{X}}\cdot\mathrm{d}I = 0,$$

$$i_{\mathrm{S}_1}\cdot\mathrm{d}I = 0. \tag{5}$$

Therefore I is a first integral of $\mathbf{X}$ and $\mathbf{S}_1$. Note that the first integral is global if $\{\mathbf{X}, \mathbf{S}_1, \mathbf{S}_2\}$ are globally independent.

Consider now the v.f. $\mathbf{X}_c$ and $\mathbf{S}_{1c}$ (the v.f. induced by $\mathbf{X}$ and $\mathbf{S}_1$ on the level sets $I = c$ of I). These v.f. can be written in the local form:

$$\mathbf{X}_c = X_1(u,v)\cdot\partial_u + X_2(u,v)\cdot\partial_v,$$

$$\mathbf{S}_{1c} = S_{11}(u,v)\cdot\partial_u + S_{12}(u,v)\cdot\partial_v, \tag{6}$$

u and v standing for a set of local coordinates on the level set $I = c$.

Define now the 1-form w_1^*

$$w_1^* = (i_{\mathrm{X}_c} w_2)\cdot(i_{\mathrm{X}_c}\cdot i_{\mathrm{S}_{1c}}\cdot w_2)^{-1}, \tag{7}$$

where w_2 stands for $\mathrm{d}u \wedge \mathrm{d}v$.

It is easy to verify that w_1^* is closed and, therefore, we can locally write

$$w_1^* = \mathrm{d}I_1^*, \tag{8}$$

I_1^* standing for a function of the variables (u, v).

On the other hand, it is clear that

$$i_{\mathrm{X}_c}\cdot\mathrm{d}I_1 = 0. \tag{9}$$

Therefore, I_1^* is a first integral of $\mathbf{X}_c$. Since $\mathbf{X}_c$ is, locally, a $\mathbb{R}^2$ v.f. we conclude that $\mathbf{X}_c$ is integrable, and this implies the local integrability of $\mathbf{X}$.

Nevertheless, this algorithm cannot be applied around an equilibrium point of $\mathbf{X}$ since the function Δ appearing in formula (3) vanishes on the equilibrium points of $\mathbf{X}$, which makes the forms w_1 and w_1^* undefined wherever $\mathbf{X}$ vanishes. Therefore the above algorithm is not valid in order to answer stability questions near equilibrium points of $\mathbf{X}$.

Just in order to circumvent this difficulty that the techniques of the following sections are introduced. As we shall see the symmetries of $\mathbf{X}$ will allow us to compute invariant sets and first integrals of $\mathbf{X}$ (when $\mathbf{X}$ is a divergence-free) with whose help stability questions around equilibrium points of $\mathbf{X}$ can be succesfully answered.

3. Symmetries and invariant sets

We show here that if $\mathbf{X}$ is divergence-free, invariant sets, families of invariant sets and first integrals of $\mathbf{X}$ can be obtained. All these mathematical structures can be useful as far as stability is concerned, as is shown with an example.

First of all, let us demonstrate that the set (Z) of points of $\mathbb{R}^3$ defined by

$$Z = \{x \mid \Delta(x) = 0\} \tag{10}$$

is invariant under $\mathbf{X}$ (note that $\Delta(\mathbf{0}) = 0$, since $\mathbf{X}(\mathbf{0}) = \mathbf{0}$).

In fact, writing Δ in the form

$$\Delta = i_{\mathbf{X}} \cdot i_{\mathbf{S}_1} \cdot i_{\mathbf{S}_2} \cdot w_3, \tag{11}$$

we get, through straightforward manipulations.

$$\mathscr{L}_{\mathbf{X}}\Delta = \operatorname{Div}\mathbf{X} \cdot \Delta, \tag{12}$$

$\operatorname{Div}\mathbf{X}$ being, as usual, defined by

$$\mathscr{L}_{\mathbf{X}} w_3 = \operatorname{Div}\mathbf{X} \cdot w_3. \tag{13}$$

Let us now discuss some consequences of Eq. (12).

(i) When the set Z defined in Eq. (10) is a differential manifold, that is when the "normal" vector $\nabla(\Delta)$ on Z never vanishes, the set Z is invariant under $\mathbf{X}$. In fact, Eq. (12) implies

$$\mathscr{L}_{\mathbf{X}}(\Delta)_{|Z} = 0, \tag{14}$$

that is, $\mathbf{X}$ is tangent to Z on any of its points. Therefore, the set Z is invariant under $\mathbf{X}$. The invariance of Z under $\mathbf{X}$ can also be shown when Z fails to be a differential manifold, but the proof shall not be given.

(ii) We are assuming in this paper that $\mathbf{X}$, $\mathbf{S}_1$ and $\mathbf{S}_2$ are analytic v. f.. Therefore, the function Δ is an analytic function. This entails [46] that the set Z is, in a neighbourhood of $\mathbf{0}$, a finite union of strata of dimension 1 and 2.
If a certain strata E is invariant under $\mathbf{X}$ we can restrict $\mathbf{X}$ to it getting a vectorfield $\mathbf{X}_{|E}$ of lower dimensionality. The instability of $\mathbf{X}_{|E}$ at $\mathbf{0}$ implies the instability of $\mathbf{X}$ at $\mathbf{0}$.

An example on this point can be found at the end of these notes.

(iii) When $\mathbf{X}$ is divergence-free Eq. (12) becomes

$$\mathscr{L}_{\mathbf{X}}(\Delta) = 0. \tag{15}$$

Therefore the function Δ is a global first integral of $\mathbf{X}$.

The reader can see in Section 6 the consequences of global first integrals in stability matters.

An illustrative example is the following. Consider the v.f.

$$\begin{aligned}
\mathbf{X} &= -x_2\partial_1 + x_1\partial_2,\\
\mathbf{S}_1 &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3,\\
\mathbf{S}_2 &= \partial_3.
\end{aligned} \tag{16}$$

In this case $\operatorname{Div}\mathbf{X} = 0$ and $\Delta = -x_1^2 - x_2^2$ is a global first integral of $\mathbf{X}$.

Note that this first integral can also be obtained using the methods of Section 4 for the pair of divergence-free v.f. $\mathbf{X}$, $\mathbf{S}_2$.

The above example is trivial but shows that the assumptions of this paragraph are not incompatible.

A more interesting example is this one:

$$\begin{aligned}
\mathbf{X} &= (x_1x_3 - 2x_1^2 + x_3^2)\partial_1 + 5x_2(x_1 - x_3)\partial_2\\
&\quad + (-x_1x_3 - x_1^2 + 2x_3^2)\partial_3,\\
\mathbf{S}_1 &= x_3\partial_1 + x_2\partial_2 + x_1\partial_3,\\
\mathbf{S}_2 &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3.
\end{aligned}$$

The reader can check that $\mathbf{X}$ is divergence-free and that $\mathbf{S}_i$ $(i = 1, 2)$ are symmetries of $\mathbf{X}$. The function Δ of Eq. (11) is $\Delta = 8x_2(x_3 - x_1)^2 \cdot (x_1 + x_3)$ and Δ is a global first integral of $\mathbf{X}$.

(iv) Assume that the strata E is invariant under $\mathbf{X}$. E is not necessarily invariant under the symmetry vectors $\mathbf{S}_1$, $\mathbf{S}_2$. Therefore, if the following conditions hold:

1. $\mathbf{X}_E$ is stable at $\mathbf{0}$,
2. $\bigcup_t \varphi_t(E) = N(\mathbf{0})$ where $N(\mathbf{0})$ is a neighbourhood of $\mathbf{0}$ (17)

then we can safely conclude that $\mathbf{X}$ is stable at $\mathbf{0}$.

Note that in Eq. (17) φ_t stands for the flow of the symmetry vector under which E is not invariant.

Assumptions (17) are not empty, and they are met in the following example:

$$\mathbf{X} = -x_1\partial_1 - x_2\partial_2 - x_3\partial_3,$$

$$\mathbf{S}_1 = -x_3\partial_2 + x_2\partial_3,$$

$$\mathbf{S}_2 = -x_2\partial_1 + x_1\partial_2. \tag{18}$$

In this case one gets $\Delta = x_2(x_1^2 + x_2^2 + x_3^2)$. The set Z is composed of just one stratum: the set $x_2 = 0$. This stratum $\{x_2 = 0\}$ is invariant under $\mathbf{X}$ but is not invariant under the symmetry vectors. It is easy to verify that the assumptions (17)(1) and (17)(2) are satisfied.

To illustrate all these matters we terminate this section with an example.

Let $\mathbf{X}$ be the v.f. associated with the system of differential equations:

$$\dot{x}_1 = F(x_3) \cdot (x_1^2 + x_2^2)^n \cdot x_1,$$

$$\dot{x}_2 = F(x_3) \cdot (x_1^2 + x_2^2)^n \cdot x_2,$$

$$\dot{x}_3 = G(x_3) \cdot (x_1^2 + x_2^2)^n, \tag{19}$$

where F and G are analytic, $G(0) = 0$, $F(0) \neq 0$ and n is a natural number.

It is easy to check that $\mathbf{X}$ is symmetric under the v.f.:

$$\mathbf{S}_1 = x_1\partial_1 + x_2\partial_2, \quad \mathbf{S}_2 = x_2\partial_1 - x_1\partial_2. \tag{20}$$

The v.f. $\mathbf{X}$ is clearly non-hyperbolic at $\mathbf{0}$ and the function Δ associated with $(\mathbf{X}, \mathbf{S}_1, \mathbf{S}_2)$ is

$$\Delta = (x_1^2 + x_2^2)^{n+1} \cdot G(x_3). \tag{21}$$

Therefore, the set Z is composed, in a neighbourhood of $\mathbf{0}$, of two strata: the x_3-axis and the plane $x_3 = 0$. The reader will check that, in this case, the strata are invariant under $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{X}$.

The restriction of $\mathbf{X}$ to these strata is

$$\mathbf{X}_{|x_3\text{-axis}} = \mathbf{0} \quad \text{(identically)}, \tag{22}$$

$$\mathbf{X}_{|x_3=0} = F(0) \cdot (x_1^2 + x_2^2)^n \cdot \mathbf{S}_1. \tag{23}$$

Therefore, if $F(0)$ is positive then $\mathbf{X}_{|x_3=0}$ will be unstable at $(0, 0)$, and $\mathbf{X}$ unstable at $(0, 0, 0)$.

Note that the above procedure fails when $G(x_3)$ never vanishes, since the strata $x_3 = 0$ is no longer contained in the set Z. In this case we cannot say anything about the behaviour of $\mathbf{X}$ near $\mathbf{0}$. But if $\mathbf{X}$ is divergence-free the methods of Section 4 (ii) can be applied to the pair $(\mathbf{X}, \mathbf{S}_2)$, since it is a couple of divergence-free v.f. .

4. First integrals of divergence-free vector fields

We give in this section two methods for the obtention of first integrals of divergence-free v.f. out of symmetry vectors.

(i) We assume in this paragraph that $\mathbf{X}$ is divergence-free with respect to Ω_3 (a $\mathbb{R}^3$ volume form) and that $\mathscr{L}_{\mathbf{S}}(\mathbf{X}) = 0$. When nothing more is added Ω_3 is the standard volume form of $\mathbb{R}^3$, that is $\Omega_3 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

In fact, taking into account the assumption $\mathscr{L}_{\mathbf{S}}(\mathbf{X}) = 0$ and the relations [45]

$$\mathscr{L}_{[\mathbf{S},\mathbf{X}]} = \mathscr{L}_{\mathbf{S}}\mathscr{L}_{\mathbf{X}} - \mathscr{L}_{\mathbf{X}}\mathscr{L}_{\mathbf{S}},$$

$$\mathscr{L}_{\mathbf{u}}\Omega_3 = (\mathrm{Div}\,\mathbf{u})\Omega_3 \tag{24}$$

we immediately get

$$\mathscr{L}_{\mathbf{S}}(\mathrm{Div}\,\mathbf{X}) - \mathscr{L}_{\mathbf{X}}(\mathrm{Div}\,\mathbf{S}) = 0 \tag{25}$$

and since $\mathbf{X}$ is divergence-free we get from (25)

$$\mathscr{L}_{\mathbf{X}}(\mathrm{Div}\,\mathbf{S}) = 0 \tag{26}$$

That is, $\mathrm{Div}\,\mathbf{S}$ is a first integral of $\mathbf{X}$. Immediate first integrals obtained from $\mathrm{Div}\,\mathbf{S}$ are

$$\mathscr{L}_{\mathbf{S}}(\mathrm{Div}\,\mathbf{S}),\ \mathscr{L}_{\mathbf{S}}\mathscr{L}_{\mathbf{S}}(\mathrm{Div}\,\mathbf{S}), \ldots\ . \tag{27}$$

Note that $\mathrm{Div}\,\mathbf{S}$ can be a trivial constant. This is the case when the components of $\mathbf{S}$ are first-degree polynomials in x_1, x_2, x_3, as happens when $\mathbf{S}$ is the generator of a translation, rotation or dilatation. On the other hand, if the components of $\mathbf{S}$ are polynomials of degree greater than one Div $\mathbf{S}$ can be a non-constant first integral. This is the case of the generators of proper conformal transformations [45].

A typical conformal symmetry is

$$\mathbf{S} = 2x_1\left(\sum_{i=1}^{3} x_i\partial_i\right) - \left(\sum_{i=1}^{3} x_i^2\right)\cdot\partial_1 \tag{28}$$

and the first integral Div $\mathbf{S}$ is the non-constant function $6x_1$.

(ii) We assume in this paragraph that the symmetry vector $\mathbf{S}$ is not parallel to $\mathbf{X}$ everywhere, that $\mathbf{X}$ is divergence-free and that the symmetry condition (2) is of the form

$$\mathscr{L}_{\mathbf{S}}\mathbf{X} = -(\mathrm{Div}\,\mathbf{S})\cdot\mathbf{X}. \tag{29}$$

Eq. (29) is, of course satisfied, if $\mathbf{S}$ is divergence-free and $(\mathbf{X},\mathbf{S})$ is a pair of commuting v.f. .

A three-parameter family of $\mathbb{R}^3$ v.f. for which Eq. (29) holds, for $\mathbf{S} = x_3\partial_1 + x_2\partial_2 + x_1\partial_3$, is

$$\begin{aligned}\mathbf{X} = (&-px_1x_2 + qx_1x_3 + px_2x_3 - (q+t)x_1^2\\ &+ tx_3^2)\partial_1 + ((2t+3q)x_1x_2 - (2t+3q)x_2x_3)\partial_2\\ &+ (-px_1x_2 - qx_1x_3 + px_2x_3 - tx_1^2\\ &+ (q+t)x_3^2)\partial_3.\end{aligned} \tag{30}$$

Note that $\mathbf{S}$ is not divergence-free ($\mathrm{Div}\,\mathbf{S} = 1$) and that $\mathscr{L}_{\mathbf{S}}(\mathbf{X}) = -\mathbf{X}$, which is just Eq. (29) when $\mathrm{Div}\,\mathbf{S} = 1$.

Let us now see that under the assumptions of Section 4(ii), that is $\mathrm{Div}\,\mathbf{X} = 0$ and assumption (29), a non-trivial first integral of $\mathbf{X}$ can be obtained.

In fact, it is easy to check that defining w_1 via

$$w_1 = i_{\mathbf{X}} i_{\mathbf{S}} \Omega_3, \tag{31}$$

the 1-form w_1 is closed ($\mathrm{d}w_1 = 0$). Therefore, we can globally write

$$w_l = \mathrm{d}I. \tag{32}$$

On the other hand, and since $i_{\mathbf{X}}\mathrm{d}I = i_{\mathbf{S}}\mathrm{d}I = 0$, the function I is a first integral common to $\mathbf{X}$ and $\mathbf{S}$. I cannot become a trivial constant as $\mathbf{X}$ and $\mathbf{S}$ were assumed to be transversal.

By following this method the reader can easily show that the v.f. $\mathbf{X}$ of Eq. (30), symmetric under $\mathbf{S} = x_3\partial_1 + x_2\partial_2 + x_1\partial_3$ possess the first integral

$$\begin{aligned}I = (q+t)x_2[x_1^3 + x_3^3 - x_1x_3(x_1+x_3)]\\ + \frac{p}{2}x_2^2(x_1 - x_3)^2.\end{aligned} \tag{33}$$

On the other hand, when $\mathbf{S}$ represents the rotations around the x_3-axis the level sets of I are revolution surfaces around the x_3-axis. A level set not meeting the x_3-axis will have a cylinder-like appearance and those level surfaces meeting the x_3-axis, in one or more points, will have a cone-like appearance near these points. These two types of level sets appear in the following example.

Let $\mathbf{X}$ be the $\mathbb{R}^3$ v.f. associated with the system of differential equations

$$\begin{aligned}&\dot{x}_1 = \left(-\frac{C'}{2} - \frac{D'}{4}\cdot u\right)x_1 - x_2\cdot B(u, x_3),\\ &\dot{x}_2 = \left(-\frac{C'}{2} - \frac{D'}{4}\cdot u\right)x_2 + x_1\cdot B(u, x_3),\\ &\dot{x}_3 = C(x_3) + D(x_3)\cdot u,\\ &u = x_1^2 + x_2^2,\\ &C(0) = 0, \qquad C'(0) \neq 0,\end{aligned} \tag{34}$$

C, D and B standing for analytic functions of its arguments.

It is easy to verify that $\mathbf{X}$ is divergence-free and that $\mathbf{S} = x_2\partial_1 - x_1\partial_2$ is a symmetry vector of $\mathbf{X}$. Note that $\mathbf{X}(\mathbf{0}) = \mathbf{0}$.

The 1-form w_1 associated to the couple $(\mathbf{X},\mathbf{S})$ is

$$\begin{aligned}w_1 = &-x_1(C + D\cdot u)\,\mathrm{d}x_1 - x_2(C + D\cdot u)\,\mathrm{d}x_2\\ &+ u\left(-\frac{C'}{2} - \frac{D'}{4}\cdot u\right)\mathrm{d}x_3.\end{aligned} \tag{35}$$

The first integral of $\mathbf{X}$ associated with w_1 via Eq. (32) is

$$I = \frac{u}{2}\left(- C(x_3) - \frac{D(x_3)}{2}u\right). \tag{36}$$

The level sets of I, being invariant under $\mathbf{S}$, are revolution surfaces around the x_3-axis. The level set $I^{-1}(0)$ of this first integral contains the x_3-axis and the surface

$$2C(x_3) + D(x_3)u = 0. \tag{37}$$

Surface (37) is a cone of vertex $\mathbf{0}$ (remember that we assumed in (34) that $C'(0) \neq 0$).

The existence of invariant cones in $\mathbb{R}^3$ v.f. was signalled by Dumortier et al. [39–44]. Our contribution here is just having obtained the analytic expression of them.

Note also that the x_3-axis is a topologically isolated part of $I^{-1}(0)$. These isolated invariant lines have also been studied in [39–44], but using other methods.

The level sets $I^{-1}(c)$, $c \neq 0$, are cylinder like. This can be seen by drawing the two-dimensional curves:

$$\frac{r^2}{2}\left(- C(x_3) - \frac{D(x_3)}{2}r^2\right) = c \tag{38}$$

and rotating them around the x_3-axis.

The first integral (36) is important in order to prove the instability of $\mathbf{X}$ at $\mathbf{0}$. This is immediate by casting $\mathbf{X}$ and I in cylindrical coordinates around the x_3-axis and having into account that $I(u, x_3)$ presents a saddle point at $u = 0$, $x_3 = 0$. The details of the proof are left to the reader.

5. A higher-dimensional example

We consider in this section the differential equation of a non-relativistic charge moving on the $x_3 = 0$ plane under the action of a magnetic field $\mathbf{B}$ orthogonal to this plane.

The differential equations of this motion are

$$\begin{aligned} \ddot{x} &= B(x, y)\dot{y}, \\ \ddot{y} &= - B(x, y)\dot{x}. \end{aligned} \tag{39}$$

The v.f. associated to Eq. (39) is

$$\mathbf{X} = \dot{x}\partial_x + \dot{y}\partial_y + B(x, y)\dot{y}\partial_{\dot{x}} - B(x, y)\dot{x}\partial_{\dot{y}}. \tag{40}$$

This v.f. is divergence-free but four dimensional; a three-dimensional v.f. (to which we can apply the results of Section 4) can immediately be obtained taking into account that $\dot{x}^2 + \dot{y}^2$ is a first integral of $\mathbf{X}$.

We show now that when $B(x, y)$ is symmetric under rotations or translations a second integral of $\mathbf{X}$ can be obtained (using the methods of the last section). Note that there are other methods in order to obtain the first integrals that follow. We get them following the methods of Section 4 just for illustrative purposes.

The second integral of $\mathbf{X}$ is obtained in this way:

(i) Assume that B is of the form $B(x^2 + y^2)$. In this case it is easy to verify that $\mathscr{L}_{\mathbf{S}}(\mathbf{X}) = 0$, $\mathbf{S}$ being the divergence-free v.f.

$$\mathbf{S} = - y\partial_x + x\partial_y - \dot{y}\partial_{\dot{x}} + \dot{x}\partial_{\dot{y}}. \tag{41}$$

Note that $\dot{x}^2 + \dot{y}^2$ is a first integral of $\mathbf{S}$.

The v.f. $\mathbf{X}^*$, $\mathbf{S}^*$, induced by $\mathbf{X}$, $\mathbf{S}$ on the level sets $\dot{x}^2 + \dot{y}^2 = k^2$ of $\dot{x}^2 + \dot{y}^2$ are

$$\begin{aligned} \mathbf{X}^* &= k\cos\theta\,\partial_x + k\sin\theta\,\partial_y - B\partial_\theta, \\ \mathbf{S}^* &= - y\partial_x + x\partial_y + \partial_\theta. \end{aligned} \tag{42}$$

The vector fields $\mathbf{X}$ and $\mathbf{S}$ are, again, divergence-free with respect to the volume form $\mathrm{d}x \wedge \mathrm{d}y \wedge k\,\mathrm{d}\theta$. By applying to them the methods of Section 4 we get the first integral

$$\begin{aligned} I &= kx\sin\theta - ky\cos\theta + \frac{1}{2}\int B(u)\,\mathrm{d}u, \\ u &= x^2 + y^2, \end{aligned} \tag{43}$$

that is

$$I = x\dot{y} - y\dot{x} + \frac{1}{2}\int B(u)\,\mathrm{d}u. \tag{44}$$

(ii) When $B = B(y)$ it is immediate that $\mathscr{L}_{\mathbf{S}}(\mathbf{X}) = 0$, $\mathbf{S}$ being the divergence-free v.f.

$$\mathbf{S} = 1 \cdot \partial_x. \tag{45}$$

Note that $\dot{x}^2 + \dot{y}^2$ is a first integral of $\mathbf{S}$.

The three-dimensional v.f. $\mathbf{X}^*$ and $\mathbf{S}^*$ induced by $\mathbf{X}$, $\mathbf{S}$ on the level sets $\dot{x}^2 + \dot{y}^2 = k^2$ are

$$\begin{aligned} \mathbf{X}^* &= k\cos\theta\,\partial_x + k\sin\theta\,\partial_y - B\partial_\theta, \\ \mathbf{S}^* &= 1 \cdot \partial_x. \end{aligned} \tag{46}$$

Since these two v.f. are divergence-free we can proceed as above, getting the first integral

$$I = k\cos\theta - \int B(y)\,\mathrm{d}y = \dot{x} - \int B(y)\,\mathrm{d}y. \tag{47}$$

6. First integrals and instability

We close this paper by showing how important the first integrals obtained in the preceding sections can be in the applications. Specifically, we now show that if $\mathbf{0}$ is an isolated zero of $\mathbf{X}$ and I is a first integral of $\mathbf{X}$ satisfying a certain technical assumption then $\mathbf{0}$ is an unstable equilibrium point.

The technical assumption is

Let $I(\mathbf{0}) = 0$ and D_{ar} be the set defined by

$$D_{ar} = I^{-1}(a) \cap \|x\| \leqslant r, \tag{48}$$

$\|\ \|$ standing for the Euclidean norm of $\mathbb{R}^3$. We assume that these sets D_{ar} are (for $a \neq 0$) diffeomorphic to a disk. That is, each D_{ar} is the deformation of a two-dimensional disk.

This assumption is satisfied if, for example, $I = x_3 - F(x_1, x_2)$, and is not satisfied if the level sets of I are sphere-like or cylinder-like.

Let us show now (by contradiction) that our hypothesis imply that $\mathbf{0}$ is an unstable equilibrium point.

Let $x(t)$ be the solution of $\mathbf{X}$ starting at x_0 when $t = 0$; $x(t)$ remains on the level surface $I(x) = I(x_0)$ and near $\mathbf{0}$ (since we are assuming stability at $\mathbf{0}$).

Therefore, when $\|x_0\|$ is small, $x(t)$ lies on a certain D_{ar}. Since D_{ar} has the structure of a topological disk, the Bendixon–Poincaré theorem [47,48] can be applied to conclude that $\mathbf{X}$ will vanish in a certain point d_{ar} in D_{ar}.

For small values of r it is clear that d_{ar} approaches $\mathbf{0}$. But this contradicts the assumption that $\mathbf{0}$ is an isolated zero of $\mathbf{X}$. Therefore $\mathbf{X}$ cannot be stable at $\mathbf{0}$.

Let us apply this criterion to an example.

Consider the $\mathbb{R}^3$ v. f. given by

$$\begin{aligned}&(x_2(1 + x_3^2) - x_1 - x_3)\cdot\partial_{x_1}\\ &\quad + (-x_2 - x_1(1 + x_3^2))\cdot\partial_{x_2}\\ &\quad + ((x_1^2 + x_2^2 + x_1x_3)\cdot(1 + x_3^2))\cdot\partial_{x_3}.\end{aligned} \tag{49}$$

It is not difficult to check that this v.f. has an isolated, non-hyperbolic, zero at $(0,0,0)$; the eigenvalues of its linear part at $(0,0,0)$ are 0 and $-1 \pm i$. Therefore, linear stability arguments are unable to decide between stability or instability at $(0,0,0)$.

On the other hand $I = (x_1^2 + x_2^2)/2 + \operatorname{arctg} x_3$ is a first integral of $\mathbf{X}$. It is not difficult to verify that the level sets of this first integral are either topological planes or topological cylinders (deformations of ordinary plane or cylinder via diffeomorphisms): one has just to introduce cylindrical coordinates around the x_3-axis and draw the level sets of the function

$$\frac{r^2}{2} + \operatorname{arctg} x_3.$$

The level sets of I turn out to be topological planes near $(0,0,0)$. Therefore, the results of this section can be applied and we conclude that $(0,0,0)$ is an unstable equilibrium point of $\mathbf{X}$.

Note that in our example arguments based on the computation of a one-dimensional center manifold through $(0,0,0)$ can also be used in order to get the instability of $\mathbf{X}$ at $\mathbf{0}$.

Nevertheless, when the dimension of the center is greater than one the criterion of this section can be useful. This is due to the fact that to ascertain the stability character of $\mathbf{X}_{\text{center}}$ (the v.f. induced by $\mathbf{X}$ on a center manifold), when dimension (Center) $\geqslant$ 2, can be a problematic issue, as $\mathbf{X}_{|\text{center}}$ is, in general, unknown, and one has to work with just an approximation of it. On the contrary, the arguments of this section are not based on approximations.

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Dynamical systems embedded into Lie algebras

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Analytical and geometrical information on certain dynamical systems **X** is obtained under the assumption that **X** is embedded into a certain real Lie algebra. [DOI: 10.1063/1.1412598]

I. INTRODUCTION

This article deals with the problem of extracting information of a three-dimensional dynamical system **X**, when **X** is embedded into a Lie algebra of 3-D vectorfields.

This approach is interesting since up to now, as we explain later in this work, the only case considered has been that in which the generators of the Lie algebra are **X** and a certain number of symmetries or pseudosymmetries of **X**. Such restriction is dropped in this article.

Let us explain this in more detail.

It is well known[1] that when a vectorfield **X** (v.f. in what follows) admits a symmetry vector, that is, a v.f. **S** satisfying

$$\mathcal{L}_{\mathbf{S}}(\mathbf{X})=0, \tag{1}$$

$\mathcal{L}_{\mathbf{S}}$ standing for the Lie derivative along the streamlines of **S**, useful consequences on the local and global structure of **X** can be obtained: existence of local and global first integrals, limit cycles of **X**,[2] etc.

Remember that (1) implies that the flow of the v.f. **S** acts on the set of solutions of the differential equations

$$\frac{d\mathbf{x}}{dt}=\mathbf{X}(\mathbf{x}). \tag{2}$$

In other words, the local flow of **S** transforms a solution of (2) into another solution of Eq. (2).

Sometimes the pair of v.f. $(\mathbf{X},\mathbf{S})$ does not satisfy Eq. (1) but the equations

$$\mathcal{L}_{\mathbf{S}}(\mathbf{X})=\lambda(\mathbf{x})\mathbf{X}, \tag{3}$$

$\lambda(\mathbf{x})$ being a function. In this case **S** is called a pseudosymmetry of **X**. The geometrical meaning of Eq. (3) is that the local flow of **S** conserves *not* the solutions of (2) but the trajectories on which these solutions lie (a trajectory of **X** is just an unparametrized solution of **X**).

Interesting geometric information on the trajectories of **X** when (3) holds can be found in Ref. 2.

Motivated by Eqs. (1) and (3) we consider in this article that **X** (a $\mathbb{R}^3$ v.f. from now on) is one of the generators of a Lie algebra $A_{2,2}$ of dimension two or $A_{3,3}$ of dimension three. That is,

$$[\mathbf{X},\mathbf{S}_1]=a_0\mathbf{X}+a_1\mathbf{S}_1,$$

$$a_0, a_1 \in \mathbb{R}, \tag{4}$$

$$\mathrm{rank}(\mathbf{X},\mathbf{S}_1)=2 \quad \text{for any } \mathbf{x} \in \mathbb{R}^3,$$

in the first case, and

$$\begin{gathered}
[\mathbf{X},\mathbf{S}_1]=a_0\mathbf{X}+a_1\mathbf{S}_1+a_2\mathbf{S}_2,\\
[\mathbf{X},\mathbf{S}_2]=b_0\mathbf{X}+b_1\mathbf{S}_1+b_2\mathbf{S}_2,\\
[\mathbf{S}_1,\mathbf{S}_2]=c_0\mathbf{X}+c_1\mathbf{S}_1+c_2\mathbf{S}_2,\\
a_i, b_i, c_i \in \mathbb{R},\\
\mathrm{rank}(\mathbf{X},\mathbf{S}_1,\mathbf{S}_2)=3 \quad \text{for any } \mathbf{x} \in \mathbb{R}^3,
\end{gathered} \tag{5}$$

in the case of an algebra of type $A_{3,3}$.

Note that $[,]$ stands for the Lie bracket of v.f. and $A_{i,j}$ $(i \geqslant j)$ stands for a Lie algebra with i generators (including **X**) and rank $(\mathbf{X},\mathbf{S}_1,..,\mathbf{S}_{i-1})=j$.

We shall say that **X** belongs to a certain Lie algebra if **X** is one of its generators. For example, **X** belongs to the Lie algebras $A_{2,2}$ and $A_{3,3}$ defined by Eqs. (4) and (5).

Note that the case of pseudosymmetries corresponds to $a_1=0$ in Eq. (4) and $a_1=a_2=b_1=b_2=0$ in Eq. (5).

We shall prove in what follows that when a dynamical system **X** belongs to a Lie algebra this information can be useful in order to get qualitative information on the orbits of **X**.

This article is organized this way. Lie algebras of type $A_{2,2}$ are briefly considered in Sec. II, where their influence on **X** is studied. The structure constants of $A_{3,3}$ algebras are reduced to a finite number of canonical forms in Sec. III. The case of a v.f. **X** embedded into an $A_{3,3}$ Lie algebra is studied in Sec. IV. Illustrative examples are given in Sec. V, and some open problems are discussed in Sec. VI.

We end this section by motivating our study with some considerations of the significance and applicability of the idea of embedding a v.f. **X** into a Lie algebra.

We shall refer to the illustrative example of $A_{2,2}$ algebras [that is, algebras with two generators and rank equal 2: see Eq. (4)]. For these algebras Eq. (4) can be interpreted in two ways:

(i) as the structure equation of a Lie transformation (local) group G acting on $\mathbb{R}^3$ of generators **X** and **S**, or
(ii) as the equations defining an involutive distribution[3,4] generated by **X** and **S**.

The fact that a_0 and a_1 in Eq. (4) are real numbers instead of functions of $\mathbf{x}=(x_1,x_2,x_3)$ is a useful piece of information that should be taken into account.
Therefore the philosophy of this article is the following:

(i) get **X** (if you can, via computer packages, etc.) be embedded into the algebras $A_{r,3}(r \geqslant 3)$ or $A_{r,2}(r \geqslant 2)$ of some Lie transformation group G. We shall speak immediately about the difficulties of this process.
(ii) apply the techniques of this article in order to get information on some structures of **X**, as first integrals, invariant sets, existence of partitions of $\mathbb{R}^3$ invariant under **X**, integrability via quadratures, etc.

The most difficult point is, of course, the finding of the concrete embedding of **X**. In fact it may even happen that (for structural reasons connected with the orbit structure of **X**, strange or complicated limit behavior of the orbits when $t \to +\infty$) the embedding process will be a failure because it does not exist at all. For example, by topological reasons it is *impossible* to get an embedding of **X** into an algebra of type $A_{3,2}$ or $A_{2,2}$ if **X** is a dynamical system with an orbit which is an asymptotic ‘‘limit cycle’’ (orbit of type S^1 acting as limit set of neighboring orbits). Nevertheless, the dynamical system **X** could be embedded into an algebra of type $A_{3,3}$.

However, we have *not* been able to find analytical conditions, geometric structures, etc. such that if **X** satisfies them, then **X** cannot be embedded into an algebra of type $A_{3,3}$. Upto today open problems are to decide

(i) whether or not a given v.f. **X** can be embedded into a finite dimensional Lie algebra, and
(ii) whether or not a given v.f. **X** can be embedded into an algebra of type $A_{n,3}(n \geqslant 3)$, where n is a fixed natural number.

In general, the problem of studying the relation between the geometry of the orbits of **X** and the type of algebra into which **X** can or cannot be embedded seems to be a very difficult one.

In conclussion, this article could be of interest to people working in differential equations, dynamical systems, etc., and to all those normally handling symmetry techniques in differential equations since we offer here a certain generalization of them yielding, under some conditions, first integrals, invariant sets, integrability via quadratures, foliations of $\mathbb{R}^3$ invariant under **X**, etc.

II. $\mathbb{R}^3$ DYNAMICAL SYSTEMS EMBEDDED INTO A LIE ALGEBRA $A_{2,2}$

Let us now develop some consequences of the fact that our dynamical system **X** is embedded into a Lie algebra of type $A_{2,2}$, that is,

$$
\begin{gathered}
[\mathbf{X},\mathbf{S}_1]=a_0\mathbf{X}+a_1\mathbf{S}_1,\\
a_0,a_1\in\mathbb{R},\\
\operatorname{rank}(\mathbf{X},\mathbf{S}_1)=2.
\end{gathered}
\tag{6}
$$

We shall now obtain from Eq. (6) consequences of several kinds concerning the orbit structure of **X**. Most of these results fail when the real constants a_0 and a_1 of (6) are substituted by real functions $a(\mathbf{x})$ and $b(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^3$. Therefore, most of these results cannot be obtained when **X** is embedded into a two-dimensional foliation instead of being embedded into a $A_{2,2}$ algebra.

From now on all the functions v.f.'s, and differential forms of this article are assumed to be analytic (C^w). See Refs. 3–5 for the theory and applications of differential forms.

A. First integrals of X

We obtain now first integrals of **X** via the construction of exact one-forms. The reader can have a look at this method when $a_0=a_1=0$ in Ref. 3.

Our assumptions are the following:
X belongs to a $A_{2,2}$ Lie algebra [see Eq. (6)] and

$$
\operatorname{Div}\mathbf{X}=-a_1,\ \operatorname{Div}\mathbf{S}_1=a_0,
\tag{7}
$$

a_0 and a_1 being the real numbers of Eq. (6) and Div **Y** standing for

$$
\begin{gathered}
\operatorname{Div}\mathbf{Y}=\frac{\partial Y_1}{\partial x_1}+\frac{\partial Y_2}{\partial x_2}+\frac{\partial Y_3}{\partial x_3},\\
\mathbf{Y}=Y_1\partial_1+Y_2\partial_2+Y_3\partial_3.
\end{gathered}
\tag{8}
$$

Div **Y** can be alternatively defined by $\mathcal{L}_Y\Omega_3=\operatorname{Div}\mathbf{Y}\cdot\Omega_3$, Ω_3 being the standard volume form $dx_1\wedge dx_2\wedge dx_3$ of $\mathbb{R}^3$.

Under these hypotheses the one-form w_1 defined by

$$
w_1=i_{\mathbf{x}}i_{\mathbf{s}_1}\Omega_3
\tag{9}
$$

is exact $(dw_1=0)$ and we can write

$$w_1 = dI, \tag{10}$$

and since $i_x w_1 = 0$ we can write

$$\mathcal{L}_{\mathbf{x}}(I) = 0. \tag{11}$$

Therefore I is a global first integral of $\mathbf{X}$.

Note that I can never become a trivial constant, as this would imply $w_1 = 0$ (identically), getting a contradiction with the rank condition appearing in Eq. (6).

B. Independent first integrals

Let us now assume that I_1, I_2 are two independent first integrals of $\mathbf{S_1}$; this situation often appears in physics[6] as $\mathbf{S_1}$ usually is a v.f. easier to handle than $\mathbf{X}$ (isometries of $\mathbb{R}^3$ considered as Euclidean space, linear or affine v.f. and so on). Under this assumption let us see that the integration of $\mathbf{X}$ can be simplified.

Under these conditions Eq. (6) implies

$$-\mathcal{L}_{\mathbf{S_1}}\mathcal{L}_{\mathbf{X}}(I_i) = a_0 \mathcal{L}_{\mathbf{X}}(I_i), \quad i = 1,2, \tag{12}$$

and when $a_0 = 0$ we get

$$\mathcal{L}_{\mathbf{X}}(I_i) = \varphi_i(I_1, I_2), \tag{13}$$

that is, $\mathbf{X}$ projects to the $\mathbb{R}^3$ v.f.

$$\mathbf{X}_2 = \varphi_1(I_1, I_2)\partial_{I_1} + \varphi_2(I_1, I_2)\partial_{I_2}, \tag{14}$$

that is

$$\begin{aligned} \frac{dI_1}{dt} &= \varphi_1(I_1, I_2), \\ \frac{dI_2}{dt} &= \varphi_2(I_1, I_2). \end{aligned} \tag{15}$$

Therefore, the integration of $\mathbf{X}$ has been simplified.

We now summarize the results of this section: We have seen that it is, in general, impossible to get geometric information on the trajectories of the $\mathbb{R}^3$ v.f. $\mathbf{X}$ just by knowing that $\mathbf{X}$ belongs to a certain Lie algebra of v.f. More information concerning the v.f. of the Lie algebra is needed: see, for example, the requirements in (7).

A similar observation can be made in relation to the study of the pseudosymmetries of $\mathbf{X}$ [see Eq. (3)]. Namely, pseudosymmetries, *per se*, are insufficient in order to get first integrals and other geometric structures related to the trajectories of $\mathbf{X}$.

What is new in this section is the fact that we have shown the possibility of getting global geometric information on the trajectories of $\mathbf{X}$ when *no* pseudosymmetries are known but we have discovered that our dynamical system $\mathbf{X}$ is a generator of an $A_{2,2}$ algebra of vectorfields.

For brevity reasons we shall not study in the following sections algebras of type $A_{3,2}$, but just algebras of type $A_{3,3}$.

III. CLASSIFICATION OF $A_{3,3}$ ALGEBRAS

A classification list of the $A_{3,3}$ algebras is given now. The proof shall not be given and will be sent on request. As we can see the classification contains 18 different types. Note that the non-written brackets between **X**, $\mathbf{S_1}$ and $\mathbf{S_2}$ vanish and have been omitted. Nevertheless, all brackets have been written in the algebra of type number one (for esthetic reasons).

Any $A_{3,3}$ algebra can be obtained from those appearing in the list by means of linear combinations of type

$$\begin{gathered}\mathbf{X}^* = \alpha_0 \mathbf{X},\\ \mathbf{S}_1^* = \beta_0 \mathbf{X} + \beta_1 \mathbf{S}_1 + \beta_2 \mathbf{S}_2,\\ \mathbf{S}_2^* = \gamma_0 \mathbf{X} + \gamma_1 \mathbf{S}_1 + \gamma_2 \mathbf{S}_2,\\ \alpha_0, \beta_0, \gamma_0 \in \mathbb{R} \quad \alpha_0 \neq 0,\\ \beta_1 \gamma_2 - \gamma_1 \beta_2 \neq 0.\end{gathered} \tag{16}$$

These linear combinations arise as the generator **X** (representing the dynamical system) must be isolated in all the algebraic manipulations; otherwise a generator $\mathbf{X}^*$ could be obtained mixing the dynamics of **X** with the dynamics of the v.f. $\mathbf{S_1}$ and $\mathbf{S_2}$. Therefore, the orbit structure of **X** would be unrelated to the orbit structure of $\mathbf{X}^*$.

The 18 types of $A_{3,3}$ algebras are

(1) $[\mathbf{X},\mathbf{S_i}]=0$, $[\mathbf{S_1},\mathbf{S_2}]=0$, $i=1,2$;
(2) $[\mathbf{X},\mathbf{S_1}]=\mathbf{X}$;
(3) $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}$;
(4) $[\mathbf{X},\mathbf{S_1}]=\mathbf{S_1}$;
(5) $[\mathbf{X},\mathbf{S_2}]=\mathbf{S_1}$, $[\mathbf{S_1},\mathbf{S_2}]=\alpha\mathbf{S_1}$, $\alpha\in\mathbb{R}$;
(6) $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{S_1}$;
(7) $[\mathbf{X},\mathbf{S_2}]=\mathbf{X}$, $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}+\alpha\mathbf{S_1}$, $\alpha\in\mathbb{R}\backslash\{0\}$;
(8) $[\mathbf{X},\mathbf{S_2}]=\mathbf{X}+\mathbf{S_1}$, $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}$;
(9) $[\mathbf{X},\mathbf{S_2}]=\mathbf{S_1}$, $[\mathbf{S_1},\mathbf{S_2}]=-\mathbf{X}$;
(10) $[\mathbf{X},\mathbf{S_2}]=\mathbf{S_1}$, $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}$;
(11) $[\mathbf{X},\mathbf{S_1}]=\mathbf{S_1}$, $[\mathbf{X},\mathbf{S_2}]=\alpha\mathbf{S_2}$, $\alpha\in\mathbb{R}\backslash\{0\}$;
(12) $[\mathbf{X},\mathbf{S_1}]=\mathbf{S_1}$, $[\mathbf{X},\mathbf{S_2}]=\mathbf{S_1}+\mathbf{S_2}$;
(13) $[\mathbf{X},\mathbf{S_1}]=\alpha\mathbf{S_1}+\mathbf{S_2}$, $[\mathbf{X},\mathbf{S_2}]=-\mathbf{S_1}+\alpha\mathbf{S_2}$, $\alpha\in\mathbb{R}\backslash\{0\}$;
(14) $[\mathbf{X},\mathbf{S_1}]=\mathbf{X}$, $[\mathbf{X},\mathbf{S_2}]=\mathbf{S_1}$, $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}+\mathbf{S_2}$;
(15) $[\mathbf{X},\mathbf{S_1}]=\mathbf{S_2}$, $[\mathbf{X},\mathbf{S_2}]=-\mathbf{S_1}$, $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}$;
(16) $[\mathbf{X},\mathbf{S_1}]=\mathbf{S_2}$, $[\mathbf{X},\mathbf{S_2}]=\mathbf{S_1}$, $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}$;
(17) $[\mathbf{X},\mathbf{S_1}]=-\mathbf{S_1}-\mathbf{S_2}$, $[\mathbf{X},\mathbf{S_2}]=\mathbf{S_2}$, $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}$; and
(18) $[\mathbf{X},\mathbf{S_1}]=-\mathbf{S_1}+\mathbf{S_2}$, $[\mathbf{X},\mathbf{S_2}]=\mathbf{S_2}$, $[\mathbf{S_1},\mathbf{S_2}]=\mathbf{X}$.

IV. INVARIANT SETS AND FIRST INTEGRALS WHEN THE DYNAMICAL SYSTEM IS EMBEDDED INTO AN $A_{3,3}$ ALGEBRA

We now show that it is possible to get first integrals, invariant sets and foliations invariant under **X** when **X** belongs to an $A_{3,3}$ algebra. Reduction of **X** to a two-dimensional v.f. is also possible (see Sec. IV C).

A. Global results

We get in this paragraph global results on **X** assuming that

$$\mathcal{L}_{\mathbf{X}} w_i = f(\mathbf{x}) w_i, \tag{17}$$

w_i being a C^∞ differential form of degree i $(i=1,2,3)$.

Define the functions Δ_i via

$$\Delta_l = i_{\mathbf{X}} i_{\mathbf{S_1}} i_{\mathbf{S_2}} w_3, \tag{18}$$

$$\Delta_2 = i_{\mathbf{X}} i_{\mathbf{S}_j} w_2, \quad j=1,2, \tag{19}$$

$$\Delta_3 = i_{\mathbf{S_1}} i_{\mathbf{S_2}} w_2, \tag{20}$$

$$\Delta_4 = i_{\mathbf{X}} w_1, \tag{21}$$

$$\Delta_5 = i_{\mathbf{S}_j} w_1, \quad j=1,2. \tag{22}$$

We then get under standard manipulations[4]

$$\mathcal{L}_{\mathbf{X}}(\Delta_i) = (f(\mathbf{x}) + K)\Delta_i, \quad K \in \mathbb{R}, \tag{23}$$

where the real number K depends on the constants a_i, b_i, c_i $(i=0,1,2)$ defining the $A_{3,3}$ algebra [see Eq. (5)].

Now, Eq. (23) implies the following.

(i) When the set $\{\Delta_i = 0\}$ is a differential manifold ($\nabla(\Delta_i) \neq \mathbf{0}$ for any $P \in \{\Delta_i = 0\}$), then the set $\{\Delta_i = 0\}$ is invariant under $\mathbf{X}$. See Example 1 in Sec. V.

(ii) When $f+K$ is a function of Δ_i (in particular when $f+K$ is a constant real number), then the sets $\{\Delta_i = \text{const}\}$ form a two-foliation invariant under $\mathbf{X}$.

(iii) When $f(\mathbf{x})$ is a trivial constant function and $f+K$ is equal to zero, then the function Δ_i is a global first integral of $\mathbf{X}$.

These results give useful information on the orbits of $\mathbf{X}$ and they have been obtained without problems in spite of the fact that $\mathbf{S_1}$ and $\mathbf{S_2}$ are, in general, not pseudosymmetries of $\mathbf{X}$.

See the examples on these results at the end of the article.

Note that the techniques of this section can be applied to *any* of the canonical algebras of the list in Sec. III.

B. Subalgebras

We now assume that our $A_{3,3}$ algebra contains two $A_{2,2}$ subalgebras satisfying

$$\begin{gathered}
[\mathbf{X},\mathbf{S_1}] = a\mathbf{X} + b\mathbf{S_1}, \\
[\mathbf{S_2},\mathbf{X}] = a'\mathbf{X} + b'\mathbf{S_1}, \\
[\mathbf{S_2},\mathbf{S_1}] = a''\mathbf{X} + b''\mathbf{S_1}, \\
a,a',a'',b,b',b'' \in \mathbb{R},
\end{gathered} \tag{24}$$

or

$$\begin{gathered}
[\mathbf{S}_1^*,\mathbf{S}_2^*] = c\mathbf{S}_1^* + d\mathbf{S}_2^*, \\
[\mathbf{X},\mathbf{S}_1^*] = c'\mathbf{S}_1^* + d'\mathbf{S}_2^*, \\
[\mathbf{X},\mathbf{S}_2^*] = c''\mathbf{S}_1^* + d''\mathbf{S}_2^*, \\
c,c',c'',d,d',d'' \in \mathbb{R},
\end{gathered} \tag{25}$$

or both [i.e., $A_{3,3}$ might contain a subalgebra satisfying Eq. (24) and another two-dimensional subalgebra satisfying Eq. (25)]. Note that $\{\mathbf{X},\mathbf{S_1}\}$ in the case of Eq. (24) and $\{\mathbf{S_1^*},\mathbf{S_2^*}\}$ in the case of Eq. (25) are ideals of dimension two of $A_{3,3}$.[7]

First of all, notice that we can apply the techniques of Sec. II to the pair $(\mathbf{X},\mathbf{S_1})$ of Eq. (24).

Note that Eqs. (24) are fulfilled by the algebras 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 13 and Eqs. (25) are satisfied by the algebras 1, 4, 5, 6, 11, and 12.

On the other hand, algebras 13–18 satisfy neither Eqs. (24) nor Eqs. (25). The reader will have no difficulty in checking all these points.

We give now the geometric meaning of Eqs. (24) and (25). Calling $\mathcal{F}_2$ and $\mathcal{F}_2^*$ the two-foliations associated with the pairs $(\mathbf{X},\mathbf{S_1})$ and $(\mathbf{S_1^*},\mathbf{S_2^*})$, Eqs. (24) and (25) can be rewritten in the form

$$\mathcal{L}_{\mathbf{S_2}}(\mathcal{F}_2)\subset\mathcal{F}_2, \tag{26}$$

and

$$\mathcal{L}_{\mathbf{X}}(\mathcal{F}_2^*)\subset\mathcal{F}_2^*. \tag{27}$$

Accordingly, $\mathcal{F}_2$ and $\mathcal{F}_2^*$ can be locally integrated via the well known formulas[8]

$$\begin{gathered}\Delta^{-1}\cdot(i_{\mathbf{X}}i_{\mathbf{S_1}}\Omega_3)=dI,\\ \Omega_3=dx_1\wedge dx_2\wedge dx_3,\\ \Delta=i_{\mathbf{X}}i_{\mathbf{S_1}}i_{\mathbf{S_2}}\Omega_3,\end{gathered} \tag{28}$$

and

$$\Delta^{-1}\cdot(i_{\mathbf{S_1^*}}i_{\mathbf{S_2^*}}\Omega_3)=dI^*, \tag{29}$$

I and I^* satisfying

$$\mathcal{L}_{\mathbf{X}}(I)=0 \tag{30}$$

and

$$\mathcal{L}_{\mathbf{X}}(I^*)=f(I^*) \tag{31}$$

for a certain function f.

The function I is, of course, a local integral of $\mathbf{X}$ and it globalizes to a $\mathbb{R}^3$ first integral of $\mathbf{X}$ when the function Δ of Eq. (28) never vanishes.

On the other hand, the geometrical meaning of Eq. (31) is that the local flow of $\mathbf{X}$ acts on the level sets of I^*. When the function f of (31) never vanishes, $\mathbf{X}$ is free from closed trajectories. If $f(I_0^*)=0$, then closed trajectories of $\mathbf{X}$ might appear on the level set $I^*=I_0^*$.

Note that I and I^* are genuine functions, not reducing to constant functions, since in an $A_{3.3}$ algebra the ranks of the pairs $(\mathbf{X},\mathbf{S})$ and $(\mathbf{S_1^*},\mathbf{S_2^*})$ cannot be lower than 2.

C. Results

We now get several results on the orbits of the dynamical system $\mathbf{X}$ assuming that a pair of first integrals common to $\mathbf{S_1}$ and $\mathbf{S_2}$ are known. For brevity's sake, the case of only a first integral I common to $\mathbf{S_1}$ and $\mathbf{S_2}$ shall not be studied.

See Ref. 6 for a similar use of a pair of first integrals of a symmetry of a $\mathbb{R}^3$ dynamical system related to the Bessel, Poisson–Boltzmann, Emden–Fowler and Fermi–Thomas equations. This approach can be justified since in most of the applications the v.f. $\mathbf{S_i}$ are simple v.f.; often they are affine, or even linear v.f., and therefore the finding of their first integrals is, in general, not difficult.

Consider that

$$\mathcal{L}_{\mathbf{S_i}}(I_j)=0, \quad i,j=1,2 \tag{32}$$

that is I_1,I_2 are independent first integrals common to $\mathbf{S_1,S_2}$. We then get via Eq. (5) $(c_0\neq 0)$

$$\mathcal{L}_{\mathbf{X}}(I_i)=\varphi_i(I_1,I_2) \quad i=1,2. \tag{33}$$

Therefore $\mathbf{X}$ can be written in the form

$$\mathbf{X}=\varphi_1(x,y)\partial_1+\varphi_2(x,y)\partial_2. \tag{34}$$

Accordingly, $\mathbf{X}$ has been reduced to a $\mathbb{R}^2$ v.f.

V. EXAMPLES

Examples 1: Consider the conformal v.f.[4]

$$\mathbf{X}=(x_1^2-x_2^2-x_3^2)\partial_1+(2x_1x_2)\partial_2+(2x_1x_3)\partial_3 \tag{35}$$

and the v.f.

$$\begin{aligned}\mathbf{S}_1&=x_3\partial_2-x_2\partial_3,\\ \mathbf{S}_2&=x_1\partial_1+x_2\partial_2+x_3\partial_3,\end{aligned} \tag{36}$$

with commutation relations

$$[\mathbf{X},\mathbf{S_1}]=\mathbf{0}, \quad [\mathbf{X},\mathbf{S_2}]=-\mathbf{X}, \quad [\mathbf{S_1},\mathbf{S_2}]=\mathbf{0}. \tag{37}$$

By application of the results obtained in Secs. IV A and IV B we get

$$\Delta_1=i_{\mathbf{X}}i_{\mathbf{S_1}}i_{\mathbf{S_2}}(dx_1\wedge dx_2\wedge dx_3)=(x_2^2+x_3^2)(-x_1^2-x_2^2-x_3^2). \tag{38}$$

On the other hand,

$$\mathcal{L}_{\mathbf{X}}(\Delta_1)=6x_1\cdot\Delta_1. \tag{39}$$

Therefore, the set $\Delta_1=0$ is invariant under $\mathbf{X}$. Note that the set $\Delta_1=0$ is just the x_1-axis.

Let us now get a local first integral of $\mathbf{X}$ by application of the methods of Sec. IV B. In fact, computing $i_{\mathbf{X}}i_{\mathbf{S_1}}(dx_1\wedge dx_2\wedge dx_3)/\Delta_1$ we get the differential form

$$\frac{w_1}{\Delta_1}=\frac{2x_1dx_1}{-x_1^2-x_2^2-x_3^2}+\frac{(-x_2dx_2-x_3dx_3)(x_1^2-x_2^2-x_3^2)}{(x_2^2+x_3^2)(-x_1^2-x_2^2-x_3^2)}, \tag{40}$$

which is locally exact $(w_1/\Delta_1=dI)$. Upon integration we get the local first integral I that can be reduced to

$$I'=\frac{x_2^2+x_3^2}{x_1^2+x_2^2+x_3^2}. \tag{41}$$

Example 2: Consider now the family of v.f.

$$\mathbf{X}=F(x_3)(x_1^2+x_2^2)^n x_1\partial_1+F(x_3)(x_1^2+x_2^2)^n x_2\partial_2+G(x_3)(x_1^2+x_2^2)^n\partial_3,\quad n=1,2,3,..., \tag{42}$$

where F and G are analytic and G vanishes on the set $Z(Z\subset\mathbb{R})$.

Let $\mathbf{S_i}$ $(i=1,2)$ be the v.f.

$$\begin{aligned}\mathbf{S_1}&=x_1\partial_1+x_2\partial_2,\\ \mathbf{S_2}&=x_2\partial_1-x_1\partial_2.\end{aligned} \tag{43}$$

The three v.f. $\mathbf{X},\mathbf{S_1},\mathbf{S_2}$ form a commutative algebra. By applying to them the techniques of Secs. IV A and IV B we get the invariant set

$$\Delta_1=G(x_3)(x_1^2+x_2^2)^{n+1}=0, \tag{44}$$

that is, the invariant sets

$$\begin{aligned}&x_1^2+x_2^2=0,\\ &x_3=z,\quad z\in Z.\end{aligned} \tag{45}$$

On the other hand, we can also write

$$\frac{w_1}{\Delta_1}=dI, \tag{46}$$

w_1 standing for the one-form

$$w_1=i_{\mathbf{x}}i_{\mathbf{s_2}}(dx_1\wedge dx_2\wedge dx_3). \tag{47}$$

We get in this way

$$I=\frac{1}{2}L(x_1^2+x_2^2)-\int\frac{F(x_3)}{G(x_3)}dx_3, \tag{48}$$

L standing for Neperian logarithm, that is, a local first integral of $\mathbf{X}$.

Example 3: We now give an example related to Sec. II B.

Let $H_i(x_1,x_2,x_3)$ be homogeneous polynomials of degrees d_1 and d_2. Define $\mathbf{X}$ and $\mathbf{S}$ viahe equations

$$\begin{aligned}\mathbf{X}&=\nabla H_1\wedge\nabla H_2+a_0(x_1\partial_1+x_2\partial_2+x_3\partial_3),\\ \mathbf{S}&=\nabla H_1\wedge\nabla H_2,\end{aligned} \tag{49}$$

$$a_0\in\mathbb{R},\quad \nabla=\text{gradient operator}.$$

The reader will check that

$$[\mathbf{X},\mathbf{S}]=b\mathbf{S},\quad b\in\mathbb{R}. \tag{50}$$

Therefore the pair $(\mathbf{X},\mathbf{S})$ forms an $A_{2,2}$ algebra.

Since H_1 and H_2 are first integrals of $\mathbf{S}$, we get from (50)

$$\begin{aligned}\mathcal{L}_{\mathbf{x}}(H_1)&=\varphi_1(H_1,H_2),\\ \mathcal{L}_{\mathbf{x}}(H_2)&=\varphi_2(H_1,H_2),\end{aligned} \tag{51}$$

that is, $\mathbf{X}$ projects to the $\mathbb{R}^2$ v.f.

$$\varphi_1 \partial_{H_1} + \varphi_2 \partial_{H_2}. \tag{52}$$

Note that the v.f. $\mathbf{X}$ of (49) is not trivial, as it is *not* a homogeneous v.f.

Note also that any first integral $I(H_1, H_2)$ of the reduced differential equations (51) is a first integral of $\mathbf{X}$.

Example 4: The considerations of Example 3 can be extended to nonhomogeneous functions in this way.

Let H_1 and H_2 be nonhomogeneous polynomials that can be transformed into homogeneous ones via a transformation of type

$$\begin{gathered} x_1 \to x_1^a, \\ x_2 \to x_2^b, \\ x_3 \to x_3^c, \\ a,b,c \in \mathbb{R}^+. \end{gathered} \tag{53}$$

For example, the pairs

$$H_1 = x_2 x_3, \quad H_2 = x_1^2 + x_2^2 + x_3$$

and

$$H_1 = x_2^2 + x_3^2, \quad H_2 = x_1^2 - x_3$$

become homogeneous under the transformations

$$x_1 \to x_1, \quad x_2 \to x_2, \quad x_3 \to {}_3^2$$

and

$$x_1 \to x_1 \quad x_2 \to {}_2^2, \quad x_3 \to x_3^2.$$

Under these circumstances the v.f. defined by

$$\begin{gathered} \mathbf{X} = \nabla H_1 \wedge \nabla H_2 + a_0 (x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3), \\ \mathbf{S} = \nabla H_1 \wedge \nabla H_2, \end{gathered} \tag{54}$$

commutes as in Eq. (50). Therefore, the conclusions in Example 3 are valid for the v.f. of Eq. (54). For example, the Lorenz dynamical system[9]

$$\mathbf{X}_L = \sigma(x_2 - x_1)\partial_1 + (-x_1 x_3 + r x_1 - x_2)\partial_2 + (x_1 x_2 - b x_3)\partial_3, \quad \sigma, r, b \in \mathbb{R}, \tag{55}$$

for the following particular values of the parameters,

$$\sigma = \tfrac{1}{2}, \quad r = 0, \quad b = 1,$$

forms an $A_{2,2}$ algebra, of the type discussed in this example, with the v.f.

$$\mathbf{S} = \nabla(x_2^2 + x_3^2) \wedge \nabla(x_1^2 - x_3) \tag{56}$$

as the reader can check.

Example 5: We end this section with a list of second order differential equations appearing in Physics (see in Ref. 6) admitting a symmetry vector **S** to which the methods of this article can be applied (see Sec. II B).

(5.1) $x^2 y_{,xx} + x y_{,x} + x^2 y = 0$:

$$\text{Associated } \mathbf{X}\text{:}\ \mathbf{X} = \frac{-xu - x^2 y}{x^2}\partial_u + u\partial_y + \partial_x \quad u = y_{,x}.$$

$$\text{Symmetry vector: } \mathbf{S} = y\partial_y + u\partial_u.$$

$$\text{Commutation relation: } [\mathbf{X},\mathbf{S}] = \mathbf{0}.$$

$$\text{First integrals of } \mathbf{S}\text{: } I_1 = x, \quad I_2 = u/y.$$

(5.2) $y_{,xx} + y_{,x}/x = e^y$.

$$\text{Associated } \mathbf{X}\text{:}\ \mathbf{X} = \left(e^y - \frac{u}{x}\right)\partial_u + u\partial_y + \partial_x, \quad u = y_{,x}.$$

$$\text{Symmetry vector: } \mathbf{X} = x\partial_x - 2\partial_y - u\partial_u.$$

$$\text{Commutation relation: } [\mathbf{X},\mathbf{S}] = \mathbf{X}.$$

$$\text{First integrals of } \mathbf{S}\text{: } I_1 = x^2 e^y, \quad I_2 = xu.$$

(5.3) $y_{,xx} + (2/x) y_{,x} + y^n = 0$.

$$\text{Associated } \mathbf{X}\text{:}\ \mathbf{X} = \left(-y^n - \frac{2u}{x}\right)\partial_u + u\partial_y + \partial_x \quad u = y_{,x}.$$

$$\text{Symmetry vector: } \mathbf{S} = x\partial_x + \frac{2y}{1-n}\partial_y + \frac{1+n}{1-n}u\partial_u.$$

$$\text{Commutation relation: } [\mathbf{X},\mathbf{S}] = \mathbf{X}.$$

$$\text{First integrals of}\mathbf{S}\text{: } I_1 = x^2 y^{n-1}, \quad I_2 = x^{n+1} u^{n-1}.$$

(5.4) $y_{,xx} = x^{-1/2} y^{3/2}$.

$$\text{Associated } \mathbf{X}\text{:}\ \mathbf{X} = (x^{-1/2} y^{3/2})\partial_u + u\partial_y + \partial_x, \quad u = y_{,x}.$$

$$\text{Symmetry vector: } \mathbf{S} = x\partial_x - 3y\partial_y - 4u\partial_u.$$

$$\text{Commutation relation: } [\mathbf{X},\mathbf{S}] = \mathbf{X}.$$

$$\text{First integrals of } \mathbf{S}\text{: } I_1 = x^3 y, \quad I_2 = x^4 u.$$

VI. FINAL REMARKS

We have seen that when a $\mathbb{R}^3$ dynamical system **X** lies inside an $A_{2,2}$, $A_{3,2}$ or $A_{3,3}$ algebra useful information on its trajectories can be obtained from this piece of information.

What happens when **X** can be embedded into a Lie algebra $A_{n,3}$ when $n > 3$? Note that now the canonical forms of Sec. III are harder to obtain. On the other hand, $A_{n,3}$ might contain ideals I containing **X** of lower dimension n', reducing the problem to an algebra $A_{n',3}$ of lower dimension. If no ideal of this type can be found, we can always apply the techniques of Sec. IV A.

Considering only contractions of $\mathbf{X}$ and $\mathbf{S_i}$ with differential forms of type w_3, we can get in this way a whole set of functions Δ_{ij}:

$$\Delta_{ij} = i_X i_{S_i} i_{S_j} w_3, \quad i,j = 1,...,n-1, \tag{57}$$

leading to the sets

$$\Delta_{ij}(x_1, x_2, x_3) = 0 \tag{58}$$

that are invariant under $\mathbf{X}$ [at least near the points P on which (58) defines a differential manifold, that is $\nabla(\Delta_{ij})(P) \neq 0$].

Therefore, when n is high we can get, via Eq. (58), a collection of more and more sets invariant under $\mathbf{X}$.

An open problem is to study if the number N of invariant sets in (58) is bounded or not when n increases and whether or not these invariant sets accumulate (when N is unbounded). Does the topology of the trajectories of $\mathbf{X}$ ''feel'' that $\mathbf{X}$ is included in an $A_{n,3}$ algebra (without proper ideals) when n is large?

Another open problem meriting a separate study is this one: Assume that $\mathbf{X}$ is included among the generators of an $A_{\infty,3}$ algebra where $A_{\infty,3}$ is an infinite Lie algebra, free from finite or infinite proper ideals containing $\mathbf{X}$. Let us call them *simple* ∞-algebras.

Equation (57) can now be written in the form

$$\Delta_{ij} = i_X i_{S_i} i_{S_j} w_3, \tag{59}$$

and, therefore, invariant sets of $\mathbf{X}$ can be obtained this way.

The question arises again of classifying topologically the v.f. $\mathbf{X}$ that can be included in a simple $A_{\infty,3}$ algebra.

A final question is this one: can a dynamical system $\mathbf{X}$ embedded into a Lie algebra $A_{n,2}$, $A_{n,3}$ or $A_{\infty,2}$, $A_{\infty,3}$ possess a strange attractor?[9]

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On the first integrals of Lotka–Volterra systems

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Abstract

A new method for obtaining time independent first integrals of Lotka–Volterra systems is given. By applying this method new integrable cases are found. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Lotka–Voltera; Integrability; Symmetry vectors

1. Introduction

In the last years much effort has been directed to obtaining local and global first integrals of 3D ($\mathbb{R}^3$) dynamical systems [1–7]. Many of these studies are centered around the Lotka–Volterra (L-V) systems [8,9]; that is, the quadratical 3D vectorfields given by:

$$\begin{aligned}\dot{x}_1 &= x_1(a_{11}x_1 + a_{12}x_2 + a_{13}x_3),\\ \dot{x}_2 &= x_2(a_{21}x_1 + a_{22}x_2 + a_{23}x_3),\\ \dot{x}_3 &= x_3(a_{31}x_1 + a_{32}x_2 + a_{33}x_3).\end{aligned} \tag{1}$$

Interaction models of biological species [8,9], certain hydrodynamic equations [10–14], autocatalytic chemical reactions [15–19], ... are based on the L-V systems.

The non-wandering points of these systems have been studied by Chenciner [19], Huang [20] and Hirsch [21].

Nevertheless the dynamics of the L-V systems is far from being understood, although certain results about the organization of the orbits are known for particular values of the parameters a_{ij} appearing in Eq. (1). For instance, Gao and Liu [22] (see also [23], where first integrals of the inhomogeneous L-V equations are computed) have recently obtained new time dependent and time independent first integrals of Eqs. (1) under certain restrictions on the coefficients a_{ij}.

Following their research line we obtain in this paper new time independent first integrals of L-V systems. Our method is based on the computation of generalized symmetry vectors $\boldsymbol{S}$ of the vectorfield (v.f.) $\boldsymbol{X}$ associated with Eq. (1).

A generalized symmetry vector $\boldsymbol{S}$ of $\boldsymbol{X}$ is a v.f. satisfying

$$[\boldsymbol{X},\boldsymbol{S}] = a(x_1,x_2,x_3)\boldsymbol{X} + b(x_1,x_2,x_3)\boldsymbol{S}, \tag{2}$$

Here, [,] stands for the Lie–Jacobi bracket of vectorfields.

Note that Eq. (2) reduces to $[\boldsymbol{X},\boldsymbol{S}] = a\boldsymbol{X}$ for the symmetry v.f. of $\boldsymbol{X}$ [24]. It is for this reason that we call $\boldsymbol{S}$ generalized symmetry vector of $\boldsymbol{X}$.

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We shall see immediately that, under some additional conditions, local first integrals of $\boldsymbol{X}$ can be obtained when a v.f. $\boldsymbol{S}$ satisfying (2) has been computed. Vectorfields of this kind have recently been considered [25–29], but using techniques different than ours.

The paper is organized as follows: the theoretical basis of our method appears in Section 2. The applications of this method to L-V systems are given in Section 3. Some open problems are presented in Section 4.

2. Theory

Let $\boldsymbol{X}$ be a 3D v.f. It is indifferent at this point whether or not $\boldsymbol{X}$ is a L-V v.f. Let $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ be 3D v.f. satisfying

$$[\boldsymbol{X},\boldsymbol{S_1}] = a_1(\boldsymbol{x})\boldsymbol{X} + b_1(\boldsymbol{x})\boldsymbol{S_1}, \tag{3a}$$

$$[\boldsymbol{S_2},\boldsymbol{S_1}] = a_2(\boldsymbol{x})\boldsymbol{X} + b_2(\boldsymbol{x})\boldsymbol{S_1}, \tag{3b}$$

$$[\boldsymbol{S_2},\boldsymbol{X}] = a_3(\boldsymbol{x})\boldsymbol{X} + b_3(\boldsymbol{x})\boldsymbol{S_1}, \tag{3c}$$

$$\Delta = \mathrm{Det}(\boldsymbol{X},\boldsymbol{S_1},\boldsymbol{S_2}) \neq 0,$$

$$\boldsymbol{x} = (x_1,x_2,x_3), \tag{3d}$$

where Δ stands for the determinant whose rows are the components of $\boldsymbol{X},\boldsymbol{S_1},\boldsymbol{S_2}$.

Note that Eq. (3a) is just Eq. (2). Therefore the v.f. $\boldsymbol{S_1}$ is a generalized symmetry of $\boldsymbol{X}$.

We now mention two important cases for which the requirements (3a), (3b), (3c) are satisfied.

1. $\boldsymbol{X},\boldsymbol{S_1},\boldsymbol{S_2}$ are 3D v.f. and $(\boldsymbol{S_1},\boldsymbol{S_2})$ is a pair of commuting symmetries of $\boldsymbol{X}$. Note that in this case we get $b_1 = b_2 = b_3 = a_2 = 0$. And also that $\boldsymbol{X}$ is not necessarily a L-V vectorfield.
2. $\boldsymbol{X}$ is a L-V v.f.; $\boldsymbol{S_1}$ is a generalized linear symmetry of $\boldsymbol{X}$; $\boldsymbol{S_2} = x_1\partial_1 + x_2\partial_2 + x_3\partial_3$ (dilatation). Under these assumptions it is immediate to check that requirements (3) are satisfied with $b_2 = b_3 = a_2 = 0$. We shall see in the next section that under certain restrictions on the coefficients a_{ij} of Eq. (1) a linear v.f. $\boldsymbol{S_1}$ satisfying the above requirements can be computed.

We now show that under assumptions (3) a local first integral of $\boldsymbol{X}$ can be obtained.

In fact, let w_1 be the one-form defined by

$$w_1 = \underline{\boldsymbol{X}}\rfloor\underline{\boldsymbol{S_1}}\rfloor w_3, \tag{4}$$

where w_3 stands for $dx_1 \wedge dx_2 \wedge dx_3$.

It is straightforward to check that w_1 is integrable (i.e. $w_1 \wedge dw_1 = 0$). As usual $\rfloor$ is the operator contracting v.f. and differential forms [30], $\underline{d}$, stands for the exterior derivative of forms and $\overline{\wedge}$ for the exterior product of differential forms.

Now, w_1 being integrable admits an integrating factor $\underline{f}$, that is a function $\underline{f}$ such that $d(fw_1) = 0$. There is no way, in general, of finding $\underline{f}$, but we shall show immediately how an integrating factor can be found when Eqs. (3) are fulfilled.

We now indicate the geometrical meaning of Eqs. (3). First of all note that by (3a) the couple of v.f. $(\boldsymbol{X},\boldsymbol{S_1})$ is a 2-distribution $\mathscr{D}_2$ [30], that is a pair of v.f. closed under the Lie–Jacobi bracket. On the other hand (3b) and (3c) can be written in the compact form:

$$\mathscr{L}_{\boldsymbol{S_2}}\mathscr{D}_2 \subset \mathscr{D}_2, \tag{5}$$

where $\mathscr{L}$ stands for the Lie derivative operator.

Eq. (5) just means that $\boldsymbol{S_2}$ is a symmetry of $\mathscr{D}_2$ (the flow of $\boldsymbol{S_2}$ induces a reshufling of the leaves, or level sets, of $\mathscr{D}_2$). This symmetry is external to $\mathscr{D}_2$ by Eq. (3d). Note that under an internal symmetry of $\mathscr{D}_2$ *each* of the level sets of $\mathscr{D}_2$ is invariant under the symmetry.

We now give the analytical expression of the integrating factor $\underline{f}$ of w_1.

$\underline{f}$ is given by

$$f = \Delta^{-1} = \left(\underline{\boldsymbol{X}}\rfloor\underline{\boldsymbol{S_1}}\rfloor\underline{\boldsymbol{S_2}}\rfloor w_3\right)^{-1}. \tag{6}$$

In fact, it is not too difficult to check that

$$d\left(\frac{\underline{\boldsymbol{X}}\rfloor\underline{\boldsymbol{S_1}}\rfloor w_3}{\Delta}\right) = 0. \tag{7}$$

One has to remember and apply repeatedly the identities:

$$\underline{\boldsymbol{Y}}\rfloor\underline{\boldsymbol{Y}}\rfloor = 0,$$

$$d\underline{\boldsymbol{Y}}\rfloor + \underline{\boldsymbol{Y}}\rfloor d = \mathscr{L}_{\boldsymbol{Y}},$$

$$[\mathscr{L}_{\boldsymbol{Y}},\underline{\boldsymbol{Z}}\rfloor] = \mathscr{L}_{\boldsymbol{Y}}\underline{\boldsymbol{Z}}\rfloor - \underline{\boldsymbol{Z}}\rfloor\mathscr{L}_{\boldsymbol{Y}},$$

$$\mathscr{L}_{[\boldsymbol{Y},\boldsymbol{Z}]} = \mathscr{L}_{\boldsymbol{Y}}\mathscr{L}_{\boldsymbol{Z}} - \mathscr{L}_{\boldsymbol{Z}}\mathscr{L}_{\boldsymbol{Y}},$$

Note that $\underline{f}$ is not defined where Δ vanishes.

Once $\underline{f}$ is known we can state

$$fw_1 = dI, \tag{8}$$

where I stands for a local function of x_1, x_2, x_3 that can be obtained from (8) via quadratures.

The function I defined in (8) is a local first integral of $\boldsymbol{X}$ since

$$\underline{\boldsymbol{X}}|(fw_1) = \underline{\boldsymbol{X}}|(dI) = 0. \tag{9}$$

Sometimes I, or certain functions of I, can globalize. This is the case when the function Δ (see Eqs. (3)) never vanishes. These matters will become clearer in the next section.

3. Applications

In this section $\boldsymbol{X}$ stands for a L-V vectorfield, $\boldsymbol{S}_1$ for a linear v.f., that is

$$\begin{aligned}\boldsymbol{S}_1 = &(b_{11}x_1 + b_{12}x_2 + b_{13}x_3)\partial_1 \\ &+ (b_{21}x_1 + b_{22}x_2 + b_{23}x_3)\partial_2 \\ &+ (b_{31}x_1 + b_{32}x_2 + b_{33}x_3)\partial_3,\end{aligned} \tag{10}$$

and $\boldsymbol{S}_2$ for the dilatation v.f. $x_1\partial_1 + x_2\partial_2 + x_3\partial_3$.

It is clear that under these assumptions we get

$$[\boldsymbol{S}_2, \boldsymbol{S}_1] = 0, \quad [\boldsymbol{S}_2, \boldsymbol{X}] = \boldsymbol{X}. \tag{11}$$

On the other hand we are looking for generalized symmetry v.f. of type (10) satisfying

$$\begin{aligned}&[\boldsymbol{X}, \boldsymbol{S}_1] - a_1\boldsymbol{X} + (B_1x_1 + B_2x_2 + B_3x_3)\boldsymbol{S}_1, \\ &a_1, B_1, B_2, B_3 \in \mathbb{R}.\end{aligned} \tag{12}$$

Note that (12) is just a particular case of Eq. (3a).

We shall immediately see that even if assumptions (10) and (12) are quite restrictive we will be able to obtain many solutions for $\boldsymbol{S}_1$ leading to new first integrals of $\boldsymbol{X}$. In fact, using the computer packages MAPLE V and MATHEMATICA we have obtained the solutions for $\boldsymbol{S}_1$ listed in Table 1.

To any of the nine $\boldsymbol{S}_1$ v.f. appearing in this Table corresponds a first integral I obtained by integration of the equation

$$\frac{\underline{\boldsymbol{X}}|\underline{\boldsymbol{S}_1}|w_3}{\underline{\boldsymbol{X}}|\underline{\boldsymbol{S}_1}|\underline{\boldsymbol{S}_2}|w_3} = dI. \tag{13}$$

The generalized symmetry vectors $\boldsymbol{S}_1$ of the Table are valid under the restrictions on a_{ij} listed in the second column of the Table.

Table 1
Table of $\boldsymbol{S}_1$ leading to new integrable cases

Different cases	$\boldsymbol{S}_1$	Restrictions on a_{ij}
Case number 1	$x_1\partial_1 + x_2\partial_2 + \frac{a_{33}}{a_{13}}x_3\partial_3$	$a_{13} = a_{23} \neq 0$
Case number 2	$x_1\partial_1 + \frac{a_{22}}{a_{32}}x_2\partial_2 + x_3\partial_3$	$a_{12} = a_{32} \neq 0$
Case number 3	$\frac{a_{11}}{a_{21}}x_1\partial_1 + x_2\partial_2 + \frac{a_{31}}{a_{21}}x_3\partial_3$	$\frac{a_{11}}{a_{13}} = \frac{a_{21}}{a_{21}} = \frac{a_{31}}{a_{33}} \neq 0$
Case number 4	$\frac{a_{11}}{a_{31}}x_1\partial_1 + \frac{a_{21}}{a_{31}}x_2\partial_2 + x_3\partial_3$	$\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} = \frac{a_{31}}{a_{32}} \neq 0$
Case number 5	$x_1\partial_1 + \frac{a_{22}}{a_{12}}x_2\partial_2 + \frac{a_{32}}{a_{12}}x_3\partial_3$	$\frac{a_{12}}{a_{13}} = \frac{a_{22}}{a_{23}} = \frac{a_{32}}{a_{33}} \neq 0$
Case number 6	$x_1\partial_1 + x_2\partial_2 + \left[\frac{a_{31} - a_{11}}{a_{13}}x_1 + \frac{a_{33}}{a_{13}}x_3\right]\partial_3$	$a_{12} = a_{32}, \quad a_{13} = a_{23} \neq 0$
Case number 7	$\frac{a_{31}}{a_{21}}x_1\partial_1 + x_2\partial_2 + \left[\frac{a_{32} - a_{22}}{a_{23}}x_2 + \frac{a_{31}}{a_{21}}x_3\right]\partial_3$	$a_{11} = a_{31}, \quad a_{13} = a_{33}, \quad \frac{a_{21}}{a_{31}} = \frac{a_{23}}{a_{13}} \neq 0$
Case number 8	$x_1\partial_1 + x_2\partial_2$	$a_{13} = a_{23}, \quad a_{33} = 0$
Case number 9	$\frac{a_{13}}{a_{23}}x_1\partial_1 + x_2\partial_2 + \left[\frac{a_{31}}{a_{23}}x_1 + \frac{a_{33}}{a_{23}}x_3\right]\partial_3$	$a_{12} = a_{32}, \quad a_{11} = 0, \quad a_{21} = 0, \quad a_{23} \neq 0$

The readers will have no problems in verifying that our restrictions on a_{ij} are, in general, different and complement those appearing in Ref. [22,23].

We are not giving the explicit expression of the first integrals associated with $\boldsymbol{S_1}$ in all the cases listed in the Table and for all the values of a_{ij} compatible with the restrictions. But in order to see how the vanishing of Δ implies the non-global character of I we give the explicit expression of I in the following case.

Consider case 1 of the Table with the following values of a_{ij}:

$$a_{11}=2,\quad a_{12}=3,\quad a_{13}=a_{23}=1,$$
$$a_{21}=4,\quad a_{22}=5,\quad a_{31}=6,$$
$$a_{32}=7,\quad a_{33}=10. \tag{14}$$

In this case we get

$$\begin{aligned}\boldsymbol{X}&=x_1(2x_1+3x_2+x_3)\partial_1+x_2(4x_1+5x_2+x_3)\partial_2\\&\quad+x_3(6x_1+7x_2+10x_3)\partial_3,\\ \boldsymbol{S_1}&=x_1\partial_1+x_2\partial_2+10x_3\partial_3,\\ \boldsymbol{S_2}&=x_1\partial_1+x_2\partial_2+x_3\partial_3.\end{aligned} \tag{15}$$

Therefore the integrating factor $\underline{f}$ is given by

$$f=\left(18x_1x_2x_3(x_1+x_2)\right)^{-1}. \tag{16}$$

The expression of fw_1 is

$$\begin{aligned}&-\frac{34x_1+43x_2}{18x_1(x_1+x_2)}dx_1\\&+\frac{14x_1+23x_2}{18x_2(x_1+x_2}\Big)dx_2+\frac{1}{9x_3}dx_3.\end{aligned} \tag{17}$$

As $fw_1=dI$ the expression of I is

$$\tfrac{1}{9}\ln|x_3|-\tfrac{43}{18}\ln|x_1|+\tfrac{7}{9}\ln|x_2|+\tfrac{1}{2}\ln|x_1+x_2|. \tag{18}$$

We see in (18) that I is not defined on the set formed by the union of the planes $x_1=0, x_2=0, x_3=0, x_1+x_2=0$. This set coincides with the set where the function $\underline{f}$ of (16) is not defined.

Let us point out that it is very difficult to say when the local first integral I of Eq. (8) can globalize or not by means of functions of type:

$$(\exp(I))^a,\left[\exp\left[(\exp(I))^a\right]\right]^b,\ldots$$
$$\text{and so on}\quad a,b,\in\mathbb{N}). \tag{19}$$

When I is of the form

$$I=\sum_{j=1}^{p}r_j\ln|c_{j1}x_1+c_{j2}x_2+c_{j3}x_3|,$$
$$c_{j1},c_{j2},c_{j3}\in\mathbb{R},\quad r_j\in Q^+,\quad p\in\mathbb{N}, \tag{20}$$

globalization is indeed possible. In fact $I_{\text{global}}=(\exp(I))^a$ for a certain $a\in\mathbb{N}$, and I_{global} becomes a polynomial.

As we see in (20), a crucial role to obtain a smooth global first integral (C^∞) is played by the fact that r_j is a rational number, this being a very sensitive arithmetical condition, which is no longer fulfilled when the coefficients a_{ij} of Eq. (1) are perturbed. Even so, global smooth first integrals can be useful for the perturbed differential equations (KAM theory) [31–36]. This fact makes them useful when the perturbed v.f. $\boldsymbol{X}$ is free from them.

When the local first integral appearing in (8) is not of type (20) its possible smooth globalization can only be decided by a case by case study. On the other hand globalization of the first integral I of Eq. (8) on the domain

$$O=\{(x_1,x_2,x_3)|x_i>0\}$$

is easier to discuss. In fact $\Delta\neq 0$ on O implies globalization of I on O (remember that O is a simply connected domain). It remains to solve the problem of knowing when Δ is free from zeros on O (note that Δ is a fourth degree polynomial). Nevertheless in cases 1, 2, 3, 4, 5 and 8 listed on the Table Δ decomposes in a product of four linear factors, and in cases 6, 7 and 9 Δ decomposes in a product of two linear factors an a quadratic one. Therefore the discussion of the sign of Δ on O can be carried out without difficulty.

4. Final remarks

On looking at the Table in this paper we observe that the numbers of restrictions on the coefficients a_{ij} is one (cases 1 and 2) and two or three in the rest of the cases 3 to 9. All the restrictions are of algebraic type, that is, they are given by polynomials in the a_{ij}. Restrictions of a similar type can be found in the literature (see Ref. [22,23]).

It is an open question to ascertain whether or not first integrals of L-V systems can be obtained by algorithms such that the coefficients a_{ij} are free to move inside open sets of $\mathbb{R}^9$. These possible open sets can be defined by inequalities of algebraic type on the a_{ij}.

It is also of certain interest to determine the structure of the limit sets w of the solutions $\boldsymbol{x}(t)$ of L-V systems when a first integral is known. Note that sometimes (like in the example of the last section) I globalizes on $\mathbb{R}^3 - \{\Delta = 0\}$ and the limit sets must be contained either in the level sets $(I = C)$ of I or in the singular set $\Delta = 0$. Note that the set $\Delta = 0$ is an invariant set of $\boldsymbol{X}$ (the proof is easy).

The classification of the w-limit sets of v.f. on compact surfaces is well known [37] but very little is known when the orbits of the dynamical system lie on unbounded surfaces. Note that the level sets of I are in general unbounded surfaces.

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Ordered behaviour in force-free magnetic fields

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Abstract

Conditions in order that the trajectories of a force-free vectorfield lie on the level sets of a given function are studied. Force-free vectorfields symmetric under translations, rotations and roto-translations are also considered. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Force-free magnetic fields are characterized by the differential equation

$$\operatorname{curl} \mathbf{B} = \lambda \mathbf{B}, \tag{1}$$

where λ may be a constant (a real number) or a function of $(x, y, z) \in \mathbb{R}^3$, and $\mathbf{B}$ is the magnetic induction vectorfield (v.f. in what follows). They were first introduced by Lundquist, Lust and Schluter [1] to allow magnetic fields and large currents to exist in stellar matter (solar corona [2], the environment of magnetic stars [3] and large domains in the magnetosphere of accreting magnetic compact objects) with vanishing Lorentz force [4]. In fact, the magnetic field formed within a reversed field pinch machine or a spheromak configuration relaxes to a minimum energy state which is well approximated by a force-free magnetic field [5].

On the other hand, the magnetic clouds ejected from the Sun, producing the major perturbations to the Earth's radiation belts during the satellite era seem to possess force-free magnetic fields which have budded from the solar magnetic field [6].

A similar equation ($\operatorname{curl} \mathbf{v} = \lambda \mathbf{v}$) concerning the velocity field of an ideal stationary fluid arises in hydrodynamics, as well as in electromagnetism and accoustics [7]. Eigenfunctions of the curl operator and helicity and geometrical implications of it are studied in Ref. [8].

The case of constant λ is particularly interesting since in this case the trajectories of the v.f. $\mathbf{v}$ or $\mathbf{B}$ can have a complicated topology [9] (they can be ergodic in open sets of R^3) due to a lack of integrability of $\mathbf{v}$ or $\mathbf{B}$, as it happens with the ABC solutions [10]

$$\begin{aligned} v = \bigl(&A \sin(z) + C \cos(y), B \sin(x) + A \cos(z), \\ &C \sin(y) + B \cos(x)\bigr), \\ A, B, &C \in \mathbb{R}, \quad ABC \neq 0. \end{aligned} \tag{2}$$

In contrast with Refs. [9–11] where the authors study chaotic force-free v.f. we consider in this Letter force-free v.f. with ordered trajectories. This ordered behaviour can be ascertained via several visualization techniques [12], using dyes and the smoke of certain gases in the fluid.

An order is introduced by forcing the trajectories to lie on the level set of a function I. I is then a

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first integral of **v** or **B**. Assuming that the function I is given, necessary conditions have been found in order that I can hold a force-free v.f. These conditions take, in general, the form of one or several third-order partial differential equations.

Interesting related references on streamline dynamics of magnetic fields and Hamiltonian representation can be found in Ref. [13].

In brief, the Letter is organized as follows: a summary of the basic facts concerning symmetries and first integrals is given in Section 2. Two geometrical configurations incompatible with force-free v.f. are given in Section 3, while geometrical configurations compatible with (non-trivial) force-free v.f. are studied in Section 4. A similar study when first integrals I and Euclidean symmetries of **B** coexist appears in Section 5. Finally, some open problems are discussed in Section 6. Appendix A develops an specific calculation related to Section 5.

2. A brief account on symmetries and first integrals of v.f.

We now give the definitions and some examples concerning the terms symmetry vector and first integral for three-dimensional v.f.

A smooth (C^∞) function $I(x, y, z)$ is called a first integral of the R^3 v.f. **v** when the streamlines of **v** are contained in the level sets

$$I(x, y, z) = c \tag{3}$$

of I.

It is easy to prove that when I and **v** are smooth condition (3) can also be written

$$\nabla I \cdot \mathbf{v} = 0, \tag{4}$$

∇ standing for the gradient operator

$$\nabla I = \left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}, \frac{\partial I}{\partial z}\right). \tag{5}$$

For example, the function $I = x^2 + y^2 + z^2$ is a first integral of the v.f.

$$\mathbf{v} = \left(yz + xz, zy - xz, -x^2 - y^2\right). \tag{6}$$

Therefore, the streamlines (or trajectories) of **v** lie on the spherical surfaces $x^2 + y^2 + z^2 = c$.

On the other hand, the v.f. **S** is called a symmetry of **v** when the local flow θ_t associated to **S** transforms, for every fixed value of t, every streamline of **v** into another streamline of **v**. It is a classical result [14] that this condition is satisfied when

$$[\mathbf{S}, \mathbf{v}] = \mu \mathbf{v}, \tag{7}$$

[,] standing for the Lie–Jacobi bracket and μ being a function of (x, y, z).

For example, the reader can check that the v.f.

$$\begin{aligned}\mathbf{v} = \big(&F(z)\left(x^2 + y^2\right)^n x, F(z)\left(x^2 + y^2\right)^n y,\\ &G(z)\left(x^2 + y^2\right)^n\big)\end{aligned} \tag{8}$$

has the two symmetry vectors

$$\begin{aligned}S_1 &= (x, y, 0),\\ S_2 &= (-y, x, 0),\end{aligned} \tag{9}$$

$F(z), G(z)$ standing for arbitrary functions of $\underline{z}$ and $\underline{n}$ being any real number.

Analogously the v.f. **v** defined by

$$\begin{aligned}v = \Bigg(&\left(-\frac{C_{,z}}{2} - \frac{D_{,z}}{4}u\right)x - yB,\\ &\left(-\frac{C_{,z}}{2} - \frac{D_{,z}}{4}u\right)y + xB, C + Du\Bigg),\\ &C(z), D(z), B(u, z),\\ u &= x^2 + y^2,\end{aligned} \tag{10}$$

has the symmetry vector $\mathbf{S} = (-y, x, 0)$. C, D, B stand for arbitrary functions of their arguments. Note that in Eqs. (10) $C_{,z}$ and $D_{,z}$ stand for dC/dz and dD/dz.

Many applications of these two concepts will appear in the following sections of the Letter. Specifically we shall study:

(i) The conditions under which a given function $I(x, y, z)$ can be a first integral of a non-trivial ($\mathbf{v} \neq \mathbf{0}$) force-free v.f. (when $\lambda \neq 0$).

(ii) Same questions when **v** is also symmetric under translations, rotations or roto-translations (helicoidal motions).

3. First integrals incompatible with force-free vectorfields

We show in this section that certain functions I *cannot* be first integrals of non-trivial force-free v.f.

That is, if I is one of these functions the force-free condition (1) ($\lambda \neq 0$) and Eq. (4) imply $\mathbf{B} = \mathbf{0}$. Calling $\mathcal{F}$ the set of these functions, we show that $I = \varphi = \tan^{-1}(y/x)$ and $I = \theta = \tan^{-1}[(x^2 + y^2)^{1/2} z^{-1}]$ belong to $\mathcal{F}$.

Note that the level sets of these functions are, respectively, half-planes through the z-axis and right circular cones centered on the z-polar axis.

The fact that $\mathcal{F}$ *is not* the empty set is in contrast with the following well known property of divergence-free v.f. (solenoidal v.f.): *any function* I admits a solenoidal v.f. $\mathbf{B}$ satisfying Eq. (4). In fact, any v.f. $\mathbf{B}$ defined by

$$\mathbf{B} = \nabla I \wedge \nabla J \tag{11}$$

is divergence-free (remember that $\operatorname{div}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{b} \times \operatorname{curl}(\mathbf{a}) - \mathbf{a} \operatorname{curl}(\mathbf{b})$) and admits I as a first integral. J stands for an arbitrary differentiable function.

It is an open problem to characterize the functions I belonging to $\mathcal{F}$. Can this characterization be achieved in terms of the derivatives of I (up to a finite or an infinite order) and a finite or infinite number of partial differential equations to which the derivatives of I should satisfy? How these partial differential equations characterizing $\mathcal{F}$ can be obtained?

We remind the readers that if a function I satisfies (see Eq. (4))

$$\Delta I \cdot B = 0 \tag{12}$$

then the trajectories of $\mathbf{B}$, that is, the curves satisfying

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}, \tag{13}$$

are contained in the level sets of I:

$$I(x, y, z) = \text{const.} \tag{14}$$

When $I(x, y, z)$ is analytic and not trivial (identically constant) the level sets of I have a very simple structure: they resemble smooth differential manifolds of dimension two except, maybe, on the points $\mathbf{x}_0 \in R^3$ such that $\nabla I(\mathbf{x}_0) = \mathbf{0}$. As an example, consider $I(x, y, z) = x^2 + y^2 - z^2$. The level sets of this function,

$$x^2 + y^2 - z^2 = C, \tag{15}$$

are differential manifolds of dimension two for $C \neq 0$. For $C = 0$ the set

$$x^2 + y^2 - z^2 = 0 \tag{16}$$

is a cone: a differential manifold except on the point $(0, 0, 0)$, where $\nabla(x^2 + y^2 - z^2) = \mathbf{0}$.

Therefore a global function satisfying Eq. (12) is an element of order concerning the trajectories of $\mathbf{B}$. What it is shown in this section is that certain types of ordering (that is, certain level sets associated to I) are not compatible with the force-free condition (1) ($\lambda \neq 0$) and, consequently, force-free magnetic fields cannot be embedded into that kind of surfaces.

Note that in all what follows the force-free condition (1) has been used with $\lambda(x, y, z)$ a constant real number.

(a) Let us first consider the case $I = \varphi = \tan^{-1}(y/x)$.

In cylindrical coordinates (r, φ, z) $\mathbf{B}$ takes the form

$$\mathbf{B} = B_r(r, \varphi, z)\mathbf{u}_r + B_z(r, \varphi, z)\mathbf{u}_z, \tag{17}$$

$(\mathbf{u}_r, \mathbf{u}_\varphi, \mathbf{u}_z)$ standing for the unitary vectors along the coordinate lines $r = C_1$, $\varphi = C_2$, $z = C_3$ (C_i = real constants).

The force-free condition (1) becomes

$$\frac{1}{r} B_{z,\varphi} = \lambda B_r, \tag{18a}$$

$$B_{r,z} - B_{z,r} = 0, \tag{18b}$$

$$-\frac{1}{r} B_{r,\varphi} = \lambda B_z. \tag{18c}$$

Note that $B_{z,\varphi}$ stands for $\partial B_z/\partial\varphi$, etc.

From (18a) and (18c) we get

$$B_{r,\varphi\varphi} + \lambda^2 r^2 B_r = 0, \tag{19}$$

and therefore

$$B_r = A_1(r, z)\cos(\lambda r\varphi) + A_2(r, z)\sin(\lambda r\varphi), \tag{20}$$

A_1 and A_2 being arbitrary functions of (r, z).

From (20) and (18c) we get

$$B_z = A_1 \sin(\lambda r\varphi) - A_2 \cos(\lambda r\varphi), \tag{21}$$

and, finally, from (18b) we get

$$\begin{aligned}(A_{2,z} - A_{1,r})\sin(\lambda r\varphi) + (A_{1,z} - A_{2,r})\cos(\lambda r\varphi) \\ - \lambda A_1\varphi\cos(\lambda r\varphi) - \lambda A_2\varphi\sin(\lambda r\varphi) = 0.\end{aligned} \tag{22}$$

The functions $\sin(\lambda r\varphi)$, $\cos(\lambda r\varphi)$, $\varphi\sin(\lambda r\varphi)$ and $\varphi\cos(\lambda r\varphi)$ are linearly independent (as functions of the variable φ), as can be checked by computing of their Wronskian.

Therefore we get from (22)

$$A_{2,z} - A_{1,r} = 0,$$
$$A_{1,z} - A_{2,r} = 0,$$
$$\lambda A_1 = 0,$$
$$\lambda A_2 = 0, \tag{23}$$

and, since $\lambda \neq 0$,

$$A_1 = A_2 = 0. \tag{24}$$

Therefore

$$B_r = B_z = 0. \tag{25}$$

Accordingly $\mathbf{B} = \mathbf{0}$. This is the only v.f. compatible with the first integral $I = \varphi$ and the force-free condition (1).

(b) Let us study now the case of $I = \theta = \tan^{-1}[(x^2 + y^2)^{1/2} z^{-1}]$.

In spherical coordinates (ρ, φ, θ) $\mathbf{B}$ will have the form

$$\mathbf{B} = B_\rho(\rho,\varphi,\theta)\mathbf{u}_\rho + B_\varphi(\rho,\varphi,\theta)\mathbf{u}_\varphi, \tag{26}$$

$(\mathbf{u}_\rho, \mathbf{u}_\varphi, \mathbf{u}_\theta)$ standing for the unitary vectors along the coordinate lines $\rho = C_1$, $\varphi = C_2$, $\theta = C_3$ (C_i are real constants).

The force-free condition (1) becomes

$$\frac{1}{\rho\sin\theta}(\sin\theta B_\varphi)_{,\theta} = \lambda B_\rho, \tag{27a}$$
$$\frac{1}{\rho\sin\theta}B_{\rho,\varphi} - \frac{1}{\rho}(\rho B_\varphi)_{,\rho} = 0, \tag{27b}$$
$$-\frac{1}{\rho}B_{\rho,\theta} = \lambda B_\varphi. \tag{27c}$$

From (27a) and (27c) we get

$$\sin\theta B_{\rho,\theta\theta} + \cos\theta B_{\rho,\theta} + \lambda^2\rho^2 \sin\theta B_\rho = 0. \tag{28}$$

The particular solutions of (28) corresponding to the initial values $B_\rho(\theta = 0) = 1$, $B_{\rho,\theta}(\theta = 0) = 0$ and $B_\rho(\theta = 0) = 0$, $B_{\rho,\theta}(\theta = 0) = 1$ have been obtained via a MAPLE V computer package and are

$$B_\rho^{(1)} = 1 - \frac{1}{4}\lambda^2\rho^2\theta^2 + \left(-\frac{1}{96}\lambda^2\rho^2 + \frac{1}{64}\lambda^4\rho^4\right)\theta^4 + o(\theta^6),$$
$$B_\rho^{(2)} = \theta + \left(-\frac{1}{3}\lambda^2\rho^2 + \frac{1}{9}\right)\theta^3 + \left(\frac{8}{675} - \frac{4}{135}\lambda^2\rho^2 + \frac{1}{45}\lambda^4\rho^4\right)\theta^5 + o(\theta^6). \tag{29}$$

Therefore we can write

$$B_\rho = A_1(\rho,\varphi)B_\rho^{(1)} + A_2(\rho,\varphi)B_\rho^{(2)}, \tag{30}$$

A_1 and A_2 being arbitrary functions of their arguments.

B_φ can be obtained through (27c),

$$B_\varphi = -\frac{1}{\lambda\rho}B_{\rho,\theta}, \tag{31}$$

but for brevity reasons its expression has not been written.

Introducing the expressions of B_ρ and B_φ into (27b) and equating the coefficients of the powers of θ^i ($0 \leqslant i \leqslant 5$) we get

$$\theta^0:\quad A_{1,\varphi} = 0, \tag{32a}$$
$$\theta^1:\quad A_{2,\varphi} + \frac{A_{2,\rho}}{\lambda} = 0, \tag{32b}$$
$$\theta^2:\quad A_{1,\rho} = 0, \tag{32c}$$
$$\theta^3:\quad \left(A_{2,\rho} - \frac{\lambda}{3}A_{2,\varphi}\right)\left(\lambda\rho^2 - \frac{1}{3\lambda}\right) + \frac{1}{6\lambda}A_{2,\rho} + \frac{A_2}{3\lambda\rho} = 0, \tag{32d}$$
$$\theta^4:\quad A_1 = 0, \tag{32e}$$
$$\theta^5:\quad A_{2,\varphi}\left(\frac{8}{675} - \frac{4}{135}\lambda^2\rho^2 + \frac{1}{45}\lambda^4\rho^4\right) - A_{2,\rho}\left(\frac{31}{540\lambda} - \frac{1}{9}\lambda^3\rho^4 - \frac{1}{54}\lambda\rho^2\right) + A_2\left(\frac{\lambda\rho}{27} - \frac{4}{9}\lambda^3\rho^3\right) = 0. \tag{32f}$$

Therefore we get $A_1 = 0$.

On the other hand, from (32b) we get

$$A_{2,\rho} = -\lambda A_{2,\varphi}. \tag{33}$$

Introducing (33) into (32d) and (32f) we get

$$\left(\frac{5}{18} - \frac{4}{3}\lambda^2\rho^2\right)A_{2,\varphi} + \frac{A_2}{3\lambda\rho} = 0, \tag{34}$$
$$\left(\frac{187}{2700} - \frac{13}{270}\lambda^2\rho^2 - \frac{4}{45}\lambda^4\rho^4\right)A_{2,\varphi} + \left(\frac{\lambda\rho}{27} - \frac{4}{9}\lambda^3\rho^3\right)A_2 = 0. \tag{35}$$

It is immediate to see that Eqs. (34) and (35) have only the common solution $A_2 = 0$. Therefore, $B_\rho = 0$, $B_\varphi = 0$ and $\mathbf{B} = \mathbf{0}$.

A much more general problem arises at this point: that of getting sufficient/necessary conditions in order that a given function $I(x, y, z)$ cannot be a first integral of a (non-trivial) force-free vectorfield.

4. First integrals compatible with force-free vectorfields

Some examples are given in this section of first integrals whose level sets are compatible with non-trivial solutions of Eq. (1). Such is the case of cylindrical, planelike and spherical level sets defined by the functions

$$\begin{aligned} &\text{(i)} \quad I(x^2 + y^2), \\ &\text{(ii)} \quad I(z), \\ &\text{(iii)} \quad I(x^2 + y^2 + z^2). \end{aligned} \tag{36}$$

(i) In cylindrical coordinates $\mathbf{B}$ will have the form

$$\mathbf{B} = B_\varphi \partial_\varphi + B_z \partial_z. \tag{37}$$

Writing Eq. (1) in these coordinates we get [15]

$$\begin{aligned} &r^2 B_{\varphi,z} = B_{z,\varphi}, \\ &-B_{z,r} = \lambda r B_\varphi, \\ &\left(r^2 B_\varphi\right)_{,r} = \lambda r B_z, \end{aligned} \tag{38}$$

λ being a real number.

Trying solutions of the form $\mathbf{B}(r)$ we get $B_\varphi = g(r)$ with $g(r)$ an arbitrary function of $\underline{r}$; the remaining two equations become

$$\begin{aligned} &B_z = \frac{1}{\lambda r}\left(r^2 g(r)\right)_{,r}, \\ &-\left(\frac{1}{\lambda r}\left(r^2 g(r)\right)_{,r}\right)_{,r} = \lambda r g(r). \end{aligned} \tag{39}$$

Note that the second of Eq. (39) is the linear differential equation

$$r\frac{d^2 g}{dr^2} + 3\frac{dg}{dr} + \lambda^2 r g = 0, \tag{40}$$

which always possesses local non-trivial solutions ($g \neq 0$) and this implies that $\mathbf{B} \neq \mathbf{0}$.

The reader can check that according to the singularity theory of linear differential equations [15] Eq. (40) has a regular analytic solution at $r = 0$ and, therefore, analytic solutions valid for all values of $\underline{r}$. These solutions are, in fact, Bessel functions of the first kind.

(ii) In a Cartesian coordinate system $\mathbf{B}$ is now of the form

$$\mathbf{B} = B_x \partial_x + B_y \partial_y \tag{41}$$

and writing Eq. (1) in Cartesian coordinates we get

$$\begin{aligned} &-B_{y,z} = \lambda B_x, \\ &B_{x,z} = \lambda B_y, \\ &B_{y,x} - B_{x,y} = 0. \end{aligned} \tag{42}$$

Let us try solutions of the form $\mathbf{B}(z)$. With this assumption Eqs. (42) become

$$\begin{aligned} &-B_{y,z} = \lambda B_x, \\ &B_{x,z} = \lambda B_y, \end{aligned} \tag{43}$$

which we write in the form

$$\begin{aligned} &B_x = -\frac{1}{\lambda} B_{y,z}, \\ &-\frac{1}{\lambda} B_{y,zz} = \lambda B_y. \end{aligned} \tag{44}$$

The second of these equations is a second-order, linear, differential equation with constant coefficients possessing non-trivial solutions ($B_y \neq 0$). These solutions are, in fact, trigonometric functions. Therefore $\mathbf{B} \neq \mathbf{0}$.

(iii) In a spherical coordinate system $\mathbf{B}$ takes now the form

$$\mathbf{B} = B_\varphi \partial_\varphi + B_\theta \partial_\theta. \tag{45}$$

Writing Eq. (1) in spherical coordinates and assuming that $\mathbf{B}$ does not depend on φ we get

$$\begin{aligned} &\rho^2 \sin^2\theta\, B_\varphi = g(\rho), \\ &-\frac{1}{\lambda \rho^2 \sin\theta} g_{,\rho} = B_\theta, \\ &\frac{1}{\lambda \rho^2 \sin\theta}\left(\rho^2 B_\theta\right)_{,\rho} = B_\varphi, \end{aligned} \tag{46}$$

$\underline{g}$ standing for an arbitrary function of ρ.

The first and second of Eqs. (46) define B_φ and B_θ and the third equation becomes the second-order differential equation

$$\frac{d^2 g}{d\rho^2} + \lambda^2 g(\rho) = 0, \tag{47}$$

whose solutions are, again, trigonometric functions.

Therefore we get again non-trivial solutions of Eq. (1) lying on the spheres $\rho = \text{const}$.

5. First integrals and Euclidean symmetries compatible with force-free vectorfields

Force-free v.f. symmetric under rotations, translations and roto-translations are studied in this section. The physical motivation of this is the following. The magnetic induction $\mathbf{B}(\mathbf{r})$ created by a current density $\mathbf{j}(x', y', z')$ is proportional to the well known expression [16]

$$\int \frac{j(r') \wedge (r - r')}{\|r - r'\|^3} \, d^3 r'. \tag{48}$$

It is easy to see that $\mathbf{B}$ inherits the Euclidean symmetries of $\mathbf{j}$; this is essentially due to the fact that Euclidean symmetries conserve volumes, the norm of vectors and the exterior product $\wedge$.

Under these conditions, and due to the fact that

$$[S_{\text{Eucl}}, B] = 0,$$
$$\operatorname{div} S_{\text{Eucl}} = 0, \qquad \operatorname{div} B = 0, \tag{49}$$

the one-form w_1 defined by

$$w_1 = (\mathbf{B} \wedge \mathbf{S}_{\text{Eucl}}) \cdot d\mathbf{r} \tag{50}$$

is closed [17] ($dw_1 = 0$) in the entire space and, hence, is exact. Therefore we can write

$$(\mathbf{B} \wedge \mathbf{S}_{\text{Eucl}}) \cdot d\mathbf{r} = dI. \tag{51}$$

The function I defined in (51) is a first integral of both $\mathbf{B}$ and $\mathbf{S}_{\text{Eucl}}$. Note that the expression $dw_1 = 0$ can be written in the form

$$\operatorname{curl}(\mathbf{B} \wedge \mathbf{S}_{\text{Eucl}}) = \mathbf{0} \tag{52}$$

on account of Eqs. (49).

Since every $\mathbf{S}_{\text{Eucl}}$ can be reduced (via an orthogonal transformation) to one of the v.f.

$$\mathbf{S} = (0, 0, 1),$$
$$\mathbf{S} = (-y, x, 0),$$
$$\mathbf{S} = (-y, x, a), \quad a \neq 0, \tag{53}$$

our following task will be that of getting necessary and sufficient conditions in order to know if there is a non-trivial force-free v.f. symmetric under one of the v.f. of Eqs. (53) and a given first integral I (remember that I is symmetric under $\mathbf{S}$, that is, $\nabla I \cdot \mathbf{S} = 0$, as just explained).

5.1. We now study the case in which $\mathbf{S}$ represents a translation along the z-axis. We shall see that I must satisfy a set of two partial differential equations in order to have a non-trivial force-free v.f. $\mathbf{B}$ compatible with I and $\mathbf{S} = \partial_z$.

Since I must be invariant under $\mathbf{S}$, I must be of type $I(x, y)$. On the other hand, Eq. (51) becomes

$$B_y = I_{,x},$$
$$B_x = -I_{,y}, \tag{54}$$

and since $\mathbf{B}$ must be independent of $\underline{z}$, we get from Eq. (1)

$$B_{z,y} = \lambda B_x,$$
$$-B_{z,x} = \lambda B_y,$$
$$B_{y,x} - B_{x,y} = \lambda B_z, \tag{55}$$

and upon substitution of (54) into (55) we get

$$B_{z,y} = -\lambda I_{,y},$$
$$B_{z,x} = -\lambda I_{,x},$$
$$I_{,xx} + I_{,yy} = \lambda B_z. \tag{56}$$

We see in Eqs. (56) that B_z is just the Laplacian of the function I times λ^{-1}. The first two equations (56) imply that I must satisfy the consistency equations

$$\left(\Delta I + \lambda^2 I\right)_{,x} = 0,$$
$$\left(\Delta I + \lambda^2 I\right)_{,y} = 0,$$
$$\Delta = \text{Laplacian}. \tag{57}$$

Note that (57) is automatic (when $\lambda = 0$) when I is an harmonic function.

It is easy to show that Eqs. (57) have non-trivial solutions $I \neq 0$ just by putting $I = f(x) + g(y)$. We obtain in this way the equations

$$f'''(x) + \lambda^2 f'(x) = 0,$$
$$g'''(y) + \lambda^2 g'(y) = 0, \tag{58}$$

having as solutions

$$f(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3,$$
$$g(y) = B_1 \sin(\lambda y) + B_2 \cos(\lambda y) + B_3,$$
$$A_1, A_2, A_3, B_1, B_2, B_3 \in \mathbb{R}. \tag{59}$$

Therefore I, and accordingly **B**, is non-trivial ($\mathbf{B} \neq \mathbf{0}$) and periodic in the variables $\underline{x}$, $\underline{y}$ (period $2\pi/\lambda$).

5.2. Let us now study the case $\mathbf{S} = (-y, x, 0)$ under which **B** must be invariant. We shall use cylindrical coordinates (r, φ, z) and we prove that $I(r, z)$ must satisfy two third-order partial differential equations in order to get a non-trivial v.f. **B** having I as a first integral. These partial differential equations admit solutions for I whose level sets are tori.

All along this section and the next one we shall expand **B** in terms of the orthonormal basis

$$\left\{\partial_r, \frac{\partial_\varphi}{r}, \partial_z\right\}. \tag{60}$$

Note that **B** is of the form $\mathbf{B}(r, z)$ because of the symmetry **S**. On the other hand, condition (51) can be written in the form

$$\det\begin{pmatrix} dr & r\,d\varphi & dz \\ B_r & B_\varphi & B_z \\ 0 & r & 0 \end{pmatrix} = dI, \tag{61}$$

from which we get

$$-rB_z = I_{,r},$$
$$rB_r = I_{,z}. \tag{62}$$

On the other hand, writing Eq. (1) for this case we get

$$-B_{\varphi,z} = \lambda B_r,$$
$$B_{r,z} - B_{z,r} = \lambda B_\varphi,$$
$$(rB_\varphi)_{,r} = \lambda r B_z, \tag{63}$$

and substituting (62) into (63) we get

$$B_{\varphi,z} = -\frac{\lambda}{r} I_{,z},$$
$$\frac{I_{,zz}}{r} + \left(\frac{I_{,r}}{r}\right)_{,r} = \lambda B_\varphi,$$
$$(rB_\varphi)_{,r} = -\lambda I_{,r}. \tag{64}$$

The second of Eqs. (64) gives B_φ in terms of I and its derivatives. Substituting this value of B_φ into the other two equations we get the third-order partial differential equations

$$\frac{1}{\lambda}\left[\frac{I_{,zzz}}{r} + \left(\frac{I_{,r}}{r}\right)_{,rz}\right] = -\frac{\lambda}{r} I_{,z}, \tag{65}$$

$$\left[\frac{r}{\lambda}\left(\frac{I_{,zz}}{r} + \left(\frac{I_{,r}}{r}\right)_{,r}\right)\right]_{,r} = -\lambda I_{,r}. \tag{66}$$

If a given I does not satisfy Eqs. (65) and (66) then the only solution to the problem of this section is the trivial one $\mathbf{B} = \mathbf{0}$.

See Appendix A for the reduction of Eqs. (65) and (66) to a single second-order partial differential equation (depending on an arbitrary constant).

Eqs. (65) and (66) admit solutions of the form

$$I = a(z) + b(r) \tag{67}$$

if $a(z)$, $b(r)$ satisfy the equations

$$a'''(z) + \lambda^2 a'(z) = 0, \tag{68}$$

$$b'''(r) - \frac{b''(r)}{r} + b'(r)\left(\lambda^2 + \frac{1}{r^2}\right) = 0. \tag{69}$$

Constant solutions of Eq. (68) correspond to streamlines of **B** ordered along the cylinder $r = \text{const}$, and constant solutions of Eq. (69) correspond to an ordering along the planes $z = \text{const}$.

Let us now see that an ordering of the streamlines of **B** along tori is also compatible with the functions I defined in (67).

In fact, choose the solution $\cos(\lambda z)$ of Eq. (68). This solution presents a strict maximum at $z = 2k\pi$ (k is an integer). On the other hand, consider the initial values

$$b(r_0) = a, \qquad b'(r_0) = 0, \qquad b''(r_0) < 0,$$
$$r_0 > 0,$$

of Eq. (69).

The solution $\hat{b}(r)$ corresponding to them will have a maximum at $r = r_0$. Therefore $I = \cos(\lambda z) + \hat{b}(r)$ has a strict maximum at $(2k\pi, r_0)$, and the level sets of $I(r, z)$ are ovals near the points $(2k\pi, r_0)$ of the (r, z) plane. These ovals become tori in three-dimensional space by rotation around the z-axis.

5.3. We now study the final case of roto-translations. The symmetry vector **S** is in this case $\mathbf{S} = (-y, x, a)$ $(a \neq 0)$. Only the case $a = 1$ shall be considered since no additional difficulties arise when $a \neq 1$.

Two third-order partial differential equations for I are again obtained, and certain solutions of them are studied.

First of all, the first integral I must take the form $I(r,u)u = z - \varphi$ since I must be invariant under **S**. Analogously **B** must be of the form $\mathbf{B}(r,u)$ since **B** and **S** commute.

On the other hand, equation $(\mathbf{B} \wedge \mathbf{S})\,d\mathbf{r} = dI$ becomes

$$\det\begin{pmatrix} dr & r\,d\varphi & dz \\ B_r & B_\varphi & B_z \\ 0 & r & 1 \end{pmatrix} = dI, \tag{70}$$

from which we get

$$I_{,r} = B_\varphi - rB_z,$$
$$I_{,u} = rB_r. \tag{71}$$

From Eq. (1) we get, after substitutions of B_r and B_z in function of B_φ and I,

$$-\frac{1}{r^2}B_{\varphi,u} + \frac{1}{r^2}I_{,ur} - B_{\varphi,u} = \frac{\lambda}{r}I_{,u},$$
$$\frac{1}{r}I_{,uu} + \frac{1}{r^2}B_\varphi - \frac{1}{r}B_{\varphi,r} - \frac{1}{r^2}I_{,r} + \frac{1}{r}I_{,rr} = \lambda B_\varphi,$$
$$\frac{1}{r}B_\varphi + B_{\varphi,r} + \frac{1}{r^2}I_{,uu} = \frac{\lambda}{r}(B_\varphi - I_{,r}). \tag{72}$$

Eliminating $B_{\varphi,r}$ between the second and third of Eqs. (72) we get

$$\frac{1}{r}\left(1 + \frac{1}{r^2}\right)I_{,uu} + \frac{\lambda - 1}{r^2}I_{,r} + \frac{1}{r}I_{,rr} = \left(\frac{\lambda - 2}{r^2} + \lambda\right)B_\varphi. \tag{73}$$

Therefore we have obtained B_φ as a function of I and its derivatives.

If we now substitute B_φ in the first and second of Eqs. (72) we get two third-order partial differential equations for I. These two equations are not written because of their length. In practical case it is better to work with the first two of Eqs. (72) and (73) directly.

Note that for $I = I(r)$ the first two equations (72) and Eq. (73) become

$$B_{\varphi,u} = 0, \tag{74}$$

$$\frac{1}{r^2}B_\varphi - \frac{1}{r}B_{\varphi,r} - \frac{I_{,r}}{r^2} + \frac{I_{,rr}}{r} = \lambda B_\varphi, \tag{75}$$

$$\frac{\lambda - 1}{r^2}I_{,r} + \frac{1}{r}I_{,rr} = \left(\frac{\lambda - 2}{r^2} + \lambda\right)B_\varphi. \tag{76}$$

Eq. (76) implies that B_φ only depends on $\underline{r}$. The first of these equations holds automatically due to the third one. After substitution of B_φ (from (76)) into (75) we get a third-order differential equation in I whose local solutions are guaranteed. Therefore Eqs. (76) and (71) define **B** once I is known.

We shall now prove that Eqs. (72) and (73) have solutions of the form

$$I = a(r) + b(r)u,$$
$$b(r) \neq 0. \tag{77}$$

Writing the classical Chandrasekhar equation [18]

$$\Delta\psi + \lambda^2\psi = 0 \tag{78}$$

in cylindrical coordinates, it is easy to see that it admits a solution of the form

$$\psi = A(r)u \tag{79}$$

if $A(r)$ satisfies the linear differential equation

$$r\frac{d^2A}{dr^2} + \frac{dA}{dr} + \lambda rA = 0. \tag{80}$$

The v.f. **B** is given by

$$B = \lambda^{-1}\operatorname{curl}\bigl(\operatorname{curl}(a\psi)\bigr) + \operatorname{curl}(a\psi),$$
$$a = (0,0,1). \tag{81}$$

After some computations **B** can be written in the form

$$B = \left(\frac{A'}{\lambda} - \frac{A}{r}\right)\partial_r + (-u'A)\frac{\partial_\varphi}{r} + \left(\frac{-u(r'A)'}{\lambda r}\right)\partial_z. \tag{82}$$

Note that **B** commute with $\mathbf{S} = (-y, x, 1) = \partial_\varphi + \partial_z$.

Therefore equation $(\mathbf{B} \wedge \mathbf{S})\,d\mathbf{r} = dI$ becomes

$$\det\begin{pmatrix} dr & r\,d\varphi & dz \\ B_r & B_\varphi & B_z \\ 0 & r & 1 \end{pmatrix} = dI,$$
$$B_r = \frac{A'}{\lambda} - \frac{A}{r},$$
$$B_\varphi = -uA',$$
$$B_z = \frac{-u(r'A)'}{\lambda r}. \tag{83}$$

We get in this way

$$I = \left(\frac{rA'}{\lambda} - A\right)u + G(r), \tag{84}$$

$G(r)$ standing for a function of A and its derivatives (after calculations it is easy to see that $G(r)$ is just a constant function).

Note that the coefficient of $\underline{u}$ in Eq. (84) cannot identically vanish (remember that A is a non-trivial solution of Eq. (80)).

Eq. (84) gives the form of a first integral I depending on $\underline{u}$.

6. Final remarks

We have shown in this Letter that certain geometrical configurations (planes through a line, circular cones through a point) are forbidden as first integrals of a non-trivial force-free v.f. Others, like tori, parallel planes, circular cylinders, spheres are not forbidden.

A first open question is: can $I = \text{const}$ represent tori with more than one handle and force-free v.f. **B** exist with I as first integral? We cannot see at present any physical consequence (on the motion of the charges) of these toruslike configurations, but perhaps some readers can.

At the mathematical level the following question arises: It is well known that the streamlines of the magnetic induction **B** can be closed loops. Stokes theorem plus Eq. (1) makes this impossible to occur for force-free v.f. having a first integral I with planelike level sets (note that planelike level sets are just topological planes; on these level sets a loop is deformable to a point). The question is that of ascertain if a force-free v.f. can have closed trajectories or not when the level sets of I are not topological planes or when **B** is free from global first integrals. Note that if the reply is negative if $\mathbf{B} \neq \mathbf{0}$ on the tori introduced in Section 5.2 the streamlines of **B** are dense on each of these tori [19].

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Appendix A

In the assumed absence of dependence on φ the first of Eqs. (64) can be written

$$B_\varphi + \frac{\lambda}{r} I = f(r), \tag{A.1}$$

where $f(r)$ stands for an arbitrary function of $\underline{r}$. Substituting (A.1) into the third of Eqs. (64) one immediately get

$$f(r) = \frac{c}{r}, \tag{A.2}$$

c being an arbitrary constant.

Therefore we can write

$$B_\varphi = \frac{c - \lambda I}{r}. \tag{A.3}$$

Substituting (A.3) into the second of Eqs. (64) we finally get the second-order differential equation

$$\frac{I_{,zz}}{r} + \left(\frac{I_{,r}}{r}\right)_{,r} = \lambda\left(\frac{c - \lambda I}{r}\right), \tag{A.4}$$

that is,

$$I_{,zz} + I_{,rr} - \frac{I_{,r}}{r} + \lambda^2 I - \lambda c = 0, \tag{A.5}$$

as we desired.

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Motion of a charge in the magnetic field created by wires: impossibility of reaching the wires

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Abstract

Using a remarkable connection between pairs of first integrals of the magnetic field $\mathbf{B}$ and first integrals of the Newton–Lorentz equation $\ddot{\mathbf{x}} = \dot{\mathbf{x}} \wedge \mathbf{B}$, it is shown that, under certain conditions, the wires creating $\mathbf{B}$ are unreachable for electric charges moving under the action of $\mathbf{B}$. Part of these mathematical results are of interest to electrical engineers, helping to keep the power lines electrically neutral.

Keywords: Magnetic field; Newton–Lorentz equation; First integral

1. Introduction

It is a well-known fact that, for appropriate initial conditions $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$, a negative unit charge, unit mass, test particle, subjected to the Coulombian electric field $\mathbf{E}$ created by the fixed charges $(q_1, \mathbf{x}_1)$, $(q_2, \mathbf{x}_2)$, $q_1, q_2 > 0$, via Newton equations

$$\ddot{\mathbf{x}} = -\mathbf{E}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{1}$$

can reach the points $\mathbf{x}_1$ or $\mathbf{x}_2$ [1].

In fact, this can be achieved simply by choosing $\dot{\mathbf{x}}_0 = \mathbf{0}$, $\mathbf{x}_0$ sufficiently near $\mathbf{x}_1$ or $\mathbf{x}_2$ and $(\mathbf{x}_i - \mathbf{x}_0) \parallel (\mathbf{x}_1 - \mathbf{x}_2)$ $(i = 1, 2)$.

This result obviously holds when the negative unit charge is substituted by a unit (positive) mass and the couple (q_1, q_2) by the pair (m_1, m_2) of (positive) masses at the fixed points $(\mathbf{x}_1, \mathbf{x}_2)$.

We show in this Letter that for the magnetic field $\mathbf{B}(\mathbf{x})$ created by a straight line wire (W_s) or a circular wire (W_c) the solutions of the Newton–Lorentz equations [1] (unit charge, unit mass test particle again)

$$\ddot{\mathbf{x}} = \dot{\mathbf{x}} \wedge \mathbf{B}(\mathbf{x}), \tag{2}$$

cannot reach the wires W_s or W_c.

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In fact, this result is valid as well when W_s represents not just a straight line wire but a finite number of straight line parallel wires and also when W_c represents a finite number of circular wires, lying on parallel planes, with collinear centers lying on a straight line L orthogonal to these planes. In any of these two cases the wires W_s or the wires W_c cannot be reached by a unit charge, unit mass particle moving according to Eq. (2).

The proof of our unreachability results is based on the knowledge of an adequate number of first integrals of Eq. (2). It will be seen that, in addition to $\dot{\mathbf{x}}^2$, Eq. (2) can possess additional first integrals I when two first integrals (I_1, I_2) of the magnetic induction vector field $\mathbf{B}$ are known (see Section 2). This result connecting, under certain conditions, the first integrals $(I_1(\mathbf{x}), I_2(\mathbf{x}))$ of $\mathbf{B}(\mathbf{x})$ and those of the Newton–Lorentz equations (2), is apparently new (see Eqs. (6) and (7)). An application of it in order to get unreachability when $\mathbf{B}$ is created either by a finite number of parallel straight line wires or by a finite number of circular wires (lying on parallel planes with collinear centers on a straight line L orthogonal to these planes) appears in Sections 3 and 4. Finally, a discussion of Eq. (6), which is basic in the obtention of the first integral (7), is given in Section 5.

Note, finally, that when Eq. (2) is substituted by the corresponding relativistic equation

$$\frac{d}{dt}(\gamma\dot{\mathbf{x}}) = \dot{\mathbf{x}} \wedge \mathbf{B}(\mathbf{x}),$$
$$\gamma = \left(1-\dot{\mathbf{x}}^2\right)^{-1/2}, \qquad c = \text{light velocity} = 1, \tag{3}$$

the above unreachability results hold, as well. This is due to the fact that $\dot{\mathbf{x}}^2$ is again a first integral of Eqs. (3) (this is immediately seen, since

$$\dot{\mathbf{x}} \cdot \frac{d}{dt}(\gamma\dot{\mathbf{x}}) = 0,$$

via quite easy computations). Therefore, γ is a positive constant, and Eqs. (3) can be written in the form:

$$\ddot{\mathbf{x}} = \dot{\mathbf{x}} \wedge \tilde{\mathbf{B}}(\mathbf{x}), \qquad \tilde{\mathbf{B}}(\mathbf{x}) = \gamma^{-1}\mathbf{B}(\mathbf{x}), \tag{4}$$

which is just Eq. (2) with $\mathbf{B}$ rescaled (by the constant factor γ^{-1}). The reader will check in Sections 3 and 4 that a factor like this has no effect on the unreachability results of these sections.

2. First integrals of Newton–Lorentz equations induced by pairs of first integrals of the magnetic induction B

We show in this section that if $I_1(\mathbf{x}), I_2(\mathbf{x})$ $(\mathbf{x} \in \mathbb{R}^3)$ are independent and orthogonal first integrals of $\mathbf{B}$, that is

$$\mathbf{B}\nabla I_1 = 0, \qquad \mathbf{B}\nabla I_2 = 0,$$
$$\nabla I_1 \nabla I_2 = 0, \qquad \operatorname{rank}(\nabla I_1, \nabla I_2) = 2, \tag{5}$$

where ∇ is gradient operator, and if

$$\frac{\ddot{\mathbf{x}} \cdot \nabla I_1}{\lambda(\nabla I_1)^2} = \frac{d}{dt}\big(A(\mathbf{x}, \dot{\mathbf{x}})\big), \tag{6}$$

then

$$I = I_2(\mathbf{x}) - A(\mathbf{x}, \dot{\mathbf{x}}), \tag{7}$$

is a first integral of Eq. (2).

Note that by orthogonality of $(I_1(\mathbf{x}), I_2(\mathbf{x}))$ we mean orthogonality of their level sets $I_1^{-1}(a)$, $I_2^{-1}(b)$ $(a, b \in \mathbb{R})$. The meaning of the factor λ in Eq. (6) will be clear immediately. Note also that the symbol d/dt in Eq. (6) stands for the derivative along the streamlines of the $\mathbb{R}^6$ vector field $\mathbf{X}_\mathrm{L}$ given by

$$\mathbf{X}_L = \dot{\mathbf{x}}\frac{\partial}{\partial \mathbf{x}} + (\dot{\mathbf{x}} \wedge \mathbf{B})\frac{\partial}{\partial \dot{\mathbf{x}}}. \tag{8}$$

Therefore, Eq. (6) can also be written in the form

$$\frac{\ddot{\mathbf{x}} \cdot \nabla I_1}{\lambda(\nabla I_1)^2} = \frac{\partial A}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial A}{\partial \dot{\mathbf{x}}}(\dot{\mathbf{x}} \wedge \mathbf{B}). \tag{9}$$

The proof of Eq. (7) now follows.

By Eq. (5) we can write

$$\mathbf{B} = \lambda(\nabla I_1 \wedge \nabla I_2), \qquad \lambda = \lambda(\mathbf{x}), \tag{10}$$

and therefore Eq. (2) can also be written in the form

$$\ddot{\mathbf{x}} = \lambda\dot{\mathbf{x}} \wedge (\nabla I_1 \wedge \nabla I_2), \tag{11}$$

or, equivalently,

$$\ddot{\mathbf{x}} = \lambda\big\{(\dot{\mathbf{x}} \cdot \nabla I_2)\nabla I_1 - (\dot{\mathbf{x}} \cdot \nabla I_1)\nabla I_2\big\}, \tag{12}$$

and by the orthogonality of the pair $(I_1(\mathbf{x}), I_2(\mathbf{x}))$ we get from Eq. (12)

$$\ddot{\mathbf{x}} \cdot \nabla I_1 = \lambda(\dot{\mathbf{x}} \cdot \nabla I_2)(\nabla I_1)^2, \tag{13}$$

that is

$$\frac{\ddot{\mathbf{x}} \cdot \nabla I_1}{\lambda(\nabla I_1)^2} = \dot{\mathbf{x}} \cdot \nabla I_2 = \frac{d}{dt}(I_2), \tag{14}$$

and by Eq. (6) Eq. (14) becomes

$$\frac{dA}{dt} = \frac{dI_2}{dt}, \tag{15}$$

that is, $I_2 - A$ is a first integral of Eq. (2) (or of the vector field in (8)), as we desired to prove.

As an application of this result, consider that $\mathbf{x} = (x, y, z)$ and that $(I_1 = z, I_2 = I_2(x, y))$ are two first integrals of $\mathbf{B}$. It is immediate to show that the magnetic induction vector field $\mathbf{B} = \nabla(z) \wedge \nabla(I_2(x, y))$ is parallel to the x–y plane (note that $\lambda = 1$) and that $(z, I_2(x, y))$ are orthogonal functions.

On the other hand, Eq. (6) holds, since $I_1 = z$ implies

$$\frac{\ddot{\mathbf{x}} \cdot \nabla I_1}{(\nabla I_1)^2} = \ddot{z}, \tag{16}$$

and therefore $A = \dot{z}$. Accordingly, the first integral I in Eq. (7) becomes

$$I = I_2(x, y) - \dot{z}. \tag{17}$$

Note that vector fields $\mathbf{B}$ of type

$$\mathbf{B} = \lambda \nabla z \wedge \nabla I_2(x, y), \tag{18}$$

include the magnetic induction created by a finite set of parallel wires parallel to the z-axis. In fact [1] $\mathbf{B}$ is in this case given by

$$\mathbf{B} = \sum_{i=1}^{N} \frac{J_i(y - y_i, -(x - x_i), 0)}{(x - x_i)^2 + (y - y_i)^2}, \tag{19}$$

$(x_i, y_i, 0)$ standing for the intersections of the straight line wires with the $z = 0$ plane and J_i for the current intensities flowing along these wires.

On the other hand, $I_1 = z$ and

$$I_2(x, y) = -\frac{1}{2} \sum_{i=1}^{N} J_i \ln\bigl((x - x_i)^2 + (y - y_i)^2\bigr), \tag{20}$$

are independent first integrals of the vector field (19).

Note that it is not difficult to get I_2, since $\mathbf{B}$ can be considered to be a plane divergence-free vector field. For these vector fields [2] it is a classical result that a first integral (our I_2) of them can always be found (via quadratures). It is precisely using this classical result that the function I_2 in (20) has been obtained from the expression of $\mathbf{B}$ in Eq. (19).

Using the first integrals $I_1 = z$ and I_2 of $\mathbf{B}$ (see Eqs. (19) and (20)) it is straightforward to check that Eq. (18) holds (with $\lambda = 1$), as we desired to prove.

Another application of the first integral I in (7) appears in Section 4, in relation with the magnetic induction $\mathbf{B}$ created by circular wires.

We now prove, in Sections 3 and 4, some unreachability results when electric charges move in the magnetic field created by certain configurations of straight line or circular wires.

3. Unreachability in the magnetic field of parallel wires

We now show that an electric charge under the action of the magnetic field created by a finite number of straight line parallel wires W_i $(i = 1, \ldots, N)$, will never reach the wires. This property holds for any initial condition $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$, $\mathbf{x}_0 \in \mathbb{R}^3 - \bigcup_{i=1}^{N} W_i$.

As a consequence of this fact the set of parallel wires constituting a power line [3] remain practically uncharged when the wires are surrounded by an atmosphere of positive or negative ions. This neutrality is, in its turn, important since charged wires would attract or repel via Coulombian forces tending to destroy the parallel wire configuration.

Note that although direct current transmission is the exception, rather than the rule, in power transmission, in a number of applications HVDC (high-voltage direct current) [4] is often the preferred option, as in:

- Undersea cables;
- Endpoint-to-endpoint long-haul bulk power transmission without intermediate taps, for example, in remote areas;
- Interconnecting unsynchronized AC systems.

The case of an infinite sequence of wires can be studied in a similar way, by substituting in Eqs. (22) and (23) the finite sum $\sum_{i=1}^{N}$ by infinite converging series. This, of course, requires some restrictions on J_i and $r_i = \sqrt{x_i^2 + y_i^2}$.

Consider the magnetic induction $\mathbf{B}$ created by the wires [1]:

$$\mathbf{B} = \sum_{i=1}^{N} J_i \frac{(y - y_i, -(x - x_i), 0)}{(x - x_i)^2 + (y - y_i)^2},$$

$$W_i = (x = x_i,\, y = y_i,\, z = z), \tag{21}$$

where J_i is current intensity flowing across the W_i wire. This vector field possesses the two first integrals:

$$I_1 = z,$$
$$I_2 = -\frac{1}{2}\sum_{i=1}^{N} J_i \ln\bigl((x-x_i)^2 + (y-y_i)^2\bigr). \tag{22}$$

It was shown in Section 2 (see Eqs. (17) and (20)) that the function

$$I = I_2 - \dot{z}$$
$$= -\frac{1}{2}\sum_{i=1}^{N} J_i \ln\bigl((x-x_i)^2 + (y-y_i)^2\bigr) - \dot{z}, \tag{23}$$

is a first integral of the Newton–Lorentz equations (2) when $\mathbf{B}$ is given by Eq. (21).

Note that when $N > 1$ angular momentum L_z is *not* conserved since the physical system is not symmetric under rotations around the z-axis.

Therefore, for initial conditions $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$, $\mathbf{x}_0 \notin \bigcup_{i=1}^{N} W_i$, and taking into account that $\dot{\mathbf{x}}^2$ is also a first integral of Eq. (2), we get:

(3.1) $\dot{z}(t)$ is bounded;

(3.2) $-\frac{1}{2}\sum_{i=1}^{N} J_i \ln\bigl((x(t)-x_i)^2 + (y(t)-y_i)^2\bigr) - \dot{z}(t) = I(\mathbf{x}_0, \dot{\mathbf{x}}_0)$.

Therefore, the function $I_2(\mathbf{x}(t))$ would be bounded. But I_2 is unbounded (see Eq. (22)) when $\mathbf{x}(t)$ approaches one of the wires indefinitely. This is a contradiction, and we conclude that the wires are unreachable.

4. Unreachability in the magnetic field of circular wires

We prove that a charged particle moving in $\mathbb{R}^3$ under the action of a magnetic field $\mathbf{B}$ created by a finite number of planar circular wires W_i, $i = 1, \ldots, N$ (their planes being parallel and their centers lying on a straight line L orthogonal to the planes) will never reach the wires. This fact holds for any initial conditions $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$, $\mathbf{x}_0 \in \mathbb{R}^3 - \bigcup_{i=1}^{N} W_i$.

Indeed, in cylindrical coordinates (r, ϕ, z), with L acting as z-axis, since $\mathbf{S} = \partial_\phi$ is a symmetry of $\mathbf{B}$ ($L_\mathbf{S}\mathbf{B} = 0$, $L_\mathbf{S}$ standing for the Lie derivative along the streamlines of $\mathbf{S}$ [6]), and as ∂_ϕ and $\mathbf{B}$ are divergence-free, by a well-known result [5,6]

$$(\partial_\phi \wedge \mathbf{B}) \cdot d\mathbf{x} = dI_2, \tag{24}$$

where I_2 is a non-trivial first integral of $\mathbf{B}$ when ∂_ϕ and $\mathbf{B}$ are non-parallel and $\wedge$ stands for the standard vector product in $\mathbb{R}^3$.

Note that Eq. (24) is linear in $\mathbf{B}$, and therefore I_2 has the remarkable property of being given by

$$I_2 = \sum_{i=1}^{N} I_2^i, \tag{25}$$

I_2^i being defined by

$$\bigl(\partial_\phi \wedge \mathbf{B}^i\bigr) \cdot d\mathbf{x} = dI_2^i, \tag{26}$$

$\mathbf{B}^i$ being the magnetic induction vector field created by each of the wires W_i.

We write now Eq. (24) in cylindrical coordinates

$$\det\begin{pmatrix} dr & r\,d\phi & dz \\ 0 & r & 0 \\ B_r & 0 & B_z \end{pmatrix} = dI_2, \tag{27}$$

since on account of the Biot–Savart law [1] $B_\phi = 0$.

We can also write Eq. (27) in the form

$$rB_z = \frac{\partial I_2}{\partial r}, \qquad -rB_r = \frac{\partial I_2}{\partial z}. \tag{28}$$

We now compute $\mathbf{B} = \operatorname{rot}\mathbf{A}$ taking into account that [1] $\mathbf{A} = A_\phi \mathbf{u}_\phi$ ($\mathbf{u}_\phi = \partial_\phi / r$) and that the expression of $\operatorname{rot}\mathbf{A}$ in cylindrical coordinates is

$$\mathbf{B} = \operatorname{rot}\mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{u}_r & r\mathbf{u}_\phi & \mathbf{u}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & rA_\phi & 0 \end{vmatrix}, \tag{29}$$

obtaining

$$B_r = -\frac{1}{r}\frac{\partial}{\partial z}(rA_\phi), \qquad B_\phi = 0,$$
$$B_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi), \tag{30}$$

and therefore, by Eq. (28) we get $I_2 = rA_\phi$. Note that $(\mathbf{u}_r, \mathbf{u}_\phi, \mathbf{u}_z)$ stand for the standard orthonormal basis associated to the cylindrical coordinates.

For a circular wire centered at $(0, 0, 0)$ and radius a_i on which a current J_i ($J_i \neq 0$) is flowing, the expression of A_ϕ is known [1] in terms of elliptic integrals and hence we get

$$I_2^i = rA_\phi^i = \frac{4J_i\sqrt{a_i r}}{k_i}\left[\left(1-\frac{k_i^2}{2}\right)K\left(k_i^2\right) - E\left(k_i^2\right)\right], \tag{31}$$

K, E and k_i defined by

$$K\left(k^2\right) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}},$$

$$E\left(k^2\right) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 x}\,dx,$$

$$k_i^2 = \frac{4a_i r}{(a_i+r)^2+z^2}. \tag{32}$$

When the center of the circular wire is $(0,0,z_i)$ I_2 is given by equations similar to (31) and (32) obtained by substituting in these formulae k_i^2 by $\tilde{k}_i^2$ given by

$$\tilde{k}_i^2 = \frac{4a_i r}{(a_i+r)^2+(z-z_i)^2}. \tag{33}$$

Note that the functions $\tilde{k}_i^2$, K and E have the following properties:

(4.1) $(\tilde{k}_i^2)|_{z\text{-axis}} = 0$;
(4.2) $(\tilde{k}_i^2)|_{\text{circular wire } (r=a_i,\, z=z_i)} = 1$;
(4.3) $\lim_\infty \tilde{k}_i^2 = 0$;
(4.4) $\tilde{k}_i^2 \leqslant 1$ and $\tilde{k}_i^2 = 1$ iff $r = a_i$, $z = z_i$;
(4.5) $K(k^2)$ and $E(k^2)$ are analytic functions on any interval $k^2 \leqslant m$ $(m<1)$ [7];
(4.6) $\lim_{\tilde{k}_i^2\to 1^-} K(\tilde{k}_i^2) = +\infty$, $\lim_{\tilde{k}_i^2\to 1^-} E(\tilde{k}_i^2) = 1$ and $K(0) = E(0) = \pi/2$.

As a consequence of these properties it follows that I_2 is a global function on $\mathbb{R}^3 - \bigcup_{i=1}^N W_i$. The key of the proof is that the term in (31) $(1-k_i^2/2)K(k_i^2) - E(k_i^2)$ vanishes for $k_i = 0$.

On the other hand (recall property (4.2)) we get from Eq. (31):

$$\lim_{\substack{r\to a_i\\ z\to z_i}} I_2^i = \begin{cases} +\infty & \text{when } J_i > 0,\\ -\infty & \text{when } J_i < 0,\end{cases}$$

$$\lim_{\substack{r\to a_i\\ z\to z_i}} I_2 = \lim_{\substack{r\to a_i\\ z\to z_i}} \left(\sum_{i=1}^N I_2^i\right) = \infty. \tag{34}$$

Let us first apply Eq. (14) to $I_1 = \phi$, $I_2 = I_2(r,z)$ in order to get the first integral $I_2(r,z) + r^2\dot{\phi}$ of Eq. (2), where $I_2 = \sum_{i=1}^N I_2^i$ and I_2^i is given by Eq. (31).

In Cartesian coordinates we have

$$\phi = \arctan\frac{y}{x}, \qquad \nabla\phi = \frac{1}{r^2}(-y,x,0),$$
$$(\nabla\phi)^2 = \frac{1}{r^2}, \qquad r^2 = x^2+y^2, \tag{35}$$

and then we get

$$\nabla\phi \wedge \nabla I_2 = \frac{1}{r}\frac{\partial I_2}{\partial z}\mathbf{u}_r - \frac{1}{r}\frac{\partial I_2}{\partial r}\mathbf{u}_z, \tag{36}$$

which by Eq. (28) becomes:

$$\nabla\phi \wedge \nabla I_2 = -B_z\mathbf{u}_z - B_r\mathbf{u}_r = -\mathbf{B}. \tag{37}$$

Since the couple (ϕ, I_2) satisfies Eqs. (5) and (10) with $\lambda = -1$, by virtue of Eq. (35), Eq. (14) can be easily transformed into the form

$$\frac{dl_z}{dt} = -\frac{dI_2}{dt}, \qquad l_z = x\dot{y} - y\dot{x}. \tag{38}$$

Therefore, $I_2 + l_z$ is a first integral of Eq. (2). In cylindrical coordinates the first integral is

$$I_2(r,z) + r^2\dot{\phi}. \tag{39}$$

Note that this first integral is a kind of generalized angular momentum around the z-axis.

Unreachability immediately follows from the presence of the first integrals $\dot{\mathbf{x}}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2$ and $I_2(r,z) + r^2\dot{\phi}$.

Indeed, the constancy of $\dot{\mathbf{x}}^2$ implies that $r\dot{\phi}$ is bounded. Therefore, near the wire $(r = a_i, z = z_i)$ the term $r^2\dot{\phi}$ is also bounded.

On the other hand, the equation $I_2(r,z) + r^2\dot{\phi} = I_2(r_0,z_0) + r_0^2\dot{\phi}_0$ $(r_0 \neq a_i$ and $z_0 \neq z_i)$ implies the boundedness of I_2 near the wire $(r = a_i, z = z_i)$, contradicting Eq. (34).

5. Discussion of Eq. (6)

We now prove that if I_2 is a first integral of $\mathbf{B}$ and I_1 is defined by [2]

$$i_{\mathbf{B}}\Omega_2 = dI_1, \qquad \Omega_2 = \frac{i_{\nabla I_2}(dx\wedge dy\wedge dz)}{\|\nabla I_2\|^2}, \tag{40}$$

that is

$$\nabla I_1 = \frac{\nabla I_2 \wedge \mathbf{B}}{\|\nabla I_2\|^2}, \tag{41}$$

then I_1 is also a first integral of $\mathbf{B}$ and we get:

(i) $\mathbf{B} = \nabla I_1 \wedge \nabla I_2$, that is $\lambda = 1$ in Eq. (10).
(ii) The 1-form $w_1 = \mathbf{B}\,d\mathbf{x}$ is closed (provided that $\operatorname{rot}\mathbf{B} = \mathbf{0}$). Therefore (locally), $dI_3 = \mathbf{B}\,d\mathbf{x}$.
(iii) The functions (I_1, I_2, I_3) define an orthogonal (local) coordinate system in $\mathbb{R}^3$.
(iv) Eq. (6) holds ($\lambda = 1$) if and only if the vector field $\partial/\partial I_1 = \partial_{I_1}$ is an Euclidean symmetry of $\mathbf{B}$.

First of all, note that by Eq. (41) the level sets of I_1 and I_2 are orthogonal ($\nabla I_1 \nabla I_2 = 0$). On the other hand, since $\operatorname{rot}\mathbf{B} = \mathbf{0}$ on ($\mathbb{R}^3$-wires), the function I_3 in (ii) is (locally) well defined. The orthogonality of the gradients of (I_1, I_2, I_3) is immediate from Eq. (41), and we obviously get $\operatorname{rank}(\nabla I_1, \nabla I_2, \nabla I_3) = 3$. Therefore, the set of functions (I_1, I_2, I_3) forms an orthogonal (local) coordinate system.

The proof of (iv) is based on projecting Eq. (6) on the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ associated to the functions (I_1, I_2, I_3).

Remembering the useful equalities:

$$\partial_{I_i} = \frac{\nabla I_i}{\|\nabla I_i\|^2},$$

$$\ddot{\mathbf{x}} = \sum_{i=1}^{3} a^i(\mathbf{I}, \dot{\mathbf{I}})\mathbf{u}_i, \quad \mathbf{I} = (I_1, I_2, I_3) \tag{42}$$

Eq. (6) becomes

$$\frac{a^1(\mathbf{I}, \dot{\mathbf{I}})}{\|\nabla I_1\|} = \sum_{i=1}^{3}\big(A_{I_i}(\mathbf{I}, \dot{\mathbf{I}})\dot{I}_i + A_{\dot{I}_i}(\mathbf{I}, \dot{\mathbf{I}})\ddot{I}_i\big), \tag{43}$$

where the subscripts denote, as usual, partial differentiation with respect to the corresponding variable.

Now, via straightforward computations we get

$$\dot{\mathbf{x}} = \sum_{i=1}^{3} \frac{\partial \mathbf{x}}{\partial I_i}\dot{I}_i,$$

$$\ddot{\mathbf{x}} = \sum_{i,j=1}^{3}\left(\frac{\partial^2 \mathbf{x}}{\partial I_i \partial I_j}\dot{I}_i\dot{I}_j + \frac{\partial \mathbf{x}}{\partial I_i}\ddot{I}_i\right)$$

$$= \sum_{i,j,k=1}^{3}\left(\frac{1}{\|\nabla I_i\|}\Gamma^i{}_{jk}\dot{I}_j\dot{I}_k\mathbf{u}_i + \frac{1}{\|\nabla I_i\|}\ddot{I}_i\mathbf{u}_i\right), \tag{44}$$

$\Gamma^i{}_{jk}$ standing for the Christoffel symbols [8], defined by

$$\Gamma^i{}_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{jm}}{\partial I_k} + \frac{\partial g_{km}}{\partial I_j} - \frac{\partial g_{jk}}{\partial I_m}\right),$$

$$g_{ij} = 0 \quad \text{if } i \neq j \text{ and } g_{ii} = \frac{1}{\|\nabla I_i\|^2}. \tag{45}$$

Therefore, Eq. (43) becomes

$$\frac{a^1(\mathbf{I}, \dot{\mathbf{I}})}{\|\nabla I_1\|} = \frac{1}{\|\nabla I_1\|^2}\left(\ddot{I}_1 + \sum_{j,k=1}^{3}\Gamma^1{}_{jk}\dot{I}_j\dot{I}_k\right)$$

$$= \sum_{i=1}^{3}\big(A_{I_i}(\mathbf{I}, \dot{\mathbf{I}})\dot{I}_i + A_{\dot{I}_i}(\mathbf{I}, \dot{\mathbf{I}})\ddot{I}_i\big). \tag{46}$$

Since Eq. (46) must hold identically in $\dot{I}_i$ and $\ddot{I}_i$ we get from (46)

$$A = \frac{\dot{I}_1}{\|\nabla I_1\|^2},$$

$$\Gamma^1_{11} = \|\nabla I_1\|^2 \frac{\partial}{\partial I_1}\left(\frac{1}{\|\nabla I_1\|^2}\right),$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}\|\nabla I_1\|^2 \frac{\partial}{\partial I_2}\left(\frac{1}{\|\nabla I_1\|^2}\right),$$

$$\Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{2}\|\nabla I_1\|^2 \frac{\partial}{\partial I_3}\left(\frac{1}{\|\nabla I_1\|^2}\right),$$

$$\Gamma^1_{22} = \Gamma^1_{23} = \Gamma^1_{32} = \Gamma^1_{33} = 0. \tag{47}$$

The reader can check now that Eqs. (45) and (47) imply

$$\frac{\partial g_{ii}}{\partial I_1} = 0, \quad i = 1, 2, 3,$$

and therefore

$$L_{\partial_{I_1}}(g) = 0. \tag{48}$$

That is, ∂_{I_1} is a Killing vector field of the Euclidean metric g defined in Eq. (45).

On the other hand,

$$\mathbf{B} = \|\mathbf{B}\|\mathbf{u}_3 = \|\nabla I_1\|\|\nabla I_2\|\|\nabla I_3\|\partial_{I_3}, \tag{49}$$

and since

$$\frac{\partial}{\partial I_1}\big(\|\nabla I_i\|^2\big) = 0$$

we get

$$[\partial_{I_1}, \mathbf{B}] = 0. \tag{50}$$

Eqs. (48) and (50) show that when (6) holds then ∂_{I_1} is an Euclidean symmetry of **B**.

6. Conclusion

For certain realistic configurations of power lines we have shown that the wires creating a magnetic field **B** are unreachable for electric charges which move under the action of **B**. This result has been obtained by using a remarkable relation between first integrals of the Newton–Lorentz equation and first integrals of the corresponding magnetic field.

As an application of the study in Section 5 consider the magnetic field created by $N > 1$ straight-line wires concurrent at $(0, 0, 0)$. We can apply to this physical system the results of this section since $I_2 = x^2 + y^2 + z^2$ is a first integral of **B** and conclude that Eq. (6) cannot hold because for $N > 1$ this system of wires is free from continuous groups of Euclidean symmetries (although it possesses a radial symmetry). Remember that continuous Euclidean symmetries are just translations, rotations and roto-translations and that an Euclidean symmetry of the wires is, automatically (via the Biot–Savart law), a symmetry of the magnetic field created by the wires.

It is an open problem to prove, or disprove, unreachability for wires geometrically located in $\mathbb{R}^3$ in positions different from that considered in this Letter or which are no longer straight lines or circles. An easier question is: can unreachability be proved for wires $\tilde{W}_i$ which are perturbations of the wires W_i studied in this Letter? By the term perturbation we mean a small deformation of the original wire. This deformation is free and need not have any particular symmetry. For instance, if W_i is a circular wire then a perturbation $\tilde{W}_i$ could be an ellipse with very small eccentricity.

The reader can have a look at Ref. [9] where Ulam pointed out the numerous open problems arising in the study of the magnetic fields created by electric currents flowing in wires such as the existence of ergodicity and knotted magnetic orbits.

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Invariant sets of second order differential equations

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Abstract

The partial differential equations defining the invariant sets of Newtonian, second order, analytic differential equations, are obtained and discussed. An example is given for which *no* codimension one invariant sets of type $x^3 = g(x^1, x^2)$ exist. Invariant sets for relativistic equations of motion and for equations of motion of the rays of light in geometrical optics are also considered, some examples are given and it is shown that the invariant sets for these equations must be planes; this is in strong contrast with the invariant sets of ordinary Newton equations $\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ allowing the presence of "curved" invariant surfaces. All these results are apparently new.

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1. Introduction

A set $I \subset \mathbb{R}^n$ is called invariant under the dynamical system $\mathbf{X}$

$$\left.\begin{aligned}
&\mathbf{X}\colon \frac{dx^i}{dt} = X^i(\mathbf{x}),\\
&\mathbf{x} \in \mathbb{R}^n,\\
&\mathbf{X} = X^i \partial_i,\\
&\partial_i = \frac{\partial}{\partial x^i},\\
&x^i = \text{Cartesian coordinates},\\
&1 \leqslant i \leqslant n,
\end{aligned}\right\} \tag{1}$$

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when for initial conditions $\mathbf{x}_0 \in I$ the solution of Eq. (1) $\mathbf{x} = \boldsymbol{\varphi}(t; \mathbf{x}_0)$, $t \in E(0)$, corresponding to $\mathbf{x}_0$, lies in I, that is

$$\left.\begin{array}{l} \boldsymbol{\varphi}(t; \mathbf{x}_0) \subset I, \\ \forall t \in E(0), \end{array}\right\} \tag{2}$$

$E(0)$ standing for the open interval around $t = 0$ for which $\boldsymbol{\varphi}(t; \mathbf{x}_0)$ is defined.

It is well known [1] that when I is a differential submanifold defined by the set of zeros of several C^∞ or C^{w} functions $f_i(\mathbf{x})$ $(i = 1, 2, \ldots, r < n)$ and

$$\operatorname{rank} \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_r \end{pmatrix}_{|I} = r, \tag{3}$$

then the invariance of I under (1) is equivalent to the equations

$$\left.\begin{array}{l} \dot{f}_1 = L_{\mathbf{X}}(f_1)_{|I} = 0, \\ \vdots \\ \dot{f}_r = L_{\mathbf{X}}(f_r)_{|I} = 0, \end{array}\right\} \tag{4}$$

$L_{\mathbf{X}}(f_i)_{|I}$ standing for the restriction of the function $L_{\mathbf{X}}(f_i)$ to the set I, ∇f_i representing the gradient vector field $(\partial f_i/\partial x^1, \ldots, \partial f_i/\partial x^n)$ and $L_{\mathbf{X}}(f_i)$ being $(\nabla f_i \cdot \mathbf{X})$. $L_{\mathbf{X}}(f_i)$ is called the Lie derivative of f_i along the streamlines of $\mathbf{X}$ [1].

Invariant sets were discussed by Fermi [2–4] and Benettin et al. [5] in relation with the problem of whether or not they disappear when an integrable Hamiltonian system $\mathbf{X}_{H_0}$ is perturbed: $\mathbf{X}_{H_0+H_{\mathrm{pert}}}$. For similar stability questions see Refs. [6–12].

For invariant manifolds passing through either a hyperbolic or non-hyperbolic zero of $\mathbf{X}$ (and its behaviour under perturbations), and for relations between invariant hyperplanes and Darboux theory of integrability for polynomial vector fields see Refs. [13–16].

We are interested in this Letter in the study of invariant sets of second order differential equations, specifically invariant sets of Newtonian equations. These invariant sets will be defined by:

$$\left.\begin{array}{l} f_1(x^1, \ldots, x^n) = 0, \\ \vdots \\ f_r(x^1, \ldots, x^n) = 0, \\ \dot{f}_1 = \nabla f_1 \cdot \dot{\mathbf{x}} = 0, \\ \vdots \\ \dot{f}_r = \nabla f_r \cdot \dot{\mathbf{x}} = 0, \\ \dot{\mathbf{x}} = (\dot{x}^1, \ldots, \dot{x}^n), \\ \operatorname{rank}(\nabla f_1, \ldots, \nabla f_r) = r < n. \end{array}\right\} \tag{5}$$

The first r equations in (5) define, via the rank condition in (5), a codimension r differential submanifold in the configuration space $\mathbb{R}^n$ of the variables $(x^1, \ldots, x^n)$. The set of all Eqs. (5) defines a codimension $2r$ differential submanifold of phase space.

Note that the first of Eqs. (5), $\{f_1 = 0, \ldots, f_r = 0\}$, have little to do with what are called in Mechanics “holonomous constraints” [17].

Note also that invariant sets of type (5) are poorly studied. No references to them have been found except for the particular case of the equation defining the geodesics in Riemannian or pseudo-Riemannian spaces.

We will always refer along this Letter to *the first* of Eqs. (5) when speaking about the invariant sets, since the second of Eqs. (5) is just a consequence of the first one obtained via differentiation with respect to the time variable t. Note also that the second of Eqs. (5) just expresses the fact that the velocity of the unit mass is a vector tangent to the submanifold $\{f_1 = 0, \ldots, f_r = 0\}$ for every $t \in E(0)$.

Concerning Newtonian equations $\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ we only consider in this Letter the cases

$$\ddot{\mathbf{x}} = \mathbf{0}, \tag{6}$$

and

$$\left.\begin{aligned} &\ddot{x}^i = \sum_{j,k=1}^{n} a^i_{jk}(\mathbf{x})\dot{x}^j\dot{x}^k, \\ &i = 1, \ldots, n, \end{aligned}\right\} \tag{7}$$

these last equations being a generalization of the differential equations of the geodesics of a Riemannian or pseudo-Riemannian space [18]. In fact, calling $g_{ij}(\mathbf{x})$ the non-degenerate metric tensor the connection coefficients $-a^i_{jk}(\mathbf{x})$ in Eq. (7) are given by

$$\left.\begin{aligned} &-a^i_{jk} = \sum_{m=1}^{n} \frac{1}{2} g^{mi}(g_{jm,k} + g_{km,j} - g_{jk,m}), \\ &g_{ij} = g_{ji}. \end{aligned}\right\} \tag{8}$$

Note that (8) implies that $a^i_{jk}(\mathbf{x}) = a^i_{kj}(\mathbf{x})$, and therefore a non-symmetric $a^i_{jk}(\mathbf{x})$ in its two lower indices *can never* be obtained through Eqs. (8) from a metric tensor.

Note that perturbing equations (6) and (7) with a "force" $\mathbf{F}(\mathbf{x})$ leads to the equations

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{9}$$

and

$$\ddot{x}^i = \sum_{j,k=1}^{n} a^i_{jk}(\mathbf{x})\dot{x}^j\dot{x}^k + F^i(\mathbf{x}). \tag{10}$$

We shall see that these perturbed equations have identical invariant sets I as the unperturbed equations (6) and (7).

The organization of the Letter is as follows:

(i) the non-linear partial differential equations for the invariant sets of type (5) for Newtonian equations of type $\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ are obtained in Section 2;
(ii) in Section 3 it is shown that the codimension one invariant sets of Eqs. (6) or (9) are hyperplanes in the configuration space (or intersection of hyperplanes when $r = 2, 3, \ldots$);
(iii) for differential equations of type (7) the partial differential equations of its invariant sets of type (5) are obtained and examples of equations of type (7) with or without invariant sets, of type (5), are studied (Section 4);
(iv) in Section 5 we get the invariant sets for the relativistic equations $d(\gamma\dot{\mathbf{x}})/dt = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$, $\gamma = (1 - \dot{\mathbf{x}}^2)^{-1/2}$, $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ being a polynomial in $\dot{\mathbf{x}}$;
(v) finally we obtain (Section 6) the invariant sets for the geometrical optics equations $\ddot{\mathbf{x}} + 2(\nabla n \cdot \dot{\mathbf{x}})\dot{\mathbf{x}} = \nabla n/n^2$, $n(\mathbf{x})$ being the refraction index of the material medium.

Note that invariant sets of dimension one of Eqs. (10) do *always exist*. They can be obtained through elimination of t in the solutions $x^i = \varphi^i(t)$ $(i = 1, \ldots, n)$ of Eqs. (10) [19].

Note, however, that higher-dimensional invariant sets of type (5) for $2 \leqslant r \leqslant n-1$ do not necessarily exist in general (even locally) since they have a very specific form (they are invariant sets in the phase space that are obtained from the configuration space submanifold $\{f_1 = 0, \ldots, f_r = 0\}$). A well known example is provided by totally geodesic submanifolds of Riemannian manifolds. In Ref. [20] examples of Riemannian manifolds without (local) totally geodesic submanifolds are given. In fact it is believed that "general" Riemannian manifolds do not have any r-dimensional totally geodesic submanifolds for $2 \leqslant r \leqslant n-1$ [20].

We finish this section by remarking that the treatment of Section 2 can be applied to differential equations of the form

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}), \tag{11}$$

$\mathbf{F}$ standing for a polynomial in $\dot{\mathbf{x}}$ of degree $N \in \mathbb{N}$ (see Eqs. (6) and (7) above, corresponding to the values $N = 0$ and $N = 2$, respectively).

Note, finally, that the techniques developed in this Letter have not very much to do with the techniques used in Ref. [21], which involve Frechet differential calculus and the theory of tangent sets. Moreover the relativistic equations (see Section 5) and the geometrical optics equations (see Section 6) are not considered in Ref. [21].

2. Invariant sets of Newtonian equations

In this section we get the non-linear partial differential equations that must be satisfied by the functions f_i in order that the codimension $2r$ submanifold defined by Eqs. (5) be invariant under the vectorfield $\mathbf{X} = \dot{\mathbf{x}}\partial_{\mathbf{x}} + \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})\partial_{\dot{\mathbf{x}}}$ associated with the Newtonian equations of motion $\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$.

It is well known [1] that the necessary and sufficient conditions for the invariance of (5) under the solutions of the Newtonian equations are:

$$\left.\begin{aligned} &L_{\mathbf{X}}(f_i)_{|(5)} = \dot{f}_{i|(5)} = 0,\\ &L_{\mathbf{X}}(\nabla f_i \cdot \dot{\mathbf{x}})_{|(5)} = 0, \end{aligned}\right\} \tag{12}$$

where $E_{|(5)} = 0$ means that E is not necessarily identically equal to zero, but E becomes zero on the manifold defined by Eqs. (5).

Now, the first set of Eqs. (12) are automatically satisfied due to the second r equations in (5); the second set of Eqs. (12) can be written in the form

$$\left.\begin{aligned} &\left[\sum_{j,k=1}^{n} f_{i,jk}\dot{x}^j\dot{x}^k + \sum_{k=1}^{n} f_{i,k}\ddot{x}^k\right]_{|(5)} = 0,\\ &f_{i,jk} = \frac{\partial^2 f_i}{\partial x^j \partial x^k}, \end{aligned}\right\} \tag{13}$$

and since $\ddot{\mathbf{x}}$ satisfies Newton equation we get the basic equation

$$\left[\sum_{j,k=1}^{n} f_{i,jk}\dot{x}^j\dot{x}^k + \sum_{k=1}^{n} f_{i,k}F^k\right]_{|(5)} = 0. \tag{14}$$

Basically Eqs. (14) are non-linear because, due to Eqs. (5) $\mathbf{x}$ and $\dot{\mathbf{x}}$ depend on the f_i and $f_{i,j}$ on the submanifold defined by Eqs. (5). Concerning Eqs. (14) we will write them more explicitly in the useful cases of Eqs. (7), (9) and (10). These three cases are studied in the following sections.

Note that the expression of the system dynamics on the invariant set is given by the restriction of the Hamiltonian vector field to the invariant submanifold, that is $\mathbf{X}_{|(5)}$. This gives rise to $2(n-r)$ ordinary differential equations defining a dynamical system on the invariant set defined by Eqs. (5). For example, if the phase space is $\mathbb{R}^4$

and the invariant set (in the configuration space) is the curve $y = g(x)$ then the induced dynamics is given by $\ddot{x} = F_x(x, g(x), \dot{x}, g'(x)\dot{x})$.

In this Letter we only study forces depending quadratically on $\dot{\mathbf{x}}$ but, as we have mentioned in the introduction, the mathematical treatment is the same for other polynomial dependences. We are going to illustrate this fact with an example for which F^k is linear in $\dot{\mathbf{x}}$, $n = 3$ and $r = 1$.

Let the Newtonian equations be defined by

$$\left.\begin{aligned} &\ddot{\mathbf{x}} = \dot{\mathbf{x}} \wedge \mathbf{B}\big(x^1, x^2, x^3\big), \\ &\mathbf{B}_{|x^3=0} \parallel \big(x^3\text{-axis}\big), \\ &\mathbf{x} \in \mathbb{R}^3. \end{aligned}\right\} \tag{15}$$

Let us see that these equations admit the invariant set $x^3 = 0$. Indeed, Eqs. (14) become in this case:

$$\big(\dot{x}^1 B_2 - \dot{x}^2 B_1\big)_{|x^3=0} = 0. \tag{16}$$

This last equation is satisfied since by hypothesis $\mathbf{B}$ is parallel to the x^3-axis on the plane $x^3 = 0$.

Note that the magnetic field of a dipole at (0, 0, 0) (the Earth magnetic field) is parallel to the dipole axis (North pole–South pole line) on the $x^3 = 0$ plane (Earth equatorial plane). Therefore an electric charge under this magnetic field, with initial velocity $\dot{\mathbf{x}}_0$ such that $\dot{x}_0^3 = 0$ will get trapped forever in the invariant plane $x^3 = 0$ just considered.

In ending this section, note that Eqs. (14) remain unchanged when the Newtonian equations are perturbed in the form

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{P}(\mathbf{x}, \dot{\mathbf{x}}),$$

$\mathbf{P}(\mathbf{x}, \dot{\mathbf{x}})$ satisfying

$$\nabla f_i \cdot \mathbf{P}(\mathbf{x}, \dot{\mathbf{x}})_{|(5)} = 0,$$

that is when $\mathbf{P}$ is tangent to the submanifold $\{f_1 = 0, \dots, f_r = 0\}$ on every point of it. Therefore $\ddot{\mathbf{x}} = \mathbf{F}$ and $\ddot{\mathbf{x}} = \mathbf{F} + \mathbf{P}$ have identical invariant sets of type (5) provided the perturbing force $\mathbf{P}(\mathbf{x}, \dot{\mathbf{x}})$ is tangent to $\{f_1 = 0, \dots, f_r = 0\}$. One of the referees of the Letter has raised the following problem: for which class of physical systems could one consider the perturbing force $\mathbf{P}$ tangential to the submanifold of interest?

3. Invariant sets for velocity free forces

We now get the explicit form of Eqs. (14) when the equations

$$f_1(\mathbf{x}) = 0, \quad \dots, \quad f_r(\mathbf{x}) = 0, \tag{17}$$

take the form

$$\left.\begin{aligned} &f_1 = x^1 - g^1(\tilde{\mathbf{x}}) = 0, \\ &\vdots \\ &f_r = x^r - g^r(\tilde{\mathbf{x}}) = 0, \\ &\tilde{\mathbf{x}} = \big(x^{r+1}, \dots, x^n\big). \end{aligned}\right\} \tag{18}$$

The functions g^i in (18) are obtained after applying the implicit function theorem to Eq. (17), and an eventual reordering of the variables $x^1, \dots, x^n$ so that $\operatorname{rank}(\partial(f_1, \dots, f_r)/\partial(x^1, \dots, x^r)) = r$.

Note that in general the reordering of $x^1, \dots, x^n$ can lead to more than one set of Eqs. (18) (see the example given in the next section).

We now write the invariant set (18) in phase space:

$$\left.\begin{array}{l} x^1 - g^1(\tilde{\mathbf{x}}) = 0, \\ \vdots \\ x^r - g^r(\tilde{\mathbf{x}}) = 0, \\ \dot{x}^1 - \nabla g^1 \cdot \dot{\tilde{\mathbf{x}}} = 0, \\ \vdots \\ \dot{x}^r - \nabla g^r \cdot \dot{\tilde{\mathbf{x}}} = 0. \end{array}\right\} \tag{19}$$

Note that the local equations (18), (19) and (20) become global when the functions $f_1, \ldots, f_r$ and $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ are analytic. That is, on account of analyticity [22] Eqs. (20) are equivalent to Eqs. (12), which are global, as we immediately prove.

Indeed, the basic equations (4) become, locally, $(\nabla f_i \cdot \mathbf{X})_{|N(P)\cap I} = 0$, $N(P)$ being a neighbourhood around a point P of I, and since the functions $\nabla f_i \cdot \mathbf{X}$ and the submanifold I are analytic [22] we can write:

$$(\nabla f_i \cdot \mathbf{X})_{|N(P)\cap I} = 0 \quad \Leftrightarrow \quad (\nabla f_i \cdot \mathbf{X})_{|I} = 0,$$

as we desired to prove.

Summarizing, the global invariant set (5) can be obtained via the use of the local analytic Eqs. (19).

Taking Eqs. (19) into account the basic equations (14) become

$$\left.\begin{array}{l} \left(-\displaystyle\sum_{j,k=r+1}^{n} g^1_{,jk}\dot{x}^j\dot{x}^k + \Big(\overbrace{1, 0, \ldots, 0}^{\text{first } r \text{ components}}, -\nabla g^1\Big) \cdot \mathbf{F}(\mathbf{x})\right)_{|(19)} = 0, \\ \vdots \\ \left(-\displaystyle\sum_{j,k=r+1}^{n} g^r_{,jk}\dot{x}^j\dot{x}^k + \Big(\overbrace{0, 0, \ldots, 1}^{\text{first } r \text{ components}}, -\nabla g^r\Big) \cdot \mathbf{F}(\mathbf{x})\right)_{|(19)} = 0, \\ g^\alpha_{,jk} = \dfrac{\partial^2 g^\alpha}{\partial x^j \partial x^k}, \end{array}\right\} \tag{20}$$

and since $\dot{x}^{r+1}, \ldots, \dot{x}^n$ are subjected to no constraints equations (20) imply:

$$\left.\begin{array}{l} g^i_{,jk|(19)} = g^i_{,jk} = 0, \\ i = 1, \ldots, r;\ r+1 \leqslant j,\ k \leqslant n, \end{array}\right\} \tag{21}$$

and therefore g^i must be an affine function in the $\tilde{\mathbf{x}}$ variables, that is

$$\left.\begin{array}{l} g^i = A^i_{r+1}x^{r+1} + \cdots + A^i_n x^n + A^i_0, \\ A^i_l \in \mathbb{R}. \end{array}\right\} \tag{22}$$

This affine dependence of g^i on the variables $(x^{r+1}, \ldots, x^n)$ is surely *not* new, but we have not traced it back in the literature.

The independence of $\dot{x}^{r+1}, \ldots, \dot{x}^n$ in Eqs. (20) also implies the following equations:

$$\left.\begin{aligned} &\Big(\big(\overbrace{1, 0, \ldots, 0}^{r\ \text{components}}, -\nabla g^1\big) \cdot \mathbf{F}(\mathbf{x})\Big)_{|(19)} = 0, \\ &\vdots \\ &\Big(\big(\overbrace{0, 0, \ldots, 1}^{r\ \text{components}}, -\nabla g^r\big) \cdot \mathbf{F}(\mathbf{x})\Big)_{|(19)} = 0. \end{aligned}\right\} \tag{23}$$

Eqs. (23) mean that $\mathbf{F}(\mathbf{x})$ must be tangent to the configuration space submanifold defined by $x^i = g^i(\tilde{\mathbf{x}})$, $i = 1, \ldots, r$ and Eqs. (22).

Note that Eqs. (23) are satisfied when $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ (force-free motion of the particle). On the other hand, Eqs. (23) are incompatible when $\mathbf{F}(\mathbf{x})$ is not tangent to any codimension r affine submanifold of $\mathbb{R}^n$; this is the case of ($n = 2$, $r = 1$)

$$\mathbf{F}(\mathbf{x}) = \big(-x^2, x^1\big). \tag{24}$$

As a final, clarifying and well-known example, let $n = 3$, $r = 1$, and $\mathbf{F}(\mathbf{x})$ be a central force, that is

$$\left.\begin{aligned} &\mathbf{F}(\mathbf{x}) \parallel \mathbf{x}, \\ &\text{or } F^i = h \cdot x^i, \end{aligned}\right\} \tag{25}$$

h standing for a function of (x^1, x^2, x^3).

Let $f_1(x^1, x^2, x^3)$ be a linear function of (x^1, x^2, x^3), then Eqs. (5) become:

$$\left.\begin{aligned} &A_1x^1 + A_2x^2 + A_3x^3 = 0, \\ &A_1\dot{x}^1 + A_2\dot{x}^2 + A_3\dot{x}^3 = 0. \end{aligned}\right\} \tag{26}$$

Note that the first of Eqs. (26) represents (in configuration space) a plane through the origin.

It is straightforward to see that Eqs. (14) reduce in this case to

$$\left(\sum_{i=1}^{3} A_i\ddot{x}^i\right)_{|(26)} = \left(h \cdot \sum_{i=1}^{3} A_i x^i\right)_{|(26)} = 0, \tag{27}$$

where the above system of equations are satisfied due to (26). Therefore any plane through the origin is an invariant set (in the configuration space) when the forces acting on the material particles are central.

The Newtonian forces

$$\left.\begin{aligned} &F^i = \frac{kx^i}{[(x^1)^2 + (x^2)^2 + (x^3)^2]^{3/2}}, \\ &k \in \mathbb{R}, \end{aligned}\right\} \tag{28}$$

are central and the above result is, of course, valid for them.

4. Invariant sets for forces quadratic in the velocity

We get now the partial differential equations obtained when the basic equations (14) are applied to forces of the following type

$$F^k = \sum_{l,m=1}^{n} a^k_{lm}(\mathbf{x})\dot{x}^l \dot{x}^m. \tag{29}$$

Indeed, substituting (29) into Eqs. (14) leads to the equations

$$\left.\begin{aligned}&\left[\sum_{j,k=1}^{n} f_{i,jl}\dot{x}^j\dot{x}^l + \sum_{k=1}^{n} f_{i,k}\left(\sum_{l,m=1}^{n} a^k_{lm}\dot{x}^l\dot{x}^m\right)\right]_{|(5)} = 0,\\ &1 \leqslant i \leqslant r,\end{aligned}\right\} \tag{30}$$

that is:

$$\left(\sum_{j,l=1}^{n}\left[f_{i,jl} + \sum_{k=1}^{n} f_{i,k}a^k_{jl}\right]\dot{x}^j\dot{x}^l\right)_{|(5)} = 0. \tag{31}$$

Using now the canonical local form (18) of Eq. (17) we get:

$$\left.\begin{aligned}&\left(\sum_{j,l=1}^{n}\left[-g^i_{,jl} + \sum_{k=1}^{n} f_{i,k}a^k_{jl}\right]\dot{x}^j\dot{x}^l\right)_{|(19)} = 0,\\ &i = 1,\dots,r,\end{aligned}\right\} \tag{32}$$

$f_{i,k}$ standing for the vector

$$(f_{i,1}, f_{i,2}, \dots, f_{i,n}) = \Big(\overbrace{0, 0, \dots, \underbrace{1}_{i\text{th place}}, 0, \dots, 0}^{\text{first } r \text{ components}}, \overbrace{-\nabla g^i}^{\text{last } n-r \text{ components}}\Big). \tag{33}$$

Note that in Eq. (33) i is fixed ($1 \leqslant i \leqslant r$).

Finally, expressing $\dot{x}^1, \dots, \dot{x}^r$ linearly (see Eqs. (19)) in terms of $\dot{x}^{r+1}, \dots, \dot{x}^n$ we get via Eqs. (32):

$$\left.\begin{aligned}&-g^i_{,jl} + \hat{a}^i_{jl} = 0,\\ &r+1 \leqslant j,\ l \leqslant n;\ 1 \leqslant i \leqslant r,\end{aligned}\right\} \tag{34}$$

$\hat{a}^i_{jl}$ standing for the coefficient of $\dot{x}^j\dot{x}^l$ ($r+1 \leqslant j, l \leqslant n$) after substituting in the term

$$\sum_{j,k,l=1}^{n} f_{i,k}a^k_{jl}(\mathbf{x})\dot{x}^j\dot{x}^l, \tag{35}$$

of Eqs. (32), $x^1, \dots, x^r, \dot{x}^1, \dots, \dot{x}^r$ by its values given by Eqs. (19).

Note that $\hat{a}^i_{jl}$ is, in general, non-linear in the functions g^i and its first derivatives since both $f_{i,k}$, $a^k_{jl|(19)}$ and $\dot{x}^j\dot{x}^l{}_{|(19)}$ contain, respectively, first-order derivatives of g^i (see Eq. (35)), the functions g^i themselves when $a^k_{jl}(\mathbf{x})$ are not free from $x^1, \dots, x^r$, and products of ∇g^i arising from terms of type $(\dot{x}^1)^2$, $\dot{x}^1\dot{x}^2$, etc. Eqs. (34) are the system of partial differential equations for the $g^1(\tilde{\mathbf{x}}), \dots, g^r(\tilde{\mathbf{x}})$ we were looking for.

We now give an example showing that (in general) the non-linear system (34) can be incompatible.

In fact, let $n=3$, $r=1$ and consider a Newtonian differential equation quadratic in $\dot{\mathbf{x}}$ of type (7) with a^i_{jk} defined by:

$$\left.\begin{array}{lll} a^1_{11}=0, & a^2_{11}=0, & a^3_{11}=\phi'(x^3), \\ a^1_{12}=a^1_{21}=0, & a^2_{12}=a^2_{21}=0, & a^3_{12}=a^3_{21}=0, \\ a^1_{13}=a^1_{31}=-\phi'(x^3), & a^2_{13}=a^2_{31}=0, & a^3_{13}=a^3_{31}=0, \\ a^1_{22}=0, & a^2_{22}=0, & a^3_{22}=\phi'(x^3), \\ a^1_{23}=a^1_{32}=0, & a^2_{23}=a^2_{32}=-\phi'(x^3), & a^3_{23}=a^3_{32}=0, \\ a^1_{33}=0, & a^2_{33}=0, & a^3_{33}=-\phi'(x^3), \end{array}\right\} \tag{36}$$

$\phi(x^3)$ being an arbitrary smooth function of x^3, different from zero.

Let us now prove that Eqs. (7) do not posses invariant sets of type $x^3=g(x^1,x^2)$ when $a^i_{jk}(\mathbf{x})$ are given by Eqs. (36).

It is straightforward to check that Eqs. (32) become in this case:

$$\begin{aligned} &\Big(\big(-g_{,x^1x^1}+\phi'(x^3)\big)(\dot{x}^1)^2-2g_{,x^1x^2}\dot{x}^1\dot{x}^2+2\phi'(x^3)g_{,x^1}\dot{x}^1\dot{x}^3+\big(-g_{,x^2x^2}+\phi'(x^3)\big)(\dot{x}^2)^2 \\ &\quad+2\phi'(x^3)g_{,x^2}\dot{x}^2\dot{x}^3-\phi'(x^3)(\dot{x}^3)^2\Big)\Big|_{\substack{x^3=g \\ \dot{x}^3=g_{,x^1}\dot{x}^1+g_{,x^2}\dot{x}^2}}=0 \end{aligned} \tag{37}$$

and Eqs. (34) are now

$$\left.\begin{array}{l} -g_{,x^1x^1}+\phi'(x^3)+\phi'(x^3)g^2_{,x^1}=0, \\ -g_{,x^1x^2}+\phi'(x^3)g_{,x^1}g_{,x^2}=0, \\ -g_{,x^2x^2}+\phi'(x^3)+\phi'(x^3)g^2_{,x^2}=0. \end{array}\right\} \tag{38}$$

Let us now discuss Eqs. (38) according to the value of $\phi'(x^3)$.

(i) If $\phi'(x^3)$ is not a constant function the second of Eqs. (38) is incompatible.

(ii) When $\phi'(x^3)=a\in\mathbb{R}$, $a\neq 0$, Eqs. (38) become

$$\left.\begin{array}{l} g_{,x^1x^1}=a\big(1+g^2_{,x^1}\big), \\ g_{,x^1x^2}=ag_{,x^1}g_{,x^2}, \\ g_{,x^2x^2}=a\big(1+g^2_{,x^2}\big). \end{array}\right\} \tag{39}$$

When $g(x^1,x^2)$ exists we must have $g_{,x^1x^1x^2}=g_{,x^1x^2x^1}$ and therefore we get from the first and second of Eqs. (39)

$$g_{,x^1}g_{,x^1x^2}=g_{,x^2}g_{,x^1x^1}. \tag{40}$$

Analogously, we must have $g_{,x^2x^2x^1}=g_{,x^1x^2x^2}$ and therefore, via the second and third of Eqs. (39) we get

$$g_{,x^2}g_{,x^1x^2}=g_{,x^1}g_{,x^2x^2}. \tag{41}$$

Writing (40) and (41) in function of the first derivatives of $g(x^1,x^2)$ (by using Eqs. (39)) we get

$$\left.\begin{array}{l} g^2_{,x^1}g_{,x^2}=\big(1+g^2_{,x^1}\big)g_{,x^2}, \\ g^2_{,x^2}g_{,x^1}=\big(1+g^2_{,x^2}\big)g_{,x^1}. \end{array}\right\} \tag{42}$$

From Eqs. (42) we get

$$\left.\begin{aligned} g_{,x^1} &= 0, \\ g_{,x^2} &= 0, \end{aligned}\right\} \tag{43}$$

and therefore $g(x^1, x^2) = \text{const}$. But a constant function does not satisfy Eqs. (39) (remember that a is a non-zero real number). We conclude that Eqs. (7) are free from invariant sets of type $x^3 = g(x^1, x^2)$ when $a^i_{jk}(\mathbf{x})$ are given by Eqs. (36).

However, these equations possess invariant sets of type $g(x^1, x^2) = 0$, $\nabla g \neq \mathbf{0}$. In fact the phase space equations of these invariant sets are

$$\left.\begin{aligned} &g\big(x^1, x^2\big) = 0, \\ &g_{,x^1}\dot{x}^1 + g_{,x^2}\dot{x}^2 = 0. \end{aligned}\right\} \tag{44}$$

Let $g_{,x^2} \neq 0$. Then

$$\dot{x}^2 = -\frac{g_{,x^1}}{g_{,x^2}}\dot{x}^1. \tag{45}$$

The equivalent of Eqs. (34) are now the equations

$$L_{\mathbf{X}}\big(g_{,x^1}\dot{x}^1 + g_{,x^2}\dot{x}^2\big)_{|(44)} = 0, \tag{46}$$

which after some computations become

$$\big(g_{,x^1x^1}\big(\dot{x}^1\big)^2 + 2g_{,x^1x^2}\dot{x}^1\dot{x}^2 - 2\phi'\big(x^3\big)g_{,x^1}\dot{x}^1\dot{x}^3 + g_{,x^2x^2}\big(\dot{x}^2\big)^2 - 2\phi'\big(x^3\big)g_{,x^2}\dot{x}^2\dot{x}^3\big)_{|(44)} = 0. \tag{47}$$

Finally, when (45) is substituted into (47) we get the equation

$$g_{,x^1x^1} - 2g_{,x^1x^2}\frac{g_{,x^1}}{g_{,x^2}} + g_{,x^2x^2}\frac{g^2_{,x^1}}{g^2_{,x^2}} = 0. \tag{48}$$

Note that Eq. (48) is free from $\phi'(x^3)$. Note also that Eq. (48) holds when $g(x^1, x^2)$ is a function of an affine function in x^1, x^2:

$$\left.\begin{aligned} &g\big(x^1, x^2\big) = F\big(ax^1 + bx^2 + c\big), \\ &a, b, c \in \mathbb{R} \quad \text{and} \quad F'(u) \neq 0. \end{aligned}\right\} \tag{49}$$

No other solutions of Eq. (48) exist since [18] the curvature k of the plane curve $g(x^1, x^2) = 0$ is given by

$$k = \left(\frac{g_{,x^1x^1}g^2_{,x^2} - 2g_{,x^1}g_{,x^2}g_{,x^1x^2} + g_{,x^2x^2}g^2_{,x^1}}{(g^2_{,x^1} + g^2_{,x^2})^{3/2}}\right)_{|g(x^1,x^2)=0}. \tag{50}$$

It is now immediate (when $\nabla g \neq \mathbf{0}$) that the expression (50) for k vanishes identically for the solutions of Eq. (48). But it is well known [18] that in this case the curve $g(x^1, x^2) = 0$ must be a straight line, and therefore $g(x^1, x^2)$ must have the structure (49), as we desired to prove.

Open remains the problem of giving equations of type (7) without invariant sets of dimension greater than one. The dimension *one* case is a special one and was discussed in Section 1.

5. Invariant sets in special relativity

For brevity, we only consider in this section the case $n=3$, $r=1$, $\mathbf{F}(x^1,x^2,x^3)$ and invariant sets of type $x^3=g(x^1,x^2)$. The relativistic differential equations replacing $\ddot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$ are now

$$\left.\begin{aligned}&\frac{d}{dt}(\gamma\dot{\mathbf{x}})=\mathbf{F}(\mathbf{x}),\\&\mathbf{x}\in\mathbb{R}^3,\quad \gamma=\left(1-\dot{\mathbf{x}}^2\right)^{-1/2}.\end{aligned}\right\}\tag{51}$$

Let

$$\left.\begin{aligned}&x^3=g\left(x^1,x^2\right),\\&\dot{x}^3=g_{,x^1}\dot{x}^1+g_{,x^2}\dot{x}^2,\end{aligned}\right\}\tag{52}$$

be invariant under Eqs. (51). It is easy to check that Eqs. (51) can also be written in the normal form:

$$\ddot{\mathbf{x}}=\left(1-\dot{\mathbf{x}}^2\right)^{1/2}\begin{pmatrix}1-(\dot{x}^1)^2 & -\dot{x}^1\dot{x}^2 & -\dot{x}^1\dot{x}^3\\ -\dot{x}^1\dot{x}^2 & 1-(\dot{x}^2)^2 & -\dot{x}^2\dot{x}^3\\ -\dot{x}^1\dot{x}^3 & -\dot{x}^2\dot{x}^3 & 1-(\dot{x}^3)^2\end{pmatrix}\mathbf{F}(\mathbf{x}),\tag{53}$$

and therefore the invariance of (52) under Eq. (53) reduces to the equation

$$\left[\ddot{x}^3-g_{,x^1x^1}\left(\dot{x}^1\right)^2-2g_{,x^1x^2}\dot{x}^1\dot{x}^2-g_{,x^2x^2}\left(\dot{x}^2\right)^2-g_{,x^1}\ddot{x}^1-g_{,x^2}\ddot{x}^2\right]_{|(52)}=0.\tag{54}$$

Now, Eq. (53) can be written in the form:

$$\left.\begin{aligned}&\ddot{x}^1=\left(1-\dot{\mathbf{x}}^2\right)^{1/2}\left[F^1(\mathbf{x})\left(1-\left(\dot{x}^1\right)^2\right)-F^2(\mathbf{x})\dot{x}^1\dot{x}^2-F^3(\mathbf{x})\dot{x}^1\dot{x}^3\right],\\&\ddot{x}^2=\left(1-\dot{\mathbf{x}}^2\right)^{1/2}\left[-F^1(\mathbf{x})\dot{x}^1\dot{x}^2+F^2(\mathbf{x})\left(1-\left(\dot{x}^2\right)^2\right)-F^3(\mathbf{x})\dot{x}^2\dot{x}^3\right],\\&\ddot{x}^3=\left(1-\dot{\mathbf{x}}^2\right)^{1/2}\left[-F^1(\mathbf{x})\dot{x}^1\dot{x}^3-F^2(\mathbf{x})\dot{x}^2\dot{x}^3+F^3(\mathbf{x})\left(1-\left(\dot{x}^3\right)^2\right)\right].\end{aligned}\right\}\tag{55}$$

Substituting Eqs. (55) into Eq. (54) and taking (52) into account we get

$$\left(1-\dot{\mathbf{x}}^2\right)^{1/2}{}_{|(52)}\left[F^3-g_{,x^1}F^1-g_{,x^2}F^2+P_2(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}})\right]_{|(52)}-\sum_{i,j=1}^{2}g_{,x^ix^j}\dot{x}^i\dot{x}^j=0,\tag{56}$$

$P_2(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}})$ being a homogeneous polynomial of second degree in $\dot{x}^1,\dot{x}^2$:

$$P_2(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}})=\sum_{i,j=1}^{2}A_{ij}\left(\mathbf{F},g_{,x^1},g_{,x^2}\right)\dot{x}^i\dot{x}^j,\tag{57}$$

A_{ij} standing for a function of its arguments $\mathbf{F}$, $g_{,x^1}$, $g_{,x^2}$. We shall see immediately that $P_{2|(52)}$ vanishes.

Now, Eq. (56) cannot hold unless

$$\left.\begin{aligned}&\mathbf{F}\cdot(-g_{,x^1},-g_{,x^2},1)=0,\\&P_2(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}})=0,\\&g_{,x^ix^j}=0,\quad i,j=1,2,\end{aligned}\right\}_{|(52)}\tag{58}$$

since otherwise $(1-(\dot{x}^1)^2-(\dot{x}^2)^2-(\dot{x}^3)^2)^{1/2}{}_{|(52)}$ would be equal to a rational function of $x^1,x^2,\dot{x}^1,\dot{x}^2$.

Note that $1-(\dot{x}^1)^2-(\dot{x}^2)^2-(g_{,x^1}\dot{x}^1+g_{,x^2}\dot{x}^2)^2$ cannot be the square of a one-degree polynomial in $\dot{x}^1,\dot{x}^2$: $(A+B\dot{x}^1+C\dot{x}^2)$, as the reader can check immediately.

The first and third of Eqs. (58) were obtained in Section 3 and imply that $\mathbf{F}$ is tangent to the manifold $x^3 = g(x^1, x^2)$ and that $g(x^1, x^2)$ *must be affine in* x^1, x^2. This same result was obtained in Section 3 in the non-relativistic case, and (to the best of our knowledge) is new.

The second of Eqs. (58), written explicitly, is

$$P_2(\tilde{\mathbf{x}}, \dot{\tilde{\mathbf{x}}})_{|(52)} = \left[-\dot{x}^3(\mathbf{F}\cdot\dot{\mathbf{x}}) + g_{,x^1}\dot{x}^1(\mathbf{F}\cdot\dot{\mathbf{x}}) + g_{,x^2}\dot{x}^2(\mathbf{F}\cdot\dot{\mathbf{x}})\right]_{|(52)} = 0, \tag{59}$$

and vanish identically via the second of Eqs. (52). Therefore only the first and third of Eqs. (58) are to be taken into account.

Note, finally, that the first of Eqs. (58) holds automatically when $\mathbf{F} \equiv \mathbf{0}$ (free motion of the particle). In this case any straight line or plane of $\mathbb{R}^3$ is invariant.

As an example, let $\mathbf{F}$ be central (see Eq. (25)). In this case only the planes π for which the first of Eqs. (58) holds

$$h\left(x^1, x^2, x^3\right)\left(x^1, x^2, x^3\right) \cdot (-g_{,x^1}, -g_{,x^2}, 1) = 0, \tag{60}$$

will be invariant under these forces. These are just the planes π through the origin. The reader will check that the last two equations in (58) are also satisfied.

In ending, note that considering the relativistic equations

$$\frac{d}{dt}(\gamma\dot{\mathbf{x}}) = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}), \tag{61}$$

$\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ being a polynomial in $\dot{\mathbf{x}}$, and invariant sets of type (52) we obviously get for the function g the equations (see Eqs. (56) and (58)):

$$\left.\begin{array}{l} \left[\mathbf{F}\cdot(-g_{,x^1}, -g_{,x^2}, 1) + P_2\right]_{|(52)} = 0, \\ g_{,x^i x^j} = 0, \quad 1 \leqslant i, j \leqslant 2. \end{array}\right\} \tag{62}$$

Decomposing $\mathbf{F}$ in the form

$$\mathbf{F} = \mathbf{F}_0(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{F}_1(\mathbf{x}, \dot{\mathbf{x}}) + \cdots + \mathbf{F}_d(\mathbf{x}, \dot{\mathbf{x}}), \tag{63}$$

$\mathbf{F}_k$ being a homogeneous polynomial of degree k $(0 \leqslant k \leqslant d)$ in $\dot{x}^1, \dot{x}^2$, Eqs. (62) can be written in the form:

$$\left.\begin{array}{l} \mathbf{F}\cdot(-g_{,x^1}, -g_{,x^2}, 1)_{|(52)} = 0, \\ g_{,x^i x^j} = 0, \quad 1 \leqslant i, j \leqslant 2. \end{array}\right\} \tag{64}$$

Remember that (see Eq. (59)) $P_{2|(52)} = 0$ identically. Taking (63) into account we get from Eqs. (64)

$$\left.\begin{array}{l} \mathbf{F}_k\cdot(-g_{,x^1}, -g_{,x^2}, 1)_{|(52)} = 0, \quad 0 \leqslant k \leqslant d, \\ g_{,x^i x^j} = 0, \quad 1 \leqslant i, j \leqslant 2. \end{array}\right\} \tag{65}$$

Therefore $g(x^1, x^2)$ must be affine in x^1, x^2, $g = Ax^1 + Bx^2 + C$, and the real numbers (A, B, C) must satisfy the equations

$$\left.\begin{array}{l} \mathbf{F}_k(\mathbf{x}, \dot{\mathbf{x}})\cdot(-A, -B, 1)_{|(52)} = 0, \\ k = 0, 1, \ldots, d. \end{array}\right\} \tag{66}$$

As an example, consider the relativistic tridimensional motion of a unit mass, unit charge particle, ruled by the equation

$$\frac{d}{dt}(\gamma\dot{\mathbf{x}}) = q\left[\mathbf{E}(\mathbf{x}) + \dot{\mathbf{x}} \wedge \mathbf{B}(\mathbf{x})\right], \tag{67}$$

$(\mathbf{E}(\mathbf{x}), \mathbf{B}(\mathbf{x}))$ being the electromagnetic field acting on the particle. In this case $\mathbf{F}_k$ are given by

$$\left.\begin{array}{l} \mathbf{F}_0(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{E}(\mathbf{x}), \\ \mathbf{F}_1(\mathbf{x}, \dot{\mathbf{x}}) = \left(B^3\dot{x}^2 - B^2\dot{x}^3, B^1\dot{x}^3 - B^3\dot{x}^1, B^2\dot{x}^1 - B^1\dot{x}^2\right) \end{array}\right\} \tag{68}$$

and therefore Eqs. (66) become

$$\left.\begin{array}{l} \left(E^1, E^2, E^3\right) \cdot (-A, -B, 1)_{|x^3=Ax^1+Bx^2+C} = 0, \\ \left(B^3\dot{x}^2 - B^2\dot{x}^3, B^1\dot{x}^3 - B^3\dot{x}^1, B^2\dot{x}^1 - B^1\dot{x}^2\right) \cdot (-A, -B, 1)_{\substack{|x^3=Ax^1+Bx^2+C \\ |\dot{x}^3=A\dot{x}^1+B\dot{x}^2}} = 0. \end{array}\right\} \tag{69}$$

One can always assume, via an adequate choosing of Cartesian coordinates, that $A = B = C = 0$; that is that the invariant plane is just the plane $x^3 = 0$, and therefore Eqs. (69) become

$$\left.\begin{array}{l} E^3{}_{|x^3=0} = 0, \\ \left(B^2\dot{x}^1 - B^1\dot{x}^2\right)_{\substack{|x^3=0 \\ |\dot{x}^3=0}} = 0. \end{array}\right\} \tag{70}$$

The physical meaning of the first of Eqs. (70) is that $\mathbf{E}$ must be parallel to the plane $x^3 = 0$. The second of Eqs. (70) just means that $B^1{}_{|x^3=0} = 0$, $B^2{}_{|x^3=0} = 0$. That is $\mathbf{B}$ must be orthogonal to the plane $x^3 = 0$.

What is important, and apparently *new* is that Eq. (61), when $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ is a polynomial of degree d in $\dot{\mathbf{x}}$ (see Eq. (63)) and the invariant set is defined by Eqs. (52), imply that "curved" invariant sets are relativistically *forbidden*. Only "not curved" invariant sets (planes in $\mathbb{R}^3$ of type $x^3 = Ax^1 + Bx^2 + C$) are allowed. Curved invariant sets are, of course, *not forbidden* for Newtonian equations $\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$.

As we can see, a strong difference between the invariant sets of equations $\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ and $d(\gamma\dot{\mathbf{x}})/dt = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ has come out.

One of the referees of the Letter has raised the following problem: is there a physical reason on why curved invariant sets are forbidden in the relativistic motion?

6. An application to geometrical optics

In this section we consider a ray of light which is moving in a material medium characterized by its refraction index $n(x, y, z)$ $(n > 0)$.

It is known that its motion is ruled by the second order differential equations [23]

$$\left.\begin{array}{l} \ddot{\mathbf{x}} + 2(\nabla n \cdot \dot{\mathbf{x}})\dot{\mathbf{x}} = \dfrac{\nabla n}{n^2}, \\ c \equiv \text{speed of light} = 1. \end{array}\right\} \tag{71}$$

If $f(x, y, z) = 0$ is an analytic invariant set for Eqs. (71), the phase space equations associated to $f = 0$ are

$$\left.\begin{array}{l} f = 0, \\ \nabla f \cdot \dot{\mathbf{x}} = 0. \end{array}\right\} \tag{72}$$

Casting Eqs. (72) in the standard form (19) and proceeding as in Sections 3 and 4 (after some computations) we obtain the equations that the function $g(x,y)$ must satisfy

$$\left.\begin{aligned} &g_{,xx}=0,\\ &g_{,xy}=0,\\ &g_{,yy}=0,\\ &(\nabla n\cdot\nabla f)_{|f=0}=0. \end{aligned}\right\} \tag{73}$$

The first three equations are obtained after application of Eqs. (34) to the equation $\ddot{\mathbf{x}}=-2(\nabla n\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}$ and the fourth equation corresponds to Eqs. (23) where the field force $\mathbf{F}(\mathbf{x})$ has been substituted by $\nabla n(\mathbf{x})/n^2(\mathbf{x})$.

Eqs. (73) imply that g must be an affine function, that is $g(x,y)=Ax+By+C$ (A,B,C real numbers), and ∇n must be tangent to the invariant set $z=g(x,y)$. Therefore we obtain the same result that was obtained for the relativistic equations of motion: curved invariant sets for the equations of motion of the rays of light are not allowed. Note that this result holds for any refraction index $n(x,y,z)$; that is the result $z=Ax+By+C$ is valid *independently* of the possible Euclidean symmetries of $n(x,y,z)$.

Note finally that the fourth equation in (73) becomes, when $f=Ax+By+C-z$,

$$(An_{,x}+Bn_{,y}-n_{,z})_{|Ax+By+C-z=0.}=0. \tag{74}$$

Eq. (74) is satisfied for:

(a) $n=n(x)$, when $n=n(x)=\text{const}$ for any values of A, B and C, and for $A=0$ when $n'(x)\neq 0$.
(b) $n=n(x^2+y^2+z^2)$, when $C=0$ (for any values of A and B).

The discussion on the solutions of Eq. (74) when $n=n(x^2+y^2,z)$ is trivial, and shall not be given.

7. Final remarks

We have studied analytic invariant sets of $\mathbb{R}^n$, the configuration space of the mass one particle. When $\mathbb{R}^n$ is substituted by *another manifold* M^n, on which a Riemannian metric g is defined, the equivalent of Newton equations $\ddot{\mathbf{x}}=\mathbf{F}(\mathbf{x},\dot{\mathbf{x}})$ is, locally, an equation of type

$$\ddot{y}^i=\sum_{j,k=1}^{n}a^i_{jk}(\mathbf{y})\dot{y}^j\dot{y}^k+G^i(\mathbf{y},\dot{\mathbf{y}}), \tag{75}$$

(y^i) being a local coordinate system near a certain point $P_0\in M$.

As long as $\mathbf{G}(\mathbf{y},\dot{\mathbf{y}})$ is a polynomial in $\dot{\mathbf{y}}$ the techniques developed above can be applied to Eq. (75) in order to get the invariant sets of it.

If the functions $a^i_{jk}(\mathbf{y})$ are obtained from a metric tensor $g_{ij}(\mathbf{y})$ via Eqs. (8) and $\mathbf{G}(\mathbf{y},\dot{\mathbf{y}})\equiv\mathbf{0}$, then Eq. (75) are the geodesics equations in the Riemannian manifold (M^n,g_{ij}). The invariant sets $I=\{\mathbf{y}\in M^n/f_1(\mathbf{y})=0,\dots,f_r(\mathbf{y})=0\}$ of Eq. (75) when $\mathbf{G}=\mathbf{0}$ are called by mathematicians *totally geodesic submanifolds* [18] because all the geodesics of the codimension r submanifold I are geodesics of the enveloping space M^n. From this point of view the treatment followed in this Letter can be applied to the study of totally geodesic submanifolds in Riemannian manifolds. In local coordinates the non-linear partial differential equations that the functions f_i must satisfy are Eqs. (34). Global results on analytic invariant sets are, as well, automatically obtained by working with analytic (C^{w}) local charts in M^n.

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Letter to the Editor

Note on a paper of J. Llibre and G. Rodríguez concerning algebraic limit cycles

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Abstract

In a recent paper of Llibre and Rodríguez (J. Differential Equations 198 (2004) 374–380) it is proved that every configuration of cycles in the plane is realizable (up to homeomorphism) by a polynomial vector field of degree at most $2(n+r)-1$, where n is the number of cycles and r the number of primary cycles (a cycle C is primary if there are no other cycles contained in the bounded region limited by C). In this letter we prove the same theorem by using an easier construction but with a greater polynomial bound (the vector field we construct has degree at most $4n-1$). By using the same technique we also construct $\mathbb{R}^3$ polynomial vector fields realizing (up to homeomorphism) any configuration of limit cycles which can be linked and knotted in $\mathbb{R}^3$. This answers a question of R. Sverdlove.

1. Introduction

In this paper we are interested in $\mathbb{R}^2$ vector fields

$$X = P_m(x, y)\partial_x + Q_m(x, y)\partial_y, \tag{1}$$

where the functions $P_m(x, y)$ and $Q_m(x, y)$ are polynomials of real variables (x, y) with real coefficients and degree not higher than m.

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The most important problem concerning planar polynomial vector fields was proposed in 1900 by Hilbert [8] (in the second part of his 16th problem) and it consists in finding the maximum number of limit cycles for the vector field (1) in terms of the degree m and studying the relative positions of these cycles. Recall that a limit cycle of the vector field (1) is an isolated periodic orbit of this vector field.

So far the 16th Hilbert's problem remains unsolved. It has been proved that the number of limit cycles of (1) must be finite [9,5] but even in the easiest case ($m = 2$) it remains open to ascertain the maximum number of limit cycles of all quadratic differential systems.

In this letter we are not interested in the 16th Hilbert's problem but in the following inverse problem: given a set C of planar cycles we wish to construct a polynomial vector field X whose limit cycles are exactly the set C (up to homeomorphism).

Let us introduce some previous definitions in order to specify the problem we are interested in. We follow here Llibre and Rodríguez [11] who have also studied the same problem.

Definition 1. A configuration of cycles is a finite set $C = \{C_1, \ldots, C_n\}$ of simple planar closed curves such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Definition 2. The curve $C_i \in C$ is primary if there is no curve $C_j \in C$ contained in the bounded region limited by C_i.

Definition 3. Two configurations of cycles C and C' are equivalent if there is a homeomorphism $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $H(C) = C'$.

Definition 4. The vector field X realizes the configuration of cycles C if the set of all limit cycles of X is equivalent to C.

Now the question can be formulated as follows: can we give a constructive method in order to find a polynomial vector field X realizing an arbitrary configuration C of cycles? In [11] Llibre and Rodríguez answer this question affirmatively but their proof is rather involved (they use the Darbouxian theory of integrability).

In the following section we prove the theorem of Llibre and Rodríguez by a different and easier method. The advantage of our method is that it can be easily extended to higher dimension, as we show in Section 3. Another advantage is that we control the stability of the limit cycles (they are stable) while the limit cycles in Llibre and Rodríguez's construction can be stable, semi-stable or unstable, and we have no control over it. Furthermore, our construction in $\mathbb{R}^2$ gives rise to hyperbolic limit cycles and hence structurally stable under small perturbations of the vector field. We are not aware whether Llibre and Rodríguez's limit cycles are structurally stable. The disadvantage is that, in general, the polynomial bound that we obtain is worse than Llibre and Rodríguez's.

Our main theorem is the following:

Theorem 1. *Let C be a configuration of n cycles. Then we have that C is realizable (as algebraic, stable and structurally stable limit cycles) by a polynomial vector field X of degree* $\leqslant 4n - 1$.

Note that the polynomial bound obtained in [11] is $2(n+r)-1$ and since $r \geqslant 1$ (r is the number of primary cycles) our bound is greater than Llibre and Rodríguez's except in the case that $r = n$, that is, when all the cycles are primary.

In Section 3 we prove a theorem analogous to Theorem 1 but in $\mathbb{R}^3$. Definitions 1, 3 and 4 extend naturally to the 3-space. Note that in this case the cycles of C are not planar and therefore they can be linked among them or even knotted. The theorem is:

Theorem 2. *Let C be a configuration of n cycles in $\mathbb{R}^3$. Then we have that C is realizable* (*as algebraic stable limit cycles*) *by a non-vanishing polynomial vector field $X \in \mathbb{R}^3$ of degree high enough.*

In the proof of Theorem 2 (Section 3) we give a specific bound of the degree of X. As far as we know, this (constructive) result is new in the literature. In fact, a vector field V with a given compact attracting set C is proved to exist in [7] but its construction implies that the dynamics on C is trivial (all the points in C are zeros of the vector field). Specifically V is a gradient field projecting a tubular neighborhood of C onto C so it cannot possess any periodic orbits. In general, most of the constructions of vector fields with given attracting set that can be found in the literature give rise to trivial dynamics on the attractor. Furthermore, it is not proved that under homeomorphism of C this vector field can become polynomical.

On the other hand Theorem 2 answers a long-standing question posed by Sverdlove [14]: what knot types can occur in polynomical systems? The answer is that all knot types are possible and we give an explicit procedure for constructing a polynomial vector field with a given knotted stable limit cycle.

2. Proof of Theorem 1

In this proof we follow the works of Sverdlove [14], Gascon et al. [6] and Winkel [16].

Let C be a configuration of n cycles in $\mathbb{R}^2$. By applying a homeomorphism H we can deform these cycles into circles of center (x_i, y_i) and radius r_i:

$$H(C_i) = \{f_i(x, y) = (x - x_i)^2 + (y - y_i)^2 - r_i^2 = 0\}. \tag{2}$$

Now let us construct the following function:

$$f(x, y) = \prod_{i=1}^{n} f_i(x, y), \tag{3}$$

where f is a polynomial of degree $2n$. Since the cycles $H(C_i)$ do not intersect among them we have that the set $\{f(x, y) = 0\}$ defines exactly the configuration $H(C)$. Note also that $(\nabla f)_{|f=0} \neq 0$.

Consider the vector field

$$X = (-f_y - ff_x)\partial_x + (f_x - ff_y)\partial_y, \tag{4}$$

where the subscripts of f denote partial differentiation with respect to the corresponding variables. The vector field defined in (4) has the following properties:

- $X_{|f=0} \neq 0$.
- $\dot{f^2} = X(f^2) = -2f^2(f_x^2 + f_y^2) \leqslant 0$ and in a neighborhood of $H(C)$ $\dot{f^2} = 0$ only on $f = 0$. f^2 is a Lyapunov function and therefore its level sets near the cycles of $H(C)$ are deformed circles [15]. These facts imply that $H(C)$ is a set of stable limit cycles of the vector field X.
- X does not possess other periodic orbits apart from $H(C)$. Assume that Γ is a periodic orbit different from $H(C_i)$ for all $i = 1, \ldots, n$. Since this orbit does not intersect any of the cycles of $H(C)$ we must have that, for example, $f_{|\Gamma} > 0$. We also require that $(\nabla f)_{|\Gamma} \neq 0$ in order that $X_{|\Gamma} \neq 0$ (see Eq. (4)). Taking into account these facts we obtain $\dot{f}_{|\Gamma} < 0$ and therefore Γ cannot be a periodic orbit. This is a contradiction.
- X is a polynomial vector field of degree at most $4n - 1$.

Finally let us prove that the limit cycles of the vector field X (see Eq. (4)) are hyperbolic, thus implying that they persist under small perturbations of X. Indeed, consider the following integral over a limit cycle of X:

$$\chi = \frac{1}{T}\int_0^T (\operatorname{div} X)_{|f=0}, \tag{5}$$

where div stands for the standard divergence operator and T is the period of the cycle. If $\chi \neq 0$ then the limit cycle is hyperbolic [1]. Taking into account Eq. (4) it is immediate to see that $(\operatorname{div} X)_{|f=0} = -(\nabla f)^2$, which does not vanish on $f = 0$, thus proving the claim.

3. Proof of Theorem 2

In this section we have a configuration C of n cycles in $\mathbb{R}^3$. As mentioned in the introduction these cycles can be untrivial knots and can be linked among them [10]. Assume that the cycles C_i are smooth enough, namely C^∞ submanifolds. Since each component C_i is diffeomorphic to S^1 then its normal bundle is trivial [12]. By the well-known Tognoli's theorem there always exists a diffeomorphism $H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ (in fact a diffeotopy) such that $H(C)$ is an algebraic set and hence an algebraic configuration of cycles [3], that is, the curves in $H(C)$ are given by

$$\begin{cases} f_m(x, y, z) = 0, \\ g_m(x, y, z) = 0, \end{cases} \tag{6}$$

where f_m and g_m are polynomials of degree at most m satisfying that

$$rk(\nabla f_m, \nabla g_m) = 2 \tag{7}$$

on the cycles of $H(C)$. Note that Tognoli's theorem guarantees that the set $H(C)$ is exactly formed by n algebraic cycles (no other compact or non-compact components appear).

The degree m is in general unknown and it probably depends on the linking and crossing numbers [10] of the cycles in C. It is evident that, for example, if all the cycles C_i lie on a certain plane, then $m = 2n$.

Our main polynomial function in this case is $F = f_m^2 + g_m^2$ whose degree is at most $2m$. Note that the configuration $H(C)$ is given by $F = 0$ and that $\nabla F = 0$ in $H(C)$ but it is different from zero in a neighborhood of the cycles of $H(C)$. F is therefore a Lyapunov function and its level sets near the cycles are deformed tori [15].

Let us construct the following vector field:

$$X = \nabla f_m \wedge \nabla g_m - F\nabla F \tag{8}$$

with $\wedge$ standing for the standard vector product in $\mathbb{R}^3$ and ∇ standing for the gradient operator.

The vector field (8) has these properties:

- $X_{|F=0} \neq 0$ since ∇f_m and ∇g_m are independent on $H(C)$.
- $\dot{F} = -F(\nabla F)^2 \leqslant 0$ and in a neighborhood of $H(C)$ we have that $\dot{F} = 0$ only on the cycles $H(C)$. Since F is a Lyapunov function we conclude that the cycles in the configuration $H(C)$ are stable limit cycles of X.
- X does not possess other periodic orbits. Assume that Γ is a periodic orbit of X which does not belong to $H(C)$. It is immediate that, for example, $F_{|\Gamma} > 0$ because otherwise Γ would intersect some cycle of $H(C)$. Since $X_{|\Gamma} \neq 0$ it is straightforward that $(\nabla F)_{|\Gamma} \neq 0$ because otherwise in a certain point of Γ the gradients of f_m and g_m would be parallel and therefore X would be zero. But these facts yield a contradiction since we would have a periodic orbit Γ such that $\dot{F}_{|\Gamma} < 0$.
- X is a polynomial vector field of degree at most $4m - 1$. As mentioned above the number m does depend on the specific configuration C. Note the difference with the planar case in which m is always $2n$. This difference is due to the many complex ways in which the cycles of C can be linked and knotted, this being a particular property of the 3-dimensional case.

It is interesting to observe that the polynomials f_m and g_m, defining the vector field X in Eq. (8), can be chosen such that $X \neq 0$ in $\mathbb{R}^3$. To show this claim note that for any given link L in $\mathbb{R}^3$ there exists a submersion $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ such that the preimage of the origin $\phi^{-1}(0)$ is L [13]. Whenever the link is algebraic the submersion ϕ can be chosen to be polynomial, say $\phi = (f_m, g_m)$, and hence the rank condition $rk(\nabla f_m, \nabla g_m) = 2$ holds in all $\mathbb{R}^3$. Since the vector fields ∇f_m, ∇g_m and $\nabla f_m \wedge \nabla g_m$ are independent it follows that $X = \nabla f_m \wedge \nabla g_m - F\nabla F$ cannot vanish at any point.

The construction of this section thus provides an algebraic vector field X, $X \neq 0$ in $\mathbb{R}^3$, whose set of limit cycles, all of them stable, is given by $H(C)$.

In ending this section we would like to pose the following open problem: are the limit cycles of the vector field (8) structurally stable, as in the 2-dimensional case?

4. Final remarks

In this letter we have proved a recent theorem of Llibre and Rodríguez by using a very different technique. Our technique is simpler and can be extended to the 3-dimensional case as was shown in Section 3. On the contrary the polynomial bound of the vector field that we construct is greater than Llibre and Rodríguez's. Our bound is therefore not sharp but note that Llibre and Rodríguez's bound is not either; see the work of Christopher [4] where polynomial vector fields of degree at most $2n$ realizing a generic class of algebraic limit cycles are constructed.

On the other hand the application of this technique to the 3-dimensional case is, to the best of our knowledge, new in the literature. We have proved that every configuration of cycles in $\mathbb{R}^3$ can be deformed into an algebraic configuration of cycles that can be realized as the limit cycles of a 3-dimensional polynomial vector field. This answers a question formulated by Sverdlove [14].

If C is a configuration of smooth cycles in $\mathbb{R}^n$, $n > 3$, Tognoli's theorem also guarantees the existence of a diffeotopy $H : \mathbb{R}^n \to \mathbb{R}^n$ such that $H(C)$ is an algebraic set of cycles (note again that the normal bundle of C is trivial [12]). $H(C)$ is expressed through the polynomials $f_m^1, \dots, f_m^{n-1}$ of degree at most m as $H(C) = \{f_m^1 = 0, \dots, f_m^{n-1} = 0\}$, $rk(\nabla f_m^1, \dots, \nabla f_m^{n-1}) = n-1$ on $H(C)$. Define now the vector field $X_t = \left[\star (\mathrm{d} f_m^1 \wedge \cdots \wedge \mathrm{d} f_m^{n-1})\right]^i$, $\star$ standing for the Hodge star operator and i standing for the index raising operator, and the function $F = \sum_{i=1}^{n-1} \left(f_m^i\right)^2$. Proceeding as in Section 3 it is immediate to prove that the vector field $X = X_t - F\nabla F$ has stable limit cycles given by the curves in $H(C)$, and it does not possess any other periodic orbits. Thus X is a polynomial vector field (of degree at most $\max\{(n-1)m-(n-1), 4m-1\}$) realizing the set C of cycles. Since $H(C)$ can be realized as the level set $\Phi^{-1}(0)$ of a polynomial submersion $\Phi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ [13] then we obtain that the polynomial vector field X does not vanish in $\mathbb{R}^n$. Note that Miyoshi's theorem [13] is proved for codimension 2; anyway, it trivially holds when the set has codimension $n-1$ in $\mathbb{R}^n$ ($n > 3$). Indeed, since the submanifold $H(C)$ can be embedded, through an ambient diffeomorphism of $\mathbb{R}^n$, into the 3-dimensional hyperplane $\{x_4 = 0, \dots, x_n = 0\}$, one only has to apply Miyoshi's theorem on this hyperplane in order to obtain a submersion $(f_m, g_m) : \mathbb{R}^3 \to \mathbb{R}^2$, and then to extend the submersion to the whole R^n in a trivial way $(f_m, g_m, x_4, \dots, x_n)$.

The degree of this vector field and the one constructed in Section 3 is surely not sharp and it remains open to connect the topological properties of the configuration C with the degree m, that is, can one give a formula expressing m in terms of the linking and crossing numbers or other topological numbers related to the configuration C?

A related question is the 16th Hilbert's problem in $\mathbb{R}^n$, $n > 2$, e.g. do there exist polynomial vector fields with an infinite number of limit cycles? An example of a 1-parameter family of polynomial vector fields in R^4, which has fixed (bounded) degree, and the number of its limit cycles tends to infinity as the parameter $\varepsilon \to 0$ has been recently constructed by Bobienski and Zoladek [2], but we are not aware of examples in the literature of $\mathbb{R}^n$ ($n \geqslant 3$) polynomial vector fields with infinitely many limit cycles. Note that the techniques in this paper allow to solve the inverse problem for a configuration of infinitely many cycles (locally finite) when the vector field X is only required to be analytic. The fact that an infinite number of algebraic sets is not algebraic prevents from constructing a polynomial vector field, thus suggesting that new ideas are necessary to tackle this question.

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Note added in proof

The author has only recently been aware of a preprint by A. Ferragut, J. Llibre and M.A. Teixeira (2005) where examples of polynomial vector fields in $\mathbb{R}^3$ with infinitely many limit cycles are constructed. As far as we know the inverse problem of constructing $\mathbb{R}^3$ polynomial vector fields realizing any infinite (locally finite) configuration of cycles is not solved.

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