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CONVERGENCIA-Γ NO PERIÓDICA

MEMORIA PARA OPTAR AL GRADO DE DOCTOR
PRESENTADA POR

Helia da Conceiçao Pereira Serrano

Bajo la dirección del doctor
Pablo Pedregal Tercero

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Tesis Doctoral
con Mención *Doctor Europeus*

Convergencia-Γ no periódica

Doctorando:
Hélia da Conceição Pereira Serrano

Director de Tesis:
Prof. Dr. Pablo Pedregal Tercero

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Resumen[1]

En esta disertación se estudia la convergencia-Γ de funcionales integrales en el contexto no periódico. En concreto, introducimos una nueva condición suficiente, designada por *Composition Gradient Property* (CGP), que nos permite calcular explícitamente la densidad de la energía límite de sucesiones de funcionales integrales no periódicos. La densidad se representa, a través de un problema de minimización, usando la medida de Young asociada a la sucesión de funciones que determinan la sucesión de funcionales. La condición CGP es una condición estrutural de la sucesión de aplicaciones, que definen la sucesión de funcionales, tal que si no se verifica, entonces el límite-Γ no se puede representar explícitamente. Se estudian algunos ejemplos interesantes.

A continuación, se estudia la convergencia-Γ de funcionales cuadráticos con perturbaciones lineales oscilantes, en los contextos no periódico y periódico con multi-escalas. En el contexto periódico con multi-escalas, obtenemos una representación completa, de los coeficientes cuadrático y lineal, de la densidad de la energía límite en dos casos distintos. En el primer caso, se considera que ambos, los coeficientes cuadrático y lineal de las energías, oscilan en la misma familia de escalas de oscilación separadas; mientras que en el segundo las oscilaciones son en distintas familias de escalas. Es importante resaltar que el coeficiente lineal homogeneizado depende de la interación entre los comportamientos oscilantes de los coeficientes cuadrático y lineal, de las densidades de las energías.

Finalmente, estudiamos la convergencia-Γ de funcionales cuyas densidades son diferentes potencias, p y q, de la norma del gradiente, que dependen de la estructura espacial laminada. Concluimos que la densidad de la energía límite es una combinación convexa de las diferentes potencias. Además, generalizamos este resultado para sucesiones de funcionales con cualquier densidad convexa con crecimiento no estándar, dependiente de dicha estrutura espacial.

[1] De acuerdo con el Artículo 14, Mención europea en el título de Doctor, del Real Decreto 56/2005 de 21 enero, por el que se regulan los estudios universitarios de Postgrado, y de acuerdo con la normativa de la Universidad Complutense, la presente Tesis Doctoral es redactada en español e inglés.

Abstract

In this dissertation we study the Γ-convergence of integral functionals, in the general non-periodic setting. Namely, a new sufficient condition, called the Composition Gradient Property (CGP), is introduced in order to compute explicitly the limit energy density of families of non-periodic integral functionals. The CGP is a structural condition on the sequence of mappings, which defines the sequence of functionals, so that if it is not satisfied, then the density of the Γ-limit cannot be explicitly represented. Under this condition, the limit energy density is fully characterized, through a minimum problem, by the Young measure associated with the sequence of functions which determines the sequence of functionals. Some examples are explored.

On the other hand, we study the Γ-convergence of quadratic functionals, with oscillating linear perturbations, in the non-periodic and multi-scale periodic settings. In the multi-scale periodic setting, we achieve an explicit characterization, of the quadratic and linear coefficients, for the limit energy density in two different situations. In the first one both, the quadratic and the linear coefficients, oscillate at the same family of separated length scales; while in the second one, they oscillate at distinct scales. We stress how the homogenized linear coefficients depend on the interaction between the oscillatory behaviours.

Finally, we study the Γ-convergence of sequences of functionals whose densities are powers of the gradient norm, with different exponents depending on a laminate spatial distribution. This analysis leads to the conclusion that the limit energy density is a convex combination of different powers. We generalize this result to sequences of functionals with general convex densities satisfying a non-standard growth condition, which depends on a laminate spatial structure.

Key words: Γ-*convergence, homogenization, multi-scale problems, weak convergence, Young measures, p-laplacian*

AMS subject classification: *35A15, 35B27, 49J40, 74K20*

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- Ω : an open bounded set in $\mathbb{R}^n$
- $\partial\Omega$: the boundary of Ω
- $|\Omega|$: the Lebesgue measure of Ω
- $\mathcal{A}(\Omega)$: the family of all open subsets of Ω
- $Q = (0,1)^n$
- $Y = (0,c_1) \times \ldots \times (0,c_n)$ for some positive numbers $c_1, \ldots, c_n$
- $a \cdot b$: the inner product of two vectors
- $a \otimes b = (a_i b_j)_{ij} \in \mathbb{R}^{d\times n}$
- $a \parallel b$: the vectors a and b are parallels
- $x = [x] + \langle x \rangle$: $[x]$ is the integer part and $\langle x \rangle$ is the fractional part
- A^T : transpose matrix
- $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$
- $\mathbb{R}^{d\times n}$: the space of $d \times n$ real matrices
- $C(\mathbb{R}^n)$: the space of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$
- $C_0^\infty(\Omega)$: the space of indefinitely differentiable functions with compact support in Ω
- $p' = \frac{p}{p-1}$: the conjugate exponent of $p > 1$ for which $\frac{1}{p} + \frac{1}{p'} = 1$
- $W_0^{1,p}(\Omega;\mathbb{R}^d)$: the closure of $C_0^\infty(\Omega;\mathbb{R}^d)$ in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^d)$, for every $1 \le p < \infty$
- $W_0^{1,\infty}(\Omega;\mathbb{R}^d)$: the closure of $C_0^\infty(\Omega;\mathbb{R}^d)$ in the weak* topology of $W^{1,\infty}(\Omega;\mathbb{R}^d)$

- $W^{1,p}_{per}(Q;\mathbb{R}^d)$: the closure of all Q-periodic functions in $C^1(Q;\mathbb{R}^d)$, in the weak topology of $W^{1,p}(Q;\mathbb{R}^d)$, for every $1 \leq p < \infty$
- $H^1_0(Y) = W^{1,2}_0(Y;\mathbb{R})$
- $H^1_{per}(Y) = W^{1,2}_{per}(Y;\mathbb{R})$
- $W^{-1,p'}(\Omega;\mathbb{R}^n)$: the dual space of $W^{1,p}_0(\Omega;\mathbb{R}^n)$
- $W^{1,p(x)}(\Omega)$: generalized Sobolev Space (see Appendix)
- $\rightarrow$: strong convergence
- $\rightharpoonup$: weak convergence
- $\rightharpoonup^\star$: weak* convergence

Chapter 1

Introduction

This dissertation focuses on the study of the Γ-convergence of integral functionals, in the non-periodic setting, and the homogenization of second-order elliptic equations (in divergence form), and p-laplacian equations.

Γ-convergence is a variational convergence on functionals. Given a family of functionals I_ε defined in the space X_ε, depending on a parameter $\varepsilon > 0$, the Γ-convergence is based on the analysis of the asymptotic behaviour of minimum problems

$$\min \{ I_\varepsilon(u) \ : \ u \in X_\varepsilon \} \tag{1.1}$$

as ε goes to 0. The Γ-limit is a functional I obtained having in mind the aim that minimizers u_ε of (1.1) converge, in an appropriate topology, to a minimizer of the limit problem

$$\min \{ I(u) \ : \ u \in X \},$$

where X is the domain of I. More precisely, the Γ-limit I is a lower bound for the family $\{I_\varepsilon\}$, in the sense that

$$I(u) \ \leq \ \liminf_{\varepsilon \searrow 0} \ I_\varepsilon(u_\varepsilon),$$

for every sequence $\{u_\varepsilon\}$ converging to u, which is attained, ie

$$I(u) \ = \ \lim_{\varepsilon \searrow 0} \ I_\varepsilon(u_\varepsilon),$$

for some sequence of minimizers u_ε of I_ε. See [17, 24, 26, 27].

A widely treated problem in the literature is the Γ-convergence of integral functionals of the form

$$I_\varepsilon(u) \ = \ \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) \ dx \tag{1.2}$$

defined in an appropriate Sobolev space, where $f(y, A)$ is periodic in the first variable, and satisfies a natural growth condition with respect to the second one. This type of integrals models various phenomena in Mathematical Physics. For instance, we may consider an elastic material, periodic in the cell with side-length ε, in a region Ω. Then $I_\varepsilon(u)$ stands for the elastic energy of such material under the deformation u. A natural question to ask is: how does the elastic material behave when the side-length ε goes to 0? The answer leads to the study of the variational convergence of the energies I_ε. In this case, it is well known that the Γ-limit is an integral functional whose density is explicitly characterized. Namely, if the energy density $f : \mathbb{R}^n \times \mathbb{R}^{d\times n} \to \mathbb{R}$ is Q-periodic in the first variable, and satisfies a growth condition of order p in the second one, then the sequence of functionals I_ε given by (1.2), and defined in $W^{1,p}(\Omega;\mathbb{R}^d)$, is Γ-convergent, as ε goes to 0, to the integral I defined by

$$I(u) \;=\; \int_\Omega f_{hom}(\nabla u(x))\; dx.$$

Here Q is the unit cube in $\mathbb{R}^n$. The limit energy density $f_{hom} : \mathbb{R}^{d\times n} \to \mathbb{R}$ is the homogenous function defined by

$$f_{hom}(A) \;=\; \lim_{T\to\infty} \inf_{v\in W_0^{1,p}(TQ,\mathbb{R}^d)} \frac{1}{T^n}\int_{TQ} f(y, A+\nabla v(y))\; dy.$$

A key point in the characterization of this limit energy density f_{hom} is the periodicity of f in the first variable, which implies that the sequence $\{f\left(\frac{\cdot}{\varepsilon}, \nabla u(\cdot)\right)\}$ oscillates periodically in Ω. See, for instance [15, 16, 18, 24, 26, 39, 42, 67].

Now, we may ask about the explicit representation of the Γ-limit when the energy density is a non-periodic function, and the integral functional I_ε depends on the parameter ε in a different way. As far as we know, the explicit characterization of the limit energy density in this general non-periodic setting was studied, for the first time, in [50] three years ago. Therefore, in this dissertation (Chapter 5) we pursue the study of the explicit representation of the limit energy density of sequences of functionals of the form

$$I_\varepsilon(u) \;=\; \int_\Omega W(a_\varepsilon(x), \nabla u(x))\; dx, \tag{1.3}$$

defined in the Sobolev space $W^{1,p}(\Omega)$, where the continuous energy density $W : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is non-periodic in the first variable, and satisfies coercivity and growth conditions of order p with respect to the second variable, for any given sequence $\{a_\varepsilon\}$.

Such study is undertaken by using Young measures as a tool. Since the notion of Young measure has been developed, in part, as a useful tool to treat minimum problems in the Calculus of Variations, it seems natural to use it in the treatment of variational convergence. Young measures are families of probability measures which, often associated with oscillating sequences, describe their oscillatory behaviour and give a representation of the limits of the composition with non-linear quantities. See [49, 64, 71].

Clearly the Γ-convergence of the sequence of functionals I_ε given by (1.3) depends on the sequence $\{a_\varepsilon\}$; in other words, to effectively compute and understand the Γ-limit one needs to analyze the structure of $\{a_\varepsilon\}$. Thus, we investigate sufficient conditions on the sequence $\{a_\varepsilon\}$ which enable to represent the limit energy density by means of the Young measure associated with $\{a_\varepsilon\}$. In [50] it was introduced a sufficient condition, called the Average Gradient Property (AGP). Roughly speaking, a sequence $\{a_\varepsilon\}$ satisfies the AGP if averages of gradients over "level sets" of a_ε are gradients themselves. Though this is a sufficient condition to represent explicitly the limit energy density, it is not easy to handle, because, in general, it is hard to verify whether $\{a_\varepsilon\}$ satisfies the AGP. More precisely, in order to verify whether the sequence $\{a_\varepsilon\}$, with associated Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$, satisfies the AGP, we should fix a point $x \in \Omega$ and follow the procedure:

- First, take a sequence $r_\varepsilon \searrow 0$ for which the rescaled sequence $\{a_\varepsilon(x + r_\varepsilon \cdot)\}$, defined in the unit ball B, generates the homogenous Young measure σ_x.

- Secondly, a covering by pairwise disjoint balls of the support of σ_x should be built for each ε, so that the sequence of piecewise constant fields V_ε^x, defined as the average, over each inverse image (by $a_\varepsilon(x + r_\varepsilon \cdot)$) of such a covering, of an arbitrary gradient ∇v, is well-defined.

- Finally, it remains to check if such sequence of fields, for any arbitrary given gradient field ∇v, is approximated by a sequence of gradients.

It is not difficult to convince oneself that this process may be cumbersome and tedious some times.

In this way, our aim is to find a new sufficient condition easy to implement. Indeed, we introduce and explore a much more tangible condition, called the Composition Gradient Property (CGP), for reasons to be understood soon. This new condition leads to a rather clear way of understanding the structure of the sequence $\{a_\varepsilon\}$, for which the density of the Γ-limit can be explicitly computed. In order to understand how the CGP comes out, let us recall briefly the process of finding the Γ-limit, through Young measures, of the sequence of functionals given in (1.3). Consider a weak convergent sequence $\{u_\varepsilon\}$ to u in $W^{1,p}(\Omega)$, such that the Young measure associated with the sequence of pairs $\{(a_\varepsilon, \nabla u_\varepsilon)\}$ may be decomposed as $\{\mu_{\lambda,x} \otimes \sigma_x\}_{x\in\Omega}$, where $\sigma = \{\sigma_x\}_{x\in\Omega}$ is the Young measure associated with $\{a_\varepsilon\}$.

Then we get the following estimate:

$$\liminf_{\varepsilon \searrow 0} \int_\Omega W(a_\varepsilon(x), \nabla u_\varepsilon(x))\, dx \;\geq\; \int_\Omega \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} W(\lambda, \rho)\, d\mu_{x,\lambda}(\rho)\, d\sigma_x(\lambda)\, dx$$
$$\geq\; \int_\Omega \int_{\mathbb{R}^m} CW\left(\lambda, \int_{\mathbb{R}^n} \rho\, d\mu_{x,\lambda}(\rho)\right) d\sigma_x(\lambda)\, dx,$$

where $CW(\lambda, \cdot)$ represents the convex envelope of $W(\lambda, \cdot)$ in $\mathbb{R}^n$, for any $\lambda \in \mathbb{R}^m$. Since σ is the Young measure associated with $\{a_\varepsilon\}$, if we define the map $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ by putting

$$\varphi(x, \lambda) \;=\; \int_{\mathbb{R}^n} \rho\, d\mu_{x,\lambda}(\rho),$$

it holds

$$\int_\Omega \int_{\mathbb{R}^m} CW\left(\lambda, \int_{\mathbb{R}^n} \rho\, d\mu_{x,\lambda}(\rho)\right) d\sigma_x(\lambda)\, dx \;=\; \lim_{\varepsilon \searrow 0} \int_\Omega CW(a_\varepsilon(x), \varphi(x, a_\varepsilon(x)))\, dx.$$

Notice that the composition sequence $\{\varphi(\cdot, a_\varepsilon(\cdot))\}$ consists in a reorganization, through averaging, of the sequence $\{\nabla u_\varepsilon\}$, over "level sets" of $\{a_\varepsilon\}$. Basically, the AGP condition is tailored to ensure that the sequence $\{\varphi(\cdot, a_\varepsilon(\cdot))\}$ may be approximated by a sequence of gradients. If such sequence does not furnish a sequence of gradients, then there is not much that can be done to determining the density of the Γ-limit, because we can not recover a sequence of gradients for which the above inequalities are in fact equalities. Therefore, we say that a sequence $\{a_\varepsilon\}$, with associated Young measure $\sigma = \{\sigma_x\}_{x \in \Omega}$ supported on $\mathbb{R}^m$, satisfies the CGP condition provided there exists a Carathéodory map $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ such that, for a.e. $x \in \Omega$,

1. $\varphi(x, \cdot)$ is one-to-one over the support of σ_x;
2. $\{\varphi(x, a_\varepsilon(x + r_\varepsilon \cdot))\}$ is "essentially a sequence of gradients", in the sense

$$\|\, \mathrm{curl}\ \varphi(x, a_\varepsilon(x + r_\varepsilon \cdot))\, \|_{W^{-1,q}(B)} \underset{\varepsilon \searrow 0}{\longrightarrow} 0,$$

   for some sequence $r_\varepsilon \searrow 0$.

So, whenever there exists an one-to-one map φ such that the composition $\{\varphi(a_\varepsilon(\cdot))\}$ may be approximated by a sequence of gradients $\{\nabla v_\varepsilon\}$, ie $\{\varphi(a_\varepsilon(\cdot)) - \nabla v_\varepsilon\}$ converges strongly to 0, in an appropriate Lebesgue space, we will say that the sequence $\{a_\varepsilon\}$ satisfies the CGP condition.

Moreover, we are able to prove that this condition implies the AGP. This implication leads to concluding that, under the CGP condition, the sequence of functionals I_ε in (1.3) is Γ-convergent (in the weak topology of $W^{1,p}(\Omega)$) to the functional

$$I(u) \;=\; \int_\Omega \overline{W}(x, \nabla u(x))\, dx,$$

where the density $\overline{W} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\overline{W}(x,\rho) \;=\; \inf_{\varphi\in\mathcal{A}_x} \left\{ \int_{\mathbb{R}^m} CW(\lambda,\varphi(\lambda))\; d\sigma_x(\lambda) \;:\; \rho = \int_{\mathbb{R}^m} \varphi(\lambda)\; d\sigma_x(\lambda) \right\}$$

with

$$\mathcal{A}_x = \left\{ \varphi : \mathbb{R}^m \to \mathbb{R}^n \text{ continuous, one to one } : \| \operatorname{curl} \varphi(a_\varepsilon(x + r_\varepsilon \cdot)) \|_{W^{-1,q}(B)} \to 0 \right\}$$

for some $q > p > 1$, whenever the sequence $\{a_\varepsilon(x + r_\varepsilon \cdot)\}$, defined in the unit ball B, generates the homogenous Young measure σ_x.

In order to clarify the CGP condition and the above explicit representation of the limit energy density, let us present two simple interesting examples. First consider a sequence of piecewise constant functions $a_\varepsilon : \Omega \to \mathbb{R}^m$ defined by

$$a_\varepsilon(x) \;=\; A_1\, \chi_{(0,t)} \left(\frac{x}{\varepsilon}\cdot \vec{n}\right) \;+\; A_2 \left(1 - \chi_{(0,t)}\left(\frac{x}{\varepsilon}\cdot\vec{n}\right)\right),$$

where $\chi_{(0,t)}(s)$ is the characteristic function of the interval $(0,t)$ over $(0,1)$, extended by periodicity to $\mathbb{R}$, and $\vec{n} \in \mathbb{R}^n$ is a unit normal vector. Notice that this sequence may be treated in the periodic setting. The sequence $\{a_\varepsilon\}$ generates the homogenous Young measure σ, supported on $\{A_1, A_2\}$, given by

$$\sigma \;=\; t\,\delta_{A_1} \;+\; (1-t)\,\delta_{A_2},$$

where δ_{A_1} is the Dirac measure concentrated at A_1. See [47, 49]. Recall that to prove $\{a_\varepsilon\}$ satisfies the CGP, it is enough to find an one-to-one, continuous field $\varphi : \{A_1, A_2\} \to \mathbb{R}^n$ such that the composition $\varphi(a_\varepsilon(\cdot))$, given by

$$\varphi(a_\varepsilon(x)) \;=\; \varphi(A_1)\; \chi_{(0,t)}\left(\frac{x}{\varepsilon}\cdot\vec{n}\right) \;+\; \varphi(A_2)\; \left(1 - \chi_{(0,t)}\left(\frac{x}{\varepsilon}\cdot\vec{n}\right)\right),$$

is "essentially a sequence of gradients". For any field φ such that $\varphi(A_1) \neq \varphi(A_2)$, and the difference $\varphi(A_1) - \varphi(A_2)$ is parallel to the normal vector $\vec{n}$, the sequence $\{\varphi(a_\varepsilon(\cdot))\}$ is indeed "essentially a sequence of gradients". See [12]. Therefore we conclude that the limit energy density of the sequence of functionals I_ε, given by

$$I_\varepsilon(u) = \int_\Omega \left[W(A_1, \nabla u(x))\, \chi_{(0,t)}\left(\frac{x}{\varepsilon}\cdot\vec{n}\right) \;+\; W(A_2, \nabla u(x))\left(1 - \chi_{(0,t)}\left(\frac{x}{\varepsilon}\cdot\vec{n}\right)\right)\right] dx,$$

is the homogenous function $\overline{W} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\overline{W}(\rho) = \min_{B_i\in\mathbb{R}^n} \left\{ t\,CW(A_1,B_1) + (1-t)\,CW(A_2,B_2) \;:\; \begin{array}{c} \rho = tB_1 + (1-t)B_2 \\ B_1 - B_2 \;\|\; \vec{n} \end{array} \right\}.$$

It is remarkable that the density $\overline{W}$ is defined through a finite dimensional minimization problem in this situation.

For the second example, consider any function $v \in W_0^{1,p}(D)$. For any open bounded set Ω, we may take a covering of Ω by pairwise disjoint sets $x_k^{(\varepsilon)} + r_k^{(\varepsilon)} D$, with $\{x_k^{(\varepsilon)}\} \subset \Omega$ and $r_k^{(\varepsilon)} \leq \varepsilon$. Then define the sequence of functions $v_\varepsilon : \Omega \to \mathbb{R}$ by putting

$$v_\varepsilon(x) \;=\; r_k^{(\varepsilon)}\, v\left(\frac{x - x_k^{(\varepsilon)}}{r_k^{(\varepsilon)}}\right) \qquad \text{if } \; x \in x_k^{(\varepsilon)} + r_k^{(\varepsilon)} D,$$

and consider the sequence of functions $a_\varepsilon : \Omega \to \mathbb{R}$ defined by $a_\varepsilon(x) = \nabla v_\varepsilon(x)$. Clearly, $\{a_\varepsilon\}$ is a sequence of gradients, and its associated Young measure σ is defined by

$$\langle \sigma, \varphi \rangle \;=\; \frac{1}{|D|} \int_D \varphi(\nabla v(y))\; dy, \qquad \text{for every } \varphi \in C_0(\mathbb{R}^n).$$

See [49]. Then we conclude that the sequence of functionals I_ε given by

$$I_\varepsilon(u) \;=\; \sum_k \int_{x_k^{(\varepsilon)} + r_k^{(\varepsilon)} D} W\left(\nabla v\left(\frac{x - x_k^{(\varepsilon)}}{r_k^{(\varepsilon)}}\right), \nabla u(x)\right)\; dx$$

Γ-converges to the functional I whose homogenous density $\overline{W} : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\overline{W}(\rho) = \inf_{w \in W^{1,p}(D)} \left\{ \frac{1}{|D|} \int_D CW(\nabla v(y), \rho + \nabla w(y))\; dy \;:\; \frac{1}{|D|} \int_D \nabla w(y)\, dy = 0 \right\}.$$

◇ ◇ ◇

The second subject matter treated in this dissertation (Chapter 6) is the homogenization of second-order linear elliptic equations, in divergence form, with oscillatory leading coefficient and source term. The homogenization theory consists in describing the macroscopic behaviour of composite materials, which are obtained by mixing microscopically several materials with different macroscopic behaviours. In order to be more precise, let us describe the model problem of thermal conductivity in a composite material. Consider two distinct isotropic materials, say 1 and 2, with thermal conductivity a_1 and a_2, respectively, so that the heat conduction is the same in any direction (a_1 and a_2 are scalar). Take the composite material obtained by mixing a layer of material 1 with a layer of material 2 in a region Ω, such that

$$\Omega \;=\; \Omega_1 \cup \Omega_2 \qquad \text{and} \qquad \Omega_1 \cap \Omega_2 = \emptyset,$$

where Ω_1 corresponds to material 1, and Ω_2 to material 2. If f stands for the heat source, then the temperature $u(x)$ of the composite in the point x of the region Ω satisfies the Dirichlet boundary problem

$$\begin{cases} -\operatorname{div}\, [\, a(x) \nabla u(x)\,] \;=\; f(x) & \text{in } \Omega \\ \qquad\qquad\qquad\quad u \;=\; 0 & \text{on } \partial\Omega, \end{cases}$$

and the flux of temperature is given by

$$a(x)\,\nabla u(x) \;=\; \begin{cases} a_1(x)\,\nabla u_1(x) & \text{in } \Omega_1 \\ a_2(x)\,\nabla u_2(x) & \text{in } \Omega_2, \end{cases}$$

where $u_i(x)$ is the temperature of material i at x, for $i = 1, 2$. The physical assumptions on this thermal problem are the continuity of temperature u and flux of temperature $a\,\nabla u$ on the interface of the two materials, ie

$$\begin{cases} u_1(x) \;=\; u_2(x) & \text{on } \partial\Omega_1 \cap \partial\Omega_2 \\ a_1(x)\,\nabla u_1(x)\,n_1 \;=\; a_1(x)\,\nabla u_1(x)\,n_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2, \end{cases}$$

where n_1 is the outward normal unit vector to $\partial\Omega_1$ and $n_2 = -n_1$.

Moreover, we may mix the two materials by taking so many fine layers as we wish so that, for each parameter $\varepsilon > 0$, with values in a sequence tending to 0, we have

$$\Omega \;=\; \Omega_1^\varepsilon \cup \Omega_2^\varepsilon,$$

where

$$\Omega_1^\varepsilon = \left\{ x \in \Omega : \chi_1\left(\frac{x}{\varepsilon}\right) = 1 \right\} \quad \text{and} \quad \Omega_2^\varepsilon = \left\{ x \in \Omega : \chi_2\left(\frac{x}{\varepsilon}\right) = 1 \right\},$$

using the characteristic function χ_i of material $i = 1, 2$, as follows.

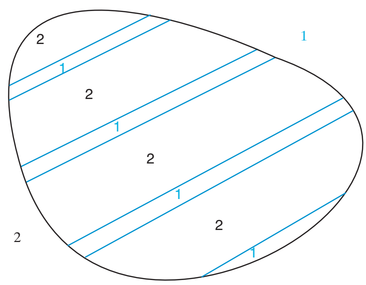

Figure 1.1: Layered composite material in a region Ω.

Then, for each ε, the temperature u_ε of the composite material satisfies the problem

$$\begin{cases} -\operatorname{div}\,[\,a_\varepsilon(x)\nabla u_\varepsilon(x)\,] \;=\; f(x) & \text{in } \Omega \\ u_\varepsilon \;=\; 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with the thermal conductivity given by

$$a_\varepsilon(x) \;=\; a_1(x)\chi_1\left(\frac{x}{\varepsilon}\right) + a_2(x)\chi_2\left(\frac{x}{\varepsilon}\right) \;=\; \begin{cases} a_1(x) & \text{in } \Omega_1^\varepsilon \\ a_2(x) & \text{in } \Omega_2^\varepsilon. \end{cases}$$

The aim of homogenization, in this simple example, is to describe the thermal conductivity of the composite material in the region Ω as ε goes to 0, ie when the two materials are so mixed microscopically, that it looks like a homogenous material from the macroscopic point of view. Mathematically, this means that the main objective is to characterize the limit u of the sequence of solutions u_ε of the Dirichlet problems (1.4), that is to find out the "homogenized" problem for which the limit u is a solution. See, for instance [9, 14, 66].

A widely treated problem in the literature is the homogenization of elliptic equations of the form

$$(P_\varepsilon) \quad \begin{cases} -\operatorname{div}\,[\,A_\varepsilon(x)\,\nabla u_\varepsilon(x)\,] & = & f(x) \quad \text{in } \Omega \\ u_\varepsilon & \in & H_0^1(\Omega), \end{cases}$$

where $\{A_\varepsilon\}$ is a sequence of matrix functions $A_\varepsilon \in [L^\infty(\Omega)]^{n\times n}$ such that $\alpha|\rho|^2 \leq A_\varepsilon(x)\rho\cdot\rho \leq \beta|\rho|^2$, for some $\beta \geq \alpha > 0$, and $f \in H^{-1}(\Omega)$. The aim is to describe the limit problem as ε goes to 0, ie to prove the existence, and to represent explicitly the "homogenized" coefficient $A_0 \in [L^\infty(\Omega)]^{n\times n}$, such that the sequence of solutions u_ε of (P_ε) converges weakly in $H_0^1(\Omega)$ to the solution of the, so-called, homogenized problem

$$(P_0) \quad \begin{cases} -\operatorname{div}\,[\,A_0(x)\,\nabla u_0(x)\,] & = & f(x) \quad \text{in } \Omega \\ u_0 & \in & H_0^1(\Omega). \end{cases}$$

It is known that, under the above assumptions, there exists a subsequence $\{A_{\varepsilon_k}\}$, and a matrix function $A_0 \in [L^\infty(\Omega)]^{n\times n}$ which does not depend on f, such that the solutions of (P_{ε_k}) converge weakly in $H_0^1(\Omega)$ to the solution of (P_0). This is a known compactness result on H-convergence of sequences of matrices A_ε. See [44, 68].

When the sequence of matrices A_ε are defined by

$$A_\varepsilon(x) \;=\; A\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. in } \Omega, \tag{1.5}$$

for some matrix function $A = [a_{ij}] \in [L^\infty(\Omega)]^{n\times n}$, so that a_{ij} is Q-periodic, for every $1 \leq i,j \leq n$, the effective coefficient A_0 has been explicitly characterized through auxiliary minimum problems. Moreover, the effective coefficient A_0 is also known in the multi-scale periodic setting, where

$$A_\varepsilon(x) \;=\; A\left(x, \frac{x}{l_1(\varepsilon)}, \ldots, \frac{x}{l_N(\varepsilon)}\right) \qquad \text{a.e. in } \Omega,$$

for some matrix function $A \in [L^\infty(\Omega \times Q \times \ldots \times Q)]^{n\times n}$ such that $A(x, y_1, \ldots, y_N)$ is Q-periodic in each variable y_k, with $1 \leq k \leq N$, and for some family of separated length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$. See, for instance [6, 7].

Another interesting problem is the homogenization of elliptic equations, with oscillatory source terms, of the type

$$(P_{f_\varepsilon}) \quad \begin{cases} -\operatorname{div}\,[\,A_\varepsilon(x)\,\nabla u_\varepsilon(x)\,] & = & f_\varepsilon(x) \quad \text{in } \Omega \\ u_\varepsilon & \in & H_0^1(\Omega), \end{cases}$$

where $\{f_\varepsilon\}$ is a sequence in $H^{-1}(\Omega)$. In the periodic case, where the matrices A_ε are given by (1.5), if the sequence $\{f_\varepsilon\}$ converges to f_0, either weakly in $L^2(\Omega)$ or strongly in $H^{-1}(\Omega)$, it is known that the sequence of solutions of (P_{f_ε}) is weak convergent in $H_0^1(\Omega)$ to the solution of the homogenized problem

$$(P_{f_0}) \quad \begin{cases} -\operatorname{div}\,[\,A_0(x)\,\nabla u_0(x)\,] &= f_0(x) \quad \text{in } \Omega \\ u_0 &\in H_0^1(\Omega). \end{cases}$$

In this situation the homogenized coefficients A_0 and f_0 are explicitly determined, the first one through auxiliary minimum problems, and the second one is the limit of $\{f_\varepsilon\}$. However, in the periodic case, when the sequence $\{f_\varepsilon\}$ converges weakly to f_0 in $H^{-1}(\Omega)$, one only has the weak convergence of a subsequence, of solutions of (P_{f_ε}), to the solution of the homogenized problem

$$(P_\star) \quad \begin{cases} -\operatorname{div}\,[\,A_0(x)\,\nabla u^\star(x)\,] &= \operatorname{div} g^\star(x) \quad \text{in } \Omega \\ u^\star &\in H_0^1(\Omega), \end{cases}$$

where $g^\star \in [L^2(\Omega)]^n$. Notice that the source term $\operatorname{div} g^\star$ is not necessarily the weak limit of $\{f_\varepsilon\}$. Moreover, it is computed in a rather elaborate way, using auxiliary periodic test functions. See [21].

In this dissertation, our goal is to give a more transparent understanding of the function $g^\star$, in the periodic and non-periodic settings, ie when both sequences $\{A_\varepsilon\}$ and $\{f_\varepsilon\}$ are periodic, and when they are not. More precisely, we are interested in studying the asymptotic behaviour of sequences of solutions u_ε of the problem

$$(P_{b_\varepsilon}) \quad \begin{cases} -\operatorname{div}\,[\,A_\varepsilon(x)\,\nabla u_\varepsilon(x)\,] &= \operatorname{div} b_\varepsilon(x) \quad \text{in } \Omega \\ u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

where $\{b_\varepsilon\}$ is a uniformly bounded sequence in $[L^\infty(\Omega)]^n$, converging only weakly, and in understanding how the interaction between the oscillatory behaviour of $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ may affect homogenization, ie the characterization of the homogenized term $g^\star$.

Our study is undertaken from a variational point of view, through the study of the Γ-convergence of quadratic energies of the type

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\, \nabla u(x)^T \,\frac{A_\varepsilon(x)}{2}\, \nabla u(x) \;+\; b_\varepsilon(x) \cdot \nabla u(x) \,\right] dx. \tag{1.6}$$

Indeed, since equations in (P_{b_ε}) can be considered as the optimality equations associated with the above quadratic functionals I_ε, the Γ-convergence of $\{I_\varepsilon\}$ to the functional I implies the convergence of the solutions of problems (P_{b_ε}) to the solution of the homogenized problem, defined through the optimality equation associated with I. Therefore, the homogenization of elliptic problems of the form (P_{b_ε}) reduces to the study of the Γ-convergence of sequences of quadratic functionals of the type (1.6).

We study the Γ-convergence of quadratic functionals I_ε, with oscillatory linear perturbations, in order to characterize explicitly the homogenized term $g^\star$, by means of the joint Young measure associated with the sequence of pairs $\{(A_\varepsilon, b_\varepsilon)\}$. Moreover, we want to understand the influence of the sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ in such term. Thus we treat separately the general one-dimensional case and the periodic multi-dimensional one, because the first one is simpler to handle, and a more explicit characterization of the homogenized source term is obtained, in the general setting.

In the general one-dimensional case (Section 6.2), we consider sequences of energies of the form

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\, \frac{a_\varepsilon(t)}{2}\, u'(t)^2 \;+\; b_\varepsilon(t)\, u'(t) \,\right] dt$$

defined in $H_0^1(\Omega)$, for any sequences $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$ weak* converging in $L^\infty(\Omega)$, so that $\{a_\varepsilon\}$ is uniformly bounded away from 0. We achieve an explicit characterization, of quadratic and linear coefficients of the limit energy density, through the joint Young measure associated with the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$, and the one associated with $\{a_\varepsilon\}$. Indeed, we prove that the sequence $\{I_\varepsilon\}$ is Γ-convergent to the functional I defined by

$$I(u) \;=\; \int_\Omega \psi(t, u'(t))\; dt,$$

where the limit energy density $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a quadratic function of the second variable, given by

$$\psi(t,\rho) \;=\; \frac{a_0(t)}{2}\,\rho^2 \;+\; a_0(t)k(t)\,\rho \;+\; \frac{a_0(t)k(t)^2}{2} \;-\; \int_{\mathbb{R}^2} \frac{\beta^2}{2\alpha}\; d\eta_t(\alpha,\beta).$$

The functions $a_0 : \Omega \to (0,\infty)$ and $k : \Omega \to \mathbb{R}$ are defined by

$$a_0(t) \;=\; \left(\int_{\mathbb{R}} \frac{1}{\alpha}\; d\sigma_t(\alpha) \right)^{-1} \qquad \text{and} \qquad k(t) \;=\; \int_{\mathbb{R}^2} \frac{\beta}{\alpha}\; d\eta_t(\alpha,\beta),$$

through the Young measure $\sigma = \{\sigma_t\}_{t\in\Omega}$ associated with $\{a_\varepsilon\}$, and the joint Young measure $\eta = \{\eta_t\}_{t\in\Omega}$ associated with the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$. Notice that the first projection of η is σ. In the particular situation in which a_ε and b_ε are given by

$$a_\varepsilon(t) = a\left(\left\langle \frac{t}{\varepsilon} \right\rangle\right) \qquad \text{and} \qquad b_\varepsilon = b\left(t, \left\langle \frac{t}{\varepsilon} \right\rangle\right),$$

(where $\langle y \rangle$ stands for the fractional part of y), we conclude

$$a_0 \equiv \int_0^1 a(y)\, dy \qquad \text{and} \qquad g^\star(t) = a_0 \int_0^1 \frac{b(t,y)}{a(y)}\, dy.$$

Remarkably, if the sequence $\{b_\varepsilon\}$ oscillates faster than $\{a_\varepsilon\}$, for instance $b_\varepsilon = b\left(t, \left\langle \frac{t}{\varepsilon^2} \right\rangle\right)$, then the homogenized function $g^\star$ is the weak* limit of $\{b_\varepsilon\}$, ie

$$g^\star(t) = \int_0^1 b(t,y)\, dy,$$

and it does not depend on the oscillatory behaviour of $\{a_\varepsilon\}$.

Thus, if u_ε is the solution of the Dirichlet boundary problem

$$\left\{ \begin{array}{rcll} -\frac{d}{dt}\,[\,a_\varepsilon(t)\, u_\varepsilon'(t)\,] & = & \frac{d}{dt}\, b_\varepsilon(t) & \text{in } \Omega \\ u_\varepsilon & \in & H_0^1(\Omega), & \end{array} \right.$$

the sequence $\{u_\varepsilon\}$ converges weakly in $H_0^1(\Omega)$ to the solution u_0 of the homogenized problem

$$(P_\star) \quad \left\{ \begin{array}{rcll} -\frac{d}{dt}\,[\,a_0(t)\, u_0'(t)\,] & = & \frac{d}{dt}\, g^\star(t) & \text{in } \Omega \\ u_0 & \in & H_0^1(\Omega), & \end{array} \right.$$

with

$$g^\star(t) = a_0(t)\, k(t) \qquad \text{a.e. in } \Omega.$$

As far as we know, this is the most complete characterization of the homogenized problem $(P_\star)$ ever achieved.

In the periodic multi-dimensional case (Section 6.3), we focus on the study of the Γ-convergence of functionals

$$I_\varepsilon(u) = \int_\Omega \left[\nabla u(x)^T \frac{A_\varepsilon(x)}{2} \nabla u(x) + b_\varepsilon(x) \cdot \nabla u(x) \right] dx \tag{1.7}$$

defined in $H_0^1(\Omega)$, where

$$A_\varepsilon(x) = A\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_N(\varepsilon)} \right\rangle\right)$$

and

$$b_\varepsilon(x) = b\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_N(\varepsilon)} \right\rangle\right),$$

for any family of N separated length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$. It is important to point out that the results we have obtained do not depend on the number of separated length scales. We assume $b \in [L^\infty(\Omega \times Q^N)]^n$, and $A = [a_{ij}] \in [L^\infty(\Omega \times Q^N)]^{n \times n}$ to be a symmetric matrix function satisfying $\alpha|\rho|^2 \leq \rho^T A \rho \leq \beta|\rho|^2$, for some $0 < \alpha \leq \beta$ and every $\rho \in \mathbb{R}^n$. In this situation both sequences, $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$, oscillate at the same family of separated length scales. For the sake of simplicity, consider the example where the oscillations are of order ε, besides

$$A_\varepsilon(x) \;=\; A\left(\left\langle \frac{x}{\varepsilon} \right\rangle\right) \qquad \text{and} \qquad b_\varepsilon(x) \;=\; b\left(x, \left\langle \frac{x}{\varepsilon} \right\rangle\right) \qquad \text{a.e. in } x \in \Omega.$$

Our conclusion is : the Γ-limit of the corresponding sequence of functionals is

$$I(u) \;=\; \int_\Omega \psi(x, \nabla u(x))\, dx,$$

defined in $H_0^1(\Omega)$, where $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi(x,\rho) \;=\; \rho^T \frac{A_0}{2} \rho \;+\; g^\star(x) \cdot \rho \;+\; c(x).$$

The constant matrix $A_0 \in \mathbb{R}^{n \times n}$ is defined by

$$A_0 \;=\; \int_Q (I_n + [\nabla w(y)])^T A(y)\, (I_n + [\nabla w(y)])\; dy,$$

where I_n is the $n \times n-$identity matrix, and $[\nabla w(y)]$ is the $n \times n-$matrix whose columns are the vectors $\nabla w_j(y)$, $1 \le j \le n$, with w_j solution of the cell problem

$$\begin{cases} -\operatorname{div}\; [\,A\,(y)\,(e_j + \nabla w_j(y))\,] \;=\; 0 \qquad \text{in } Q \\ \qquad\qquad\qquad\qquad w_j \;\in\; H^1_{per}(Q), \end{cases}$$

for some basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. The linear coefficient $g^\star : \Omega \to \mathbb{R}^n$ is defined by

$$g^\star(x) \;=\; \int_Q (I_n + [\nabla w_j(y)])^T\, [\,A(y) \nabla_y z(x,y) \;+\; b(x,y)\,]\; dy,$$

where $z(x,\cdot)$ solves

$$\begin{cases} -\operatorname{div}_y\; [\,A\,(y)\,\nabla_y z(x,y)\,] \;=\; \operatorname{div}_y\, b\,(x,y) \qquad \text{in } Q \\ \qquad\qquad\qquad\qquad z(x,\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

and

$$c(x) \;=\; \int_Q \nabla_y z(x,y)^T \left[\frac{A(y)}{2} \nabla_y z(x,y) \;+\; b(x,y) \right] dy.$$

Thus, clearly, the sequence of solutions of

$$\begin{cases} -\operatorname{div}\; \left[\,A\left(\left\langle \frac{x}{\varepsilon} \right\rangle\right) \nabla u_\varepsilon(x)\,\right] \;=\; \operatorname{div} b\left(x, \left\langle \frac{x}{\varepsilon} \right\rangle\right) \qquad \text{in } \Omega \\ \qquad\qquad\qquad\qquad u_\varepsilon \;\in\; H^1_0(\Omega) \end{cases}$$

converges weakly in $H_0^1(\Omega)$ to the solution of

$$\begin{cases} -\operatorname{div}\; [\,A_0\, \nabla u_0(x)\,] \;=\; \operatorname{div}\, g^\star(x) \qquad \text{in } \Omega \\ \qquad\qquad\qquad u_0 \;\in\; H^1_0(\Omega). \end{cases}$$

Notice that the field $g^\star$ is not the weak* limit of the sequence $\{b_\varepsilon\} = \{b\left(\cdot, \langle\frac{\cdot}{\varepsilon}\rangle\right)\}$.

Once we have pursued the previous study considering sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillating at the same family of separated length scales, one may ask : what happens if $\{A_\varepsilon\}$ oscillates at a family of length scales distinct from the one where the oscillations of $\{b_\varepsilon\}$ take place ? Is the homogenized source term $g^\star$ the same as before ? In order to answer these questions, we also study the Γ-convergence of functionals I_ε given by (1.7) where

$$A_\varepsilon(x) \;=\; A\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_N(\varepsilon)} \right\rangle\right)$$

and

$$b_\varepsilon(x) \;=\; b\left(x, \left\langle \frac{x}{l_{N+1}(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_M(\varepsilon)} \right\rangle\right),$$

for any family of M separated length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon), l_{N+1}(\varepsilon), \ldots, l_M(\varepsilon)\}$ (Section 6.4). Here the sequence $\{A_\varepsilon\}$ oscillates at different length scales, distinct from the one of $\{b_\varepsilon\}$. Let us illustrate the results obtained, by considering the simpler case of two distinct separated length scales : ε and ε^2. Namely, in case

$$A_\varepsilon(x) \;=\; A\left(\left\langle \frac{x}{\varepsilon^2} \right\rangle\right) \qquad \text{and} \qquad b_\varepsilon(x) \;=\; b\left(x, \left\langle \frac{x}{\varepsilon} \right\rangle\right),$$

we conclude that the limit energy density is given by

$$\psi(x,\rho) \;=\; \rho^T \frac{A_0}{2} \rho \;+\; b_0(x) \cdot \rho \;+\; d(x).$$

The effective matrix A_0 is the same as in the previous example, while the linear coefficient $b_0 : \Omega \to \mathbb{R}^n$ is now the weak* limit of $\{b_\varepsilon\}$, namely

$$b_0(x) \;=\; \int_Q b(x, y_2)\; dy_2.$$

Besides

$$d(x) = \int_Q \int_Q \nabla_{y_2} v(x, y_1, y_2)^T \left[\frac{A(y_1)}{2} \nabla_{y_2} v(x, y_1, y_2) \;+\; b(x, y_2) \right] dy_1\; dy_2,$$

where the function $v(x, y_1, \cdot)$ is the solution of the cell problem

$$\begin{cases} -\operatorname{div}_{y_2} \left[A(y_1) \nabla_{y_2} v(x, y_1, y_2) \right] \;=\; \operatorname{div}_{y_2} b(x, y_2) & \text{in } Q \\ \qquad\qquad v(x, y_1, \cdot) \;\in\; H^1_{per}(Q), & \end{cases}$$

for a.e. $(x, y_1) \in \Omega \times Q$. Therefore the sequence of solutions of problem

$$\begin{cases} -\operatorname{div} \left[A\left(\langle \frac{x}{\varepsilon^2} \rangle\right) \nabla u_\varepsilon(x) \right] \;=\; \operatorname{div} b\left(x, \langle \frac{x}{\varepsilon} \rangle\right) & \text{in } \Omega \\ \qquad\qquad u_\varepsilon \;\in\; H^1_0(\Omega) & \end{cases}$$

converges weakly in $H_0^1(\Omega)$ to the solution of the homogenized problem

$$\begin{cases} -\operatorname{div}\ [A_0 \nabla u_0(x)] &= \operatorname{div} b_0(x) \quad \text{in } \Omega \\ u_0 &\in H_0^1(\Omega). \end{cases}$$

Surprisingly, we reach the conclusion that the homogenized source term depends on the interaction between the oscillatory behaviour of the sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$. Indeed, when $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillate at the same length scales, it follows that the homogenized source term $g^\star$ is defined also through the sequence $\{A_\varepsilon\}$. However, when there is no such interaction between the oscillations, the homogenized term b_0 is the weak* limit of $\{b_\varepsilon\}$.

After having completed this study on homogenization of elliptic equations with periodic oscillatory coefficients, a natural and more challenging question arises: how may one characterize explicitly the homogenized coefficients, of problems

$$\begin{cases} -\operatorname{div}\ [A_\varepsilon(x) \nabla u_\varepsilon(x)] &= \operatorname{div} b_\varepsilon(x) \quad \text{in } \Omega \\ u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

considering general non-periodic coefficients A_ε and b_ε? Therefore, to answer this question, we have started to study the Γ-convergence of quadratic functionals given by (1.7), in the general non-periodic setting (Section 6.5). Recall that we have introduced previously the notion of Composition Gradient Property (CGP) to treat the general non-periodic Γ-convergence of functionals given by (1.3). So, following the same ideas, we reintroduce the CGP condition for sequences of pairs $\{(A_\varepsilon, b_\varepsilon)\}$. Namely, we say that the sequence of pairs $(A_\varepsilon, b_\varepsilon) : \Omega \to \mathbb{R}^{n\times n} \times \mathbb{R}^n$, with associated Young measure $\eta = \{\eta_x\}_{x\in\Omega}$, satisfies the CGP if and only if there exists a Carathéodory map $\phi : \Omega \times \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for a.e. $x \in \Omega$,

i) $\phi(x,\cdot,\cdot)$ is one-to-one over the support of η_x;

ii) $\{\phi(x, A_\varepsilon(x + r_\varepsilon\cdot), b_\varepsilon(x + r_\varepsilon\cdot))\}$ is "essentially a sequence of gradients", in the sense
$$\|\operatorname{curl} \phi(x, A_\varepsilon(x + r_\varepsilon\cdot), b_\varepsilon(x + r_\varepsilon\cdot))\|_{W^{-1,q}(B)} \underset{\varepsilon\searrow 0}{\longrightarrow} 0,$$
for some sequence $r_\varepsilon \searrow 0$.

Thus, under the hypotheses that the sequences $\{A_\varepsilon\} \subset [L^\infty(\Omega)]^{n\times n}$ and $\{b_\varepsilon\} \subset [L^\infty(\Omega)]^n$ satisfy:

(H1) A_ε is a symmetric matrix function, $c_1|\rho|^2 \le \rho^T A_\varepsilon(x)\rho \le c_2|\rho|^2$, with $c_2 \ge c_1 > 0$,

(H2) $\{b_\varepsilon\}$ is uniformly bounded in $[L^\infty(\Omega)]^n$,

(H3) $\{(A_\varepsilon, b_\varepsilon)\}$ satisfies the CGP, we are able to prove that the sequence of functionals

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\, \nabla u(x)^T \,\frac{A_\varepsilon(x)}{2}\, \nabla u(x) \;+\; b_\varepsilon(x)\cdot\nabla u(x) \,\right] dx$$

defined in $H_0^1(\Omega)$, Γ-converges to the functional I given by

$$I(u) \;=\; \int_\Omega \psi(x,\nabla u(x))\; dx.$$

The density $\psi : \Omega\times\mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi(x,\rho) \;=\; \inf_{\varphi\in\mathcal{A}_x} \left\{ \int_{\mathbb{R}^{n\times n+n}} \left[\, \varphi(\Lambda,\beta)^T \,\frac{\Lambda}{2}\, \varphi(\Lambda,\beta) \;+\; \beta\cdot\varphi(\Lambda,\beta) \,\right] d\eta_x(\Lambda,\beta) \;:\; \rho = \int_{\mathbb{R}^{n\times n+n}} \varphi(\Lambda,\beta)\, d\eta_x(\Lambda,\beta) \right\}$$

where

$$\mathcal{A}_x = \left\{ \varphi : \mathbb{R}^{n\times n}\times\mathbb{R}^n \to \mathbb{R}^n \;:\; \|\,\mathrm{curl}\;\varphi(A_\varepsilon(x+r_\varepsilon\cdot), b_\varepsilon(x+r_\varepsilon\cdot))\,\|_{W^{-1,q}(B)} \to 0 \right\}$$

for any $q > 2$, whenever the sequence $\{(A_\varepsilon(x+r_\varepsilon\cdot), b_\varepsilon(x+r_\varepsilon\cdot))\}$, defined in the unit ball for some sequence $r_\varepsilon \searrow 0$, generates the homogenous Young measure η_x. Obviously, in this general setting, it is not easy to write explicitly the limit energy density ψ as a quadratic function of the second variable. However, we have succeeded in identifying some sequences of pairs $\{(A_\varepsilon, b_\varepsilon)\}$ for which some type of representation is possible. The explicit representation of ψ, as a quadratic function, depends on the characterization itself of the admissible fields φ in the set $\mathcal{A}_x$, ie the fields φ for which the composition sequence $\{\varphi(A_\varepsilon(x+r_\varepsilon\cdot), b_\varepsilon(x+r_\varepsilon\cdot))\}$ is "essentially a sequence of gradients".

An interesting case, which may also be considered in the periodic setting, occurs when the sequence of pairs $\{(A_\varepsilon, b_\varepsilon)\}$ has a laminate structure, ie the sequence oscillates between distinct constant values in alternate layers of the domain Ω. Namely, to be more explicit, let $A_\varepsilon(x) = a_\varepsilon(x) I_n$ in Ω, where I_n is the $n\times n-$identity matrix, and consider the sequence of pairs $(a_\varepsilon, b_\varepsilon) : \Omega \to (0,+\infty)\times\mathbb{R}^n$ defined by

$$(a_\varepsilon(x), b_\varepsilon(x)) \;=\; (a_1,b_1)\chi_{(0,t(x))}\left(\frac{x}{\varepsilon}\cdot\vec{n}\right) \;+\; (a_2,b_2)\left(1-\chi_{(0,t(x))}\left(\frac{x}{\varepsilon}\cdot\vec{n}\right)\right).$$

(For a.e. $x\in\Omega$, $\chi_{(0,t(x))}(s)$ is the characteristic function of the interval $(0,t(x))$ over $(0,1)$, extended by periodicity to $\mathbb{R}$.) So, the sequence $\{(a_\varepsilon, b_\varepsilon)\}$ alternates between the values (a_1,b_1) and (a_2,b_2) in layers, with width $t(x)$ and $(1-t(x))$, normal to the unit vector $\vec{n}\in\mathbb{R}^n$. Notice that this sequence satisfies the CGP condition. Indeed, since the Young measure associated with it is given by

$$\eta_x \;=\; t(x)\,\delta_{(a_1,b_1)} + (1-t(x))\,\delta_{(a_2,b_2)} \qquad \text{a.e. in } \Omega,$$

then, for any continuous field $\phi : \Omega \times (0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ with the vector $\phi(x,a_1,b_1) - \phi(x,a_2,b_2)$ parallel to $\vec{n}$, the sequence $\{\phi(\cdot, a_\varepsilon(\cdot), b_\varepsilon(\cdot))\}$ is "essentially a sequence of gradients".

Thus we are interested in the Γ-convergence of the family of functionals I_ε given by

$$\begin{aligned} I_\varepsilon(u) &= \int_\Omega \left[\frac{a_\varepsilon(x)}{2} |\nabla u(x)|^2 + b_\varepsilon(x) \cdot \nabla u(x) \right] dx \\ &= \int_{\Omega_{\varepsilon,t(x)}} \left[\frac{a_1}{2} |\nabla u(x)|^2 + b_1 \cdot \nabla u(x) \right] dx + \int_{\Omega^c_{\varepsilon,t(x)}} \left[\frac{a_2}{2} |\nabla u(x)|^2 + b_2 \cdot \nabla u(x) \right] dx, \end{aligned}$$

with

$$\Omega_{\varepsilon,t(x)} = \left\{ x \in \Omega : \chi_{(0,t(x))}\left(\frac{x}{\varepsilon} \cdot \vec{n}\right) = 1 \right\}, \ \Omega^c_{\varepsilon,t(x)} = \left\{ x \in \Omega : \chi_{(0,t(x))}\left(\frac{x}{\varepsilon} \cdot \vec{n}\right) = 0 \right\}.$$

It follows, from the previous analysis, that the limit energy density of the sequence $\{I_\varepsilon\}$ is the function $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\psi(x,\rho) = \frac{a_0(x)}{2} |\rho|^2 + g^\star(x) \cdot \rho + c(x),$$

with

$$\begin{aligned} a_0(x) &= \frac{a_1\, a_2}{(1-t(x))\, a_1 + t(x)\, a_2}, \\ g^\star(x) &= \frac{t(x)\, a_2}{(1-t(x))\, a_1 + t(x)\, a_2}\, b_1 + \frac{(1-t(x))\, a_1}{(1-t(x))\, a_1 + t(x)\, a_2}\, b_2, \end{aligned}$$

and

$$c(x) = \frac{(t(x)-1)t(x)}{(1-t(x))\, a_1 + t(x)\, a_2}\, |b_1 - b_2|^2.$$

Therefore, an immediate consequence, of the explicit characterization of the limit energy density, is the characterization of the homogenized problem

$$\begin{cases} -\operatorname{div}\ [a_0(x)\, \nabla u_0(x)] = \operatorname{div} g^\star(x) & \text{in } \Omega \\ u_0 \in H^1_0(\Omega) \end{cases}$$

itself, whose solution u_0 is the weak limit of the sequence of solutions of

$$\begin{cases} -\operatorname{div}\ [a_\varepsilon(x) \nabla u_\varepsilon(x)] = \operatorname{div} b_\varepsilon(x) & \text{in } \Omega \\ u_\varepsilon \in H^1_0(\Omega). \end{cases}$$

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Once we have finished our study on linear homogenization, ie Γ-convergence of quadratic functionals, one may ask about the non-linear homogenization of p-laplacians and non-homogenous $p(x)$-laplacians. Thus, the third subject matter studied in this dissertation is the homogenization of laminate composite materials whose state equation is a combination of p-laplacians.

Recently, there appeared a great interest in equations of the type

$$- \operatorname{div} \left[p(x) \, |\nabla u(x)|^{p(x)-2} \, \nabla u(x) \right] \; = \; 0,$$

where $p : \Omega \to (1, +\infty)$ is a (piecewise) continuous function. These problems are used to model new situations in Mathematical Physics. More precisely, the $p(x)$-laplacian equation is used to model electrorheological fluids, ie special non-Newtonian fluids, which change their mechanical properties in the presence of electromagnetic fields. See [60, 65]. The study of this type of equations leads to the analysis of minimum problems, with energies of the form

$$I(u) \; = \; \int_\Omega |\nabla u(x)|^{p(x)} \; dx,$$

defined in the generalized Sobolev spaces

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^1(\Omega) : \int_\Omega |\eta u(x)|^{p(x)} dx < \infty, \int_\Omega |\eta \nabla u(x)|^{p(x)} dx < \infty, \quad \eta > 0 \right\},$$

under the appropriate assumptions on the function p. See, for instance [34, 72, 73]. Therefore, we may ask about the homogenization of problems of the type

$$\begin{cases} - \operatorname{div} \; \left[a_j(x) \, |\nabla u_j(x)|^{a_j(x)-2} \, \nabla u_j(x) \right] & = \; 0 \quad \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad u_j & = \; 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\{a_j\}$ is a sequence of (piecewise) continuous functions $a_j : \Omega \to (1, +\infty)$.

In this dissertation (Chapter 7), we are interested in studying the Γ-convergence of composite materials with energies of the type

$$I_j(u) \; = \; \int_\Omega \; f(a_j(x)) \, |\nabla u(x)|^{a_j(x)} \; dx,$$

defined in the generalized Sobolev space $W^{1,a_j(x)}(\Omega)$, for each $j \in \mathbb{N}$. We assume that the sequence of functions $a_j : \Omega \to (1, +\infty)$ defined by

$$a_j(x) \; = \; p \; \chi_{(0,t)}(jx \cdot \vec{n}) + q \left(1 - \chi_{(0,t)}(jx \cdot \vec{n}) \right), \qquad t \in (0, 1), \tag{1.8}$$

with $1 < p \leq q < \infty$, stands for a laminate normal to $\vec{n}$, and $f(p)$ and $f(q)$ bounded away from 0. Moreover, no restrictions on p and q are assumed. Thus, the main interest in this type of energies is the fact that we are dealing with a combination, depending on ε, of different powers, defined in intermediate classes of functions between the Sobolev spaces $W^{1,q}(\Omega)$ and $W^{1,p}(\Omega)$. It is known that if the exponents of all materials are the same, e.g. $p = 2 = q$, then the resulting homogenized density will be also a power-law material with the same exponent. See [34]. So, it arises the following question: in case $p \neq q$, how does the limit energy look like, as ε goes to 0? Is the limit energy density either a power of order p or one of order q?

Due to the laminate structure of the domain Ω, we conclude that the sequence $\{I_\varepsilon\}$ is Γ-convergent (with respect to the weak topology of $W^{1,p}(\Omega)$) to the functional

$$I(u) \;=\; \int_\Omega \psi_t(\nabla u(x))\; dx,$$

where the homogenized density $\psi : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi_t(\rho) = \min_{A,B\in\mathbb{R}^n} \left\{\; t\, f(p)\, |A|^p + (1-t)\, f(q)\, |B|^q \;:\; \rho = tA + (1-t)B,\; \vec{n} \parallel B - A \;\right\}.$$

Therefore, the resulting limit energy is neither a power of order p nor one of order q, but is instead a combination of both. This analysis may be translated into the homogenization of Dirichlet problems with $a_j(x)$-laplacian of the type

$$\begin{cases} -\operatorname{div}\; \left[a_j(x)\, f(a_j(x))\, |\nabla u_j(x)|^{a_j(x)-2}\, \nabla u_j(x) \right] & = \;0 \quad \text{in } \Omega \\ \hfill u_j & = \;0 \quad \text{on } \partial\Omega, \end{cases}$$

which may even be written as

$$\begin{cases} -\operatorname{div}\; \left[p\, f(p)\, |\nabla u_j^1(x)|^{p-2}\, \nabla u_j^1(x) \right] & = \;0 \quad \text{in } \Omega_j \\ -\operatorname{div}\; \left[q\, f(q)\, |\nabla u_j^2(x)|^{q-2}\, \nabla u_j^2(x) \right] & = \;0 \quad \text{in } \Omega_j^c \\ \hfill \nabla u_j^1 - \nabla u_j^2 & \parallel \;\vec{n} \quad \text{on } \partial\Omega_j \cap \partial\Omega_j^c \\ \hfill u_j^1,\; u_j^2 & = \;0 \quad \text{on } \partial\Omega, \end{cases}$$

with

$$\Omega_j \;=\; \left\{\; x \in \Omega \;:\; \chi_{(0,t)}(jx\cdot\vec{n}) \;=\; 1 \;\right\},\;\; \Omega_j^c \;=\; \left\{\; x \in \Omega \;:\; \chi_{(0,t)}(jx\cdot\vec{n}) \;=\; 0 \;\right\}.$$

Indeed, the homogenized problem is given by

$$\begin{cases} -\operatorname{div}\; \nabla\psi_t(\nabla u(x)) & = \;0 \quad \text{in } \Omega \\ \hfill u & = \;0 \quad \text{on } \partial\Omega. \end{cases}$$

Moreover, we study the Γ-convergence of general functionals of the type

$$I_j(u) \;=\; \int_\Omega W(a_j(x), \nabla u(x))\; dx$$

defined in $W^{1,a_j(x)}(\Omega)$, where $\{a_j\}$ is given by (1.8), and the continuous density $W : (1,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ is convex in the second variable, and satisfies the non-standard $a_j(x)$-growth condition

$$c\,|\rho|^{a_j(x)} \;\leq\; W(a_j(x),\rho) \;\leq\; C(1+|\rho|^{a_j(x)}), \qquad \text{for a.e. } x \in \Omega, \text{ every } \rho \in \mathbb{R}^n.$$

Under these non-standard assumptions, and without any restriction on the values of a_j, we conclude that the limit energy density $\psi_t : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi_t(\rho) = \min_{A,B \in \mathbb{R}^n} \Big\{ t\,W(p,A) + (1-t)\,W(q,B) \;:\; \rho = tA + (1-t)B, \;\; \vec{n} \parallel B - A \Big\}.$$

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We have thus described briefly the main problems, and their motivations, studied throughout this dissertation, as well as our main contributions to the Γ-convergence and homogenization theories. Let us describe the structure of this dissertation, which is divided into two main parts. The first part is a brief compilation of well known results, most of them cited in the second part. It may be considered as an introduction to the different problems analysed in the second part, and it was written taking into account the readers' convenience. This part is divided into three chapters. The first chapter is dedicated to well known notions and results on Γ-convergence theory, while the second one focuses on well known results concerning homogenization of second-order elliptic equations (in divergence form). The third chapter is an overview on Young measure theory, which is the main tool used to face the problems studied in the second part.

The second part is also divided into three chapters. The first one presents our main contributions to the characterization of the Γ-convergence of non-linear, non-periodic integral functionals, as explained at the beginning of this introduction. This work is also written in [55]. The second chapter presents our results on the explicit representation of homogenized problems of second-order elliptic equations with oscillating source term, in the periodic and non-periodic settings. This work also appears in [56]. Finally, the third chapter presents our contributions to the non-linear homogenization of p-laplacian equations in a laminate structure, also contained in [57].

Part I

Preliminaries

Capítulo 2

Convergencia-Γ de funcionales integrales

La definición formal de convergencia-Γ fue introducida en el trabajo [26], por De Giorgi y Franzoni, fruto de vários estudios desarrollados a partir de un ejemplo simple que De Giorgi estudió a mitad de los 60. El ejemplo consistía en comprender: ¿qué ocurre con las soluciones de una familia de ecuaciones diferenciales ordinarias dependentes de un entero k, de la forma $\frac{d}{dt}(a_k(t)u'(t)) = f(t)$, donde los coeficientes a_k son funciones periódicas tomando dos valores, α y β, alternativamente en intervalos disjuntos de longitud 2^{-k}, cuando hacemos k tender a infinito? En este contexto surgió primeramente la noción de convergencia-G, introducida por Spagnolo en [67], es decir la convergencia de las funciones de Green para operadores de la forma $A_k = \sum_{ij} \frac{\partial}{\partial x_j}\left(a_{ij}^k(x)\frac{\partial}{\partial x_i}\right)$, definida como la convergencia débil de las sucesiones de operadores inversos $\{A_k^{-1}\}$. Posteriormente, se centraron en el carácter variacional de la convergencia-G y, en vez de considerar una sucesión de ecuaciones diferenciales, consideraron una sucesión de problemas minimizantes para funcionales del Cálculo de Variaciones. Así la determinación del minimizante del funcional límite es la idea subyacente a la convergencia-Γ de una familia de funcionales integrales. Para más detalles, vea [25].

Desde entonces la teoría de la convergencia-Γ ha sido desarrollada, principalmente, en conexión con muchas de sus aplicaciones, como, por ejemplo, la homogeneización, es decir el estudio asinptótico de ecuaciones diferenciales en medios heterogéneos con una estrutura periódica. Vea [15, 16, 17, 18, 24, 39, 42].

Básicamente, este capítulo es un breve resumen de los resultados ya conocidos, sobre vários aspectos de la convergencia-Γ, con el objetivo de contextualizar los resultados nuevos que se introducirán en la segunda parte de esta disertación.

2.1. La noción de convergencia-Γ

La convergencia-Γ es una convergencia variacional, como se puede comprender a continuación.

Definición 2.1.1 *Sea* (X, d) *un espacio métrico. Se dice que la sucesión de funciones* $I_j : X \to \overline{\mathbb{R}}$ *converge-*$\Gamma(d)$ *en* X *para* $I : X \to \overline{\mathbb{R}}$ *si, dado cualquier* $u \in X$*,*

1. (desigualdad del lím inf) *para toda la sucesión* $\{u_j\} \subset X$ *tal que* $d(u_j, u) \to 0$*,*

$$I(u) \;\leq\; \liminf_{j\to\infty} \; I_j(u_j);$$

2. (desigualdad del lím sup) *existe una sucesión* $\{u_j\} \subset X$ *tal que* $d(u_j, u) \to 0$ *e*

$$I(u) \;\geq\; \limsup_{j\to\infty} \; I_j(u_j).$$

La función I *se designa por límite-*$\Gamma(d)$ *de* $\{I_j\}$*, y se escribe* $I = \Gamma(d) - \lim_j I_j$.

La desigualdad del límite superior se puede substituir por la igualdad

$$I(u) \;=\; \lim_{j\to\infty} I_j(u_j),$$

porque tenemos las desigualdades

$$I(u) \;\leq\; \liminf_{j\to\infty} \; I_j(u_j) \;\leq\; \limsup_{j\to\infty} \; I_j(u_j) \;\leq\; I(u).$$

También se definen los límites inferior-$\Gamma(d)$ y superior-$\Gamma(d)$.

Definición 2.1.2 *Sea* $I_j : X \to \overline{\mathbb{R}}$ *y* $u \in X$*. El límite inferior-*$\Gamma(d)$ *de* $\{I_j\}$ *en* u *se define como*

$$\Gamma(d) - \liminf_{j\to\infty} I_j(u) \;=\; \inf \left\{ \liminf_{j\to\infty} I_j(u_j) \;:\; \{u_j\} \subset X,\; d(u_j, u) \to 0 \right\}.$$

El límite superior-$\Gamma(d)$ *de* $\{I_j\}$ *en* u *se define como*

$$\Gamma(d) - \limsup_{j\to\infty} I_j(u) \;=\; \inf \left\{ \limsup_{j\to\infty} I_j(u_j) \;:\; \{u_j\} \subset X,\; d(u_j, u) \to 0 \right\}.$$

Si existe el límite-$\Gamma(d)$ de la sucesión $\{I_j\}$ en X, entonces

$$\Gamma(d) - \liminf_{j\to\infty} I_j(u) \;=\; \Gamma(d) - \lim_{j\to\infty} I_j(u) \;=\; \Gamma(d) - \limsup_{j\to\infty} I_j(u)$$

para cualquier $u \in X$.

Nota 2.1.1 *1. Una sucesión constante $\{I\}$ converge-$\Gamma(d)$ a la envoltura semicontinua inferior de I definida por*

$$\overline{I}(u) \;=\; \inf\left\{\liminf_{j\to\infty} I_j(u_j) \;:\; \{u_j\}\subset X,\; d(u_j,u)\to 0\right\}.$$

2. Las funciones $\Gamma(d)-\liminf_j I_j$ y $\Gamma(d)-\limsup_j I_j$ son semicontinuas inferiormente[1] *en X.*

3. Si G es una función continua e $I=\Gamma(d)-\lim_j I_j$, entonces $I+G=\Gamma(d)-\lim_j(I_j+G)$.

La compacidad y la convergencia de los problemas de mínimo son otras propriedades importantes de la convergencia-Γ.

Teorema 2.1.1 (Compacidad) (Vea [24]) *Sea (X,d) un espacio métrico separable*[2]*, e $I_j : X\to\overline{\mathbb{R}}$ una función, $\forall j\in\mathbb{N}$. Entonces existe una sucesión creciente $\{j_k\}\subset\mathbb{N}$ tal que el límite-$\Gamma(d)$ de la sucesión $\{I_{j_k}\}$ existe para todo $u\in X$.*

Teorema 2.1.2 (Convergencia de los problemas de mínimo) (Vea [24]) *Sea $\{I_j\}$ una sucesión d-equicoerciva (ie, existe un conjunto compacto $K\subset X$ tal que $\inf_X I_j=\inf_K I_j,\ \forall j\in\mathbb{N}$). Entonces*

$$\min_{u\in X}\ \Gamma(d){-}\liminf_{j\to\infty} I_j(u) \;=\; \liminf_{j\to\infty}\ \inf_{u\in X} I_j(u),$$

y, si $\{I_j\}$ es convergente-$\Gamma(d)$,

$$\min_{u\in X}\ \Gamma(d){-}\lim_{j\to\infty} I_j(u) \;=\; \lim_{j\to\infty}\ \inf_{u\in X} I_j(u).$$

Si $\{u_j\}$ es convergente en X, y $\lim_j I_j(u_j)=\lim_j\inf_{u\in X} I_j(u)$, entonces $\{u_j\}$ converge al punto de mínimo del límite-$\Gamma(d)$.

La d-equicoercividad de la sucesión $\{I_j\}$ es una condición suficiente para la convergencia de los mínimos de I_j al mínimo del límite-$\Gamma(d)$. Luego, una cuestión importante en la convergencia-Γ es comprender cómo elegir correctamente el espacio métrico (X,d). La topología de la métrica d deve ser suficientemente débil para obtenermos la d-equicoercividad de la sucesión $\{I_j\}$, pero suficientemente fuerte para que converja-$\Gamma(d)$.

Típicamente se siguen tres pasos fundamentales para demostrar la convergencia-Γ :

[1] f es semicontinua inferiormente en $u\in X$ si, para toda la sucesión $\{u_j\}$ convergente a u, $f(u)\le\liminf_j f(u_j)$.

[2] Un espacio métrico es separable si existe un subconjunto numerable y denso.

(i) probar un resultado de compacidad que garanta la existencia de una subsucesión que converja-Γ a un funcional límite abstracto;

(ii) probar un resultado de representación integral que permita escribir el funcional límite como un integral;

(iii) probar una representación explícita del integrando límite que no dependa de la subsucesión.

Si consideramos una sucesión de funcionales integrales definidos en un espacio de Sobolev, entonces se aplica el método directo de la convergencia-Γ para probar un resultado de compacidad general, y luego obtener la representación integral del límite-Γ. Para tal se consideran funcionales integrales dependientes de los conjuntos de integración, además de las funciones en el espacio de Sobolev, lo que se designa por método de localización. Así, en la próxima sección, consideramos funcionales integrales $I : L^p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty]$ de la forma

$$I(u,O) = \begin{cases} \int_O f(x,\nabla u(x))\,dx & \text{si } u \in W^{1,p}(O;\mathbb{R}^d) \\ \\ +\infty & \text{si } u \in L^p(O;\mathbb{R}^d) \setminus W^{1,p}(O;\mathbb{R}^d), \end{cases}$$

tal que

$$c|A|^p \le f(x,A) \le C(1+|A|^p) \qquad \forall\, x \in \Omega,\ A \in \mathbb{R}^{d\times n},$$

y presentamos algunos resultados de representación integral de límites-Γ. Se considera la convergencia-Γ de sucesiones de funcionales respecto a la topología fuerte en $L^p(\Omega;\mathbb{R}^d)$. De hecho, el funcional $I(\cdot,\Omega)$ es semicontinuo inferiormente respecto a la topología débil en $W^{1,p}(\Omega;\mathbb{R}^d)$ si y solamente si es semicontinuo inferiormente respecto a la topología fuerte en $L^p(\Omega;\mathbb{R}^d)$.

2.2. Representación integral de límites-Γ en espacios de Sobolev

Los teoremas de representación integral para funcionales no lineales $I(u,O)$, dependientes de una función u y de un conjunto abierto O, son útiles para demostrar que el límite-Γ de una sucesión de funcionales integrales es también un integral funcional. En concreto, en el próximo teorema, Buttazzo y Dal Maso obtuvieron condiciones suficientes para que un funcional abstracto admita una representación integral. Vea [20] y las referencias contenidas.

Teorema 2.2.1 (Vea [20]) *Sea $1 \le p < \infty$, e $I : W^{1,p}(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty)$ un funcional satisfaciendo las siguientes condiciones:*

i) I es local, ie $I(u,O) = I(v,O)$ si $u = v$ c.s. en $O \in \mathcal{A}(\Omega)$;

ii) *I es una medida en $\mathcal{A}(\Omega)$, ie, para todo $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, la función de conjunto $I(u,\cdot)$ es la restrición de una medida de Borel en $\mathcal{A}(\Omega)$;*

iii) *I satisface una condición de crecimiento de orden p, es decir existe $c > 0$ y $a \in L^1(\Omega)$ tal que*

$$I(u,O) \;\leq\; c\int_O [\, a(x) + |\nabla u(x)|^p \,]\; dx,$$

para todo $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ y $O \in \mathcal{A}(\Omega)$;

iv) *I es invariante por translación en u, es decir $I(u+z,O) = I(u,O)$ para todo $z \in \mathbb{R}^d$, $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ y $O \in \mathcal{A}(\Omega)$;*

v) *para todo $O \in \mathcal{A}(\Omega)$, $I(\cdot,O)$ es semicontinuo inferiormente respecto a la topología débil en $W^{1,p}(\Omega;\mathbb{R}^d)$.*

Entonces existe una función $f : \Omega \times \mathbb{R}^{d\times n} \to [0,+\infty)$ tal que

a) *f es una función de Carathéodory;*

b) *f satisface la condición de crecimiento de orden p, es decir para todo $x \in \Omega$ y $A \in \mathbb{R}^{d\times n}$ se verifica*

$$0 \;\leq\; f(x,A) \;\leq\; c\,(a(x) + |A|^p);$$

c) *para todo $O \in \mathcal{A}(\Omega)$ y $u \in W^{1,p}(\Omega;\mathbb{R}^d)$*

$$I(u,O) \;=\; \int_O f(x,\nabla u(x))\; dx.$$

Nota que el integrando f, en el teorema anterior, es cuasiconvexo[3] en la segunda variable. De hecho, el funcional I, cuyo integrando f es una función de Carathéodory que satisface la condición de crecimiento de orden p, es semicontinuo inferior débilmente en $W^{1,p}(\Omega;\mathbb{R}^d)$ si y solamente si f es cuasiconvexa en la segunda variable (vea [1, Statemente II.5]).

De acuerdo con el próximo resultado, siempre que una familia de funcionales integrales, definidos en el espacio de Sobolev $W^{1,p}(\Omega;\mathbb{R}^d)$, tenga crecimiento de orden p, existe una subsucesión de funcionales que converge-$\Gamma(L^p)$ (en la topología fuerte de $L^p(\Omega;\mathbb{R}^d)$) a un funcional integral.

[3] La función continua $f : \mathbb{R}^{d\times n} \to \mathbb{R}$ es cuasiconvexa si, para cualquier $A \in \mathbb{R}^{d\times n}$, $f(A) \leq \frac{1}{|D|}\int_D f(A + \nabla w(z))\,dz$ para algun conjunto abierto $D \subset \mathbb{R}^n$, y cualquier $w \in C_0^\infty(D;\mathbb{R}^d)$.

Teorema 2.2.2 (Vea [24]) *Sea* $\{I_j\}$ *una sucesión de funcionales definidos en* $W^{1,p}(\Omega;\mathbb{R}^d)$ *por*

$$I_j(u,O) \;=\; \int_O f_j(x,\nabla u(x))\;dx,$$

para cualquier $O\in\mathcal{A}(\Omega)$, *donde las funciones borelianas*[4] $f_j:\Omega\times\mathbb{R}^{d\times n}\to[0,+\infty)$ *satisfacen la condición de crecimiento*

$$c|A|^p \;\le\; f_j(x,A) \;\le\; C(1+|A|^p)$$

para todo $x\in\Omega$ *y* $A\in\mathbb{R}^{d\times n}$. *Entonces existe una subsucesión* $\{I_{j_k}\}$ *y una función de Carathéodory* $f:\Omega\times\mathbb{R}^{d\times n}\to[0,+\infty)$ *satisfaciendo la misma condición de crecimiento que* f_j , *tal que* $\{I_{j_k}\}$ *es convergente-*$\Gamma(L^p)$ *al funcional* I *definido por*

$$I(u,O) \;=\; \int_O f(x,\nabla u(x))\;dx$$

para todo $u\in W^{1,p}(\Omega;\mathbb{R}^d)$ *y* $O\in\mathcal{A}(\Omega)$.

A continuación se presenta un resultado de representación integral de límites-Γ de sucesiones de funcionales integrales satisfaciendo una condición de crecimiento con exponentes p y q.

Teorema 2.2.3 (Vea [18]) *Sea* Ω *con frontera lipschitziana, y sea* $p\le q<p^*$, *donde* p^* *es el exponente de Sobolev*[5] *de* p. *Sea* $\{G_\varepsilon\}$ *una sucesión de funcionales* $G_\varepsilon: L^p(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$ *definidos por*

$$G_\varepsilon(u,O) \;=\; \begin{cases} \int_O g_\varepsilon(x,\nabla u(x))\;dx & \text{si } u\in W^{1,p}(\Omega;\mathbb{R}^d)\\[2ex] +\infty & \text{si } u\in L^p(\Omega;\mathbb{R}^d)\setminus W^{1,p}(\Omega;\mathbb{R}^d),\end{cases}$$

donde las funciones borelianas $g_\varepsilon:\Omega\times\mathbb{R}^{d\times n}\to[0,+\infty)$ *satisfacen la condición de crecimiento no estándar de orden* p *y* q, *es decir*

(H4) existen constantes $0<c\le C$ *tal que*

$$c|A|^p \;\le\; g_\varepsilon(x,A) \;\le\; C(1+|A|^q), \qquad \forall\, x\in\Omega,\; A\in\mathbb{R}^{d\times n},\; \varepsilon>0. \quad (2.1)$$

Si la sucesión $\{G_\varepsilon\}$ *satisface la estimación fundamental en* L^q, *cuando* $\varepsilon\searrow 0$[6], *entonces existe una subsucesión* $\{G_{\varepsilon_k}\}$ *y una función de Carathéodory* g :

[4] f es una función boreliana si el funcional $I(u)=\int_\Omega f(x,\nabla u(x))\,dx$ está bien definido.

[5] $p^*=\frac{np}{n-p}$ si $p<n$; $p^*=\infty$ si $p\ge n$.

[6] $\{G_\varepsilon\}$ satisface la estimación fundamental en L^q, cuando $\varepsilon\searrow 0$, si para todo $U,\overline{U},V\in\mathcal{A}(\Omega)$, con $\overline{U}\subset\subset U$, y $\sigma>0$ existen $M_\sigma>0$ y $\varepsilon_\sigma>0$ tal que para todo $u,v\in L^q(\Omega;\mathbb{R}^d)$ y $\varepsilon<\varepsilon_\sigma$ existe una función *cut-off* φ entre $\overline{U}$ y U tal que $G_\varepsilon(\varphi u+(1-\varphi)v,\overline{U}\cup V)\le(1+\sigma)(G_\varepsilon(u,U)+G_\varepsilon(v,V))+M_\sigma\int_{(U\cap V)\setminus\overline{U}}|u-v|^q dx+\sigma$.

$\Omega \times \mathbb{R}^{d\times n} \to [0,+\infty)$, *que verifica la condición (2.1), tal que* $\{G_{\varepsilon_k}\}$ *converge-*$\Gamma(L^p)$ *al funcional* $G : W^{1,p}(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty)$ *definido por*

$$G(u,O) \;=\; \int_O g(x,\nabla u(x))\; dx.$$

Por otra parte, se pueden considerar funcionales integrales definidos en los espacios de Sobolev generalizados $W^{1,p(x)}(\Omega;\mathbb{R}^d)$, donde $p:\Omega\to(1,+\infty)$ es una función continua satisfaciendo la estimación sobre el modulo de continuidad

$$\forall O\subset\mathcal{A}(\Omega)\ \exists\gamma_O>0\ :\ |p(x)-p(y)|\le\frac{\gamma_O}{|\log|x-y||}\ \ \forall x,y\in O,\ 0<|x-y|<\frac{1}{2}. \tag{2.2}$$

Teorema 2.2.4 (Vea [22]) *Sea* $p:\Omega\to(1,+\infty)$ *una función continua, con* $p(x)\ge p>1$ *para todo* $x\in\Omega$, *tal que satisface la estimación (2.2). Sea* $g_\varepsilon:\Omega\times\mathbb{R}^{d\times n}\to[0,+\infty)$ *una función boreliana satisfaciendo la condición de crecimiento*

$$c\,|A|^{p(x)}\ \le\ g_\varepsilon(x,A)\ \le\ C\,(1+|A|^{p(x)})\ \ \textit{para c.t. } x\in\Omega,\ \textit{todo } A\in\mathbb{R}^{d\times n}, \tag{2.3}$$

con $C\ge c>0$, *para cualquier* $\varepsilon>0$, *y* $G_\varepsilon: L^1(\Omega,\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$ *el funcional definido por*

$$G_\varepsilon(u,O)\;=\;\begin{cases}\int_O g_\varepsilon(x,\nabla u(x))\;dx & si\ u\in W^{1,p(x)}_{loc}(O;\mathbb{R}^d)\\[2ex] +\infty & si\ u\in L^1(\Omega;\mathbb{R}^d)\setminus W^{1,p(x)}_{loc}(O;\mathbb{R}^d).\end{cases}$$

Entonces existe una subsucesión $\{\varepsilon(k)\}$, *y una función de Carathéodory* $g:\Omega\times\mathbb{R}^{d\times n}\to[0,+\infty)$ *cuasiconvexa en la segunda variable, satisfaciendo la condición (2.3), tal que* $\{G_{\varepsilon(k)}\}$ *es convergente-*$\Gamma(L^1)$ *(en la topología fuerte de* $L^1(\Omega;\mathbb{R}^d)$*) al funcional*

$$G(u,O)\;=\;\begin{cases}\int_O g(x,\nabla u(x))\;dx & si\ u\in W^{1,p(x)}_{loc}(O;\mathbb{R}^d)\\[2ex] +\infty & si\ u\in L^1(\Omega;\mathbb{R}^d)\setminus W^{1,p(x)}_{loc}(O;\mathbb{R}^d).\end{cases}$$

2.3. Convergencia-Γ periódica

En general, la representación explícita del integrando límite no es conocida, excepto en el caso periódico. En esta sección nos centramos en la convergencia-Γ de familias de funcionales integrales con integrandos periódicos. Sea $\{F_\varepsilon\}$ una sucesión de funcionales F_ε definidos en $W^{1,p}(\Omega;\mathbb{R}^d)$ por

$$F_\varepsilon(u)\;=\;\int_\Omega f\left(\frac{x}{\varepsilon},\nabla u(x)\right)\;dx, \tag{2.4}$$

con $\varepsilon>0$, donde la función boreliana $f:\mathbb{R}^n\times\mathbb{R}^{d\times n}\to[0,+\infty)$ satisface las siguientes hipótesis:

(H1) $f(\cdot, A)$ es Q-periódica, para todo $A \in \mathbb{R}^{d\times n}$,

(H2) existen constantes $0 < c \leq C$ tal que, para $p \geq 1$,

$$c|A|^p \ \leq \ f(y, A) \ \leq \ C(1 + |A|^p), \qquad \forall\, y \in \mathbb{R}^n,\ A \in \mathbb{R}^{d\times n}.$$

En el caso escalar, $d = 1$, si además, para c.t. $x \in \Omega$, $f(x, \cdot) \in C^1(\mathbb{R}^n)$ es convexa, inicialmente Marcellini demostró en [39] que el integrando, del límite-Γ de la sucesión $\{F_\varepsilon\}$, es la función convexa $f_{hom} : \mathbb{R}^n \to \mathbb{R}$, con crecimiento de orden p, definida por

$$f_{hom}(A) \ = \ \inf_{v \in W^{1,p}_{per}(Q)} \int_Q f(y, A + \nabla v(y))\, dy.$$

Todavía, en el caso vectorial, $d > 1$, si $f(x, \cdot)$ es no convexa, Müller y Braides probaron, en [42] y [16] respectivamente, que esta representación no es necesariamente válida.

Teorema 2.3.1 (Vea [18]) *La sucesión de funcionales F_ε en (2.4), con f satisfaciendo (H1) y (H2), converge-$\Gamma(L^p)$ al funcional F definido en $W^{1,p}(\Omega; \mathbb{R}^d)$ por*

$$F(u) \ = \ \int_\Omega f_{hom}(\nabla u(x))\, dx,$$

donde la función $f_{hom} : \mathbb{R}^{d\times n} \to [0, +\infty)$ es cuasiconvexa y definida por

$$f_{hom}(A) \ = \ \lim_{T\to\infty} \ \inf_{v \in W^{1,p}_0(TQ;\mathbb{R}^d)} \frac{1}{T^n} \int_{TQ} f(y, A + \nabla v(y))\, dy.$$

Así, la función homogeneizada f_{hom} es el límite, cuando la anchura de las celdas tende a infinito, del ínfimo de las variaciones en cualquier celda. Además, el ínfimo se puede considerar en el conjunto más grande de las funciones periódicas, es decir

$$f_{hom}(A) \ = \ \lim_{T\to\infty} \ \inf_{v \in W^{1,p}_{per}(Q;\mathbb{R}^d)} \int_Q f(Ty, A + \nabla v(y))\, dy\,;$$

y el límite en la anchura se puede tomar como el ínfimo,

$$f_{hom}(A) \ = \ \inf_{T\in\mathbb{N}} \ \inf_{v \in W^{1,p}_{per}(TQ;\mathbb{R}^d)} \frac{1}{T^n} \int_{TQ} f(y, A + \nabla v(y))\, dy.$$

Teorema 2.3.2 (Vea [18]) *Si, además de las hipótesis (H1) y (H2), la función f satisface la condición*

(H3) $f(y, \cdot)$ es convexa en $\mathbb{R}^{d\times n}$, para todo $y \in \mathbb{R}^n$,

entonces la sucesión de funcionales F_ε *definidos en (2.4) converge-*$\Gamma(L^p)$ *a* F *donde la función* f_{hom} *está definida a través del problema en la celda unitaria, es decir*

$$f_{hom}(A) \;=\; \inf_{v\in W^{1,p}_{per}(Q;\mathbb{R}^d)} \int_Q f(y, A+\nabla v(y))\; dy, \qquad \forall\; A\in\mathbb{R}^{d\times n}. \tag{2.5}$$

La definición del integrando f_{hom} como el ínfimo de las variaciones periódicas en la celda unitária no es válida para el caso vectorial no convexo, como destaca el contraejemplo en [42], porque este valor es superior (y no igual) al ínfimo de las mismas variaciones pero en conjuntos de T copias de la celda unitaria, cuando hacemos T tender a infinito. En realidad, cuando $f(y,\cdot)$ es no convexa es necesario considerar variaciones periódicas en celdas de anchura T, independientemente de si las condiciones en la frontera son periódicas o zero.

2.4. Convergencia-Γ periódica con condiciones de crecimiento no estándar

En el caso periódico, la representación explícita del integrando límite no depende de las condiciones de crecimiento.

Teorema 2.4.1 (Vea [18]) *Sea* Ω *con frontera lipschitziana, y sea* $p\le q<p^*$*, donde* p^* *es el exponente de Sobolev. Sea* $g:\mathbb{R}^n\times\mathbb{R}^{d\times n}\to[0,+\infty)$ *una función boreliana tal que*

(H1) $g(\cdot,A)$ *es Q-periódica, para todo* $A\in\mathbb{R}^{d\times n}$*,*

(H4) existen constantes $0<c\le C$ *tal que*

$$c|A|^p \;\le\; g(y,A) \;\le\; C(1+|A|^q), \qquad \forall\; y\in\mathbb{R}^n,\; A\in\mathbb{R}^{d\times n}.$$

Entonces la sucesión de funcionales $G_\varepsilon : L^p(\Omega;\mathbb{R}^d)\to[0,+\infty]$ *definidos por*

$$G_\varepsilon(u) \;=\; \begin{cases} \int_\Omega g\left(\frac{x}{\varepsilon},\nabla u(x)\right)\; dx & \text{si } u\in W^{1,p}(\Omega;\mathbb{R}^d) \\ \\ +\infty & \text{si } u\in L^p(\Omega;\mathbb{R}^d)\setminus W^{1,p}(\Omega;\mathbb{R}^d) \end{cases}$$

es convergente-$\Gamma(L^p)$ *al funcional* G *definido en* $W^{1,p}(\Omega;\mathbb{R}^d)$ *por*

$$G(u) \;=\; \int_\Omega g_{hom}(\nabla u(x))\; dx,$$

donde

$$g_{hom}(A) \;=\; \lim_{T\to\infty}\; \inf_{v\in W^{1,p}_0(TQ;\mathbb{R}^d)} \frac{1}{T^n}\int_{TQ} g(y, A+\nabla v(y))\; dy, \qquad \forall\; A\in\mathbb{R}^{d\times n}.$$

Nota 2.4.1 *Si los funcionales G_ε solo toman valores finitos en un espacio más pequeño que $W^{1,p}(\Omega;\mathbb{R}^d)$, es decir*

$$G_\varepsilon(u) = \begin{cases} \int_\Omega g\left(\frac{x}{\varepsilon}, \nabla u(x)\right)\, dx & si\ u \in W^{1,r}(\Omega;\mathbb{R}^d),\ r \geq p, \\ \\ +\infty & si\ u \in L^p(\Omega;\mathbb{R}^d) \setminus W^{1,r}(\Omega;\mathbb{R}^d), \end{cases}$$

entonces la sucesión converge-$\Gamma(L^p)$ *a G, donde el integrando*

$$g_{hom}(A) = \lim_{T\to\infty} \inf_{v\in W_0^{1,r}(TQ;\mathbb{R}^d)} \frac{1}{T^n} \int_{TQ} g(y, A + \nabla v(y))\, dy, \qquad \forall\ A \in \mathbb{R}^{d\times n}.$$

Teorema 2.4.2 (Vea [18]) *Si, además de las hipótesis del Teorema 2.4.1, la función g satisface la condición*

(H3) $g(y,\cdot)$ es convexa en $\mathbb{R}^{d\times n}$, para todo $y \in \mathbb{R}^n$,

entonces la sucesión de funcionales G_ε converge-$\Gamma(L^p)$ *a G donde*

$$g_{hom}(A) = \inf_{v\in W_{per}^{1,p}(Q;\mathbb{R}^d)} \int_Q g(y, A + \nabla v(y))\, dy, \qquad \forall\ A \in \mathbb{R}^{d\times n}. \tag{2.6}$$

Chapter 3

Homogenization of elliptic equations

The study of composite materials, macroscopic properties of crystalline or polymer structures, in Mechanics, Physics, Chemistry and Engineering, led to the study of procedures to pass from a microscopic description to a macroscopic one of the behaviour of periodic structures. In general, the physical parameters, as conductivity, elasticity coefficients, of composite materials are discontinuous and oscillate between different values of each component. Though these parameters oscillate very rapidly, when the components are intimately mixed so that the microscopic structure is very complicated, from a macroscopic point of view, the composite material tends to behave as a homogenous material.

The homogenization theory consists in describing the limit behaviour of composite materials when the parameter ε, which gives the fineness of the microscopic structure, tends to 0. It was introduced by Sanchez-Palencia in [66], with the study of thermal problems in composite materials, and remarkable developed, besides him and many others, by Bensoussan, Lions and Papanicolaou in [14], Tartar and Murat in [44], De Giorgi and Spagnolo in [27], ...

To make it more clear, let us describe the homogenization of the stationary heat equation of an ε-periodic material. Consider the conductivity coefficients $a_{i,j} \in L^\infty(\mathbb{R}^n)$ such that

- $a_{i,j}$ is Q-periodic, for every $1 \leq i, j \leq n$,
- $\sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq c|\xi|^2$, for every $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$.

Assume the composite material occupies a region Ω and its temperature at the boundary $\partial\Omega$ is constant $u = 0$. Then, for every external heat source f, the temperature u_ε satisfies the equation

$$\begin{cases} -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{i,j}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}(x)\right) &= f(x) \quad \text{in} \quad \Omega \\ u_\varepsilon &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

From the mathematical point of view, this problem is well-posed and, for every $f \in L^2(\Omega)$, admits a unique solution $u_\varepsilon \in H_0^1(\Omega)$. The sequence of solutions u_ε of the previous equations, as $\varepsilon \to 0$, converges weakly in $H_0^1(\Omega)$ to the solution u of the homogenized equation

$$\begin{cases} -\sum_{i,j} a_{i,j}^{hom}\frac{\partial^2 u}{\partial x_i \partial x_j}(x) &= f(x) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where the constant coefficients $a_{i,j}^{hom}$, which are called the homogenized (or effective) coefficients, are defined through the coefficients $a_{i,j}$ and do not depend on f and Ω. The homogenized equation gives the macroscopic behaviour of a homogenous material quite similar to the composite.

3.1. The classical convergence result for periodic structures

Consider the previous classical problem

$$(CP_\varepsilon) \qquad \begin{cases} -\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x) &= f(x) \quad \text{in} \quad \Omega \\ u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

where the matrix function $A = [a_{ij}] \in [L^\infty(Y)]^{n\times n}$ satisfies, for some $0 < \alpha < \beta$,

i) a_{ij} is Y-periodic,[1] for every $1 \le i, j \le n$,

ii) $A(y)\xi\cdot\xi \ge \alpha|\xi|^2$, for a.e. $y \in Y$ and every $\xi \in \mathbb{R}^n$,

iii) $|A(y)\xi| \le \beta|\xi|$, for a.e. $y \in Y$ and every $\xi \in \mathbb{R}^n$,

and $f \in H^{-1}(\Omega)$. Here Y is the reference cell

$$Y = (0, c_1) \times \ldots \times (0, c_n),$$

for some positive numbers $c_1, \ldots, c_n$.

Notice that condition ii), ie

$$\sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \ \ge\ \alpha \sum_{i=1}^{n} \xi_i^2,$$

[1] a_{ij} is Y-periodic iff $a_{ij}(y + kc_h e_h) = a_{ij}(y)$, for a.e. $y \in Y$, every $k \in \mathbb{Z}$ and $1 \le h \le n$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

is the uniform ellipticity condition for the operator

$$-\operatorname{div} A(y)\nabla = -\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) : H_0^1(Y) \longrightarrow H^{-1}(Y),$$

and, in particular, implies the invertibility of $A(y)$, for a.e. $y \in Y$. Besides, together with condition *ii*), it implies $\alpha|\xi|^2 \leq A(y)\xi\cdot\xi \leq \beta|\xi|^2$. Since $L^2(\Omega)$ is dense in $H^{-1}(\Omega)$, the problem (CP_ε) has a unique solution, too.

As we remarked before, the homogenized problem associated with (CP_ε) consists in studying which homogenized equation the limit of the sequence of solutions u_ε as $\varepsilon \to 0$ satisfies.

Theorem 3.1.1 (See [14]) *If u_ε is the solution of (CP_ε), under the previous assumptions, then*

- $u_\varepsilon \rightharpoonup u_0$ *in* $H_0^1(\Omega)$
- $A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0$ *in* $[L^2(\Omega)]^n$,

where u_0 is the unique solution of the homogenized equation

$$(CP_0) \qquad \begin{cases} -\operatorname{div} A_0 \nabla u_0(x) = f(x) & \text{in } \Omega \\ u_0 \in H_0^1(\Omega). \end{cases}$$

The effective coefficient $A_0 = [a^0_{i,j}] \in \mathbb{R}^{n\times n}$ is a constant matrix given by

$$A_0\,\xi = \frac{1}{|Y|}\int_Y A(y)\nabla w^\xi(y)\, dy, \qquad \text{for every } \xi \in \mathbb{R}^n,$$

where w^ξ is the solution of

$$\begin{cases} \int_Y A(y)\nabla w^\xi(y)\cdot\nabla v(y)\, dy = 0, & \forall\, v \in H^1_{per}(Y) : \ \frac{1}{|Y|}\int_Y v(y)\, dy = 0, \\ w^\xi - \xi\cdot y \in H^1_{per}(Y), \quad \frac{1}{|Y|}\int_Y \left(w^\xi(y) - \xi\cdot y\right) dy = 0, \end{cases}$$

or, equivalently, by

$$A_0^T\,\xi = \frac{1}{|Y|}\int_Y A^T(y)\left(\xi + \nabla z^\xi(y)\right)\, dy, \qquad \text{for every } \xi \in \mathbb{R}^n,$$

where z^ξ is the solution of

$$\begin{cases} -\operatorname{div} A^T(y)\left(\xi + \nabla z^\xi(y)\right) = 0 & \text{in } Y \\ z^\xi \in H^1_{per}(Y), \quad \frac{1}{|Y|}\int_Y z^\xi(y)\, dy = 0. \end{cases}$$

This classical result, which was initially proved by Sanchez-Palencia in [66] and Bensoussan, Lions and Papanicolaou, see [14], may be proved by different methods, such as

1. the method of asymptotic expansions,
2. the energy method of Tartar,
3. the two-scale convergence method of Nguetseng and Allaire.

The method of asymptotic expansions consists in searching for a solution of type

$$u_\varepsilon(x) \;=\; u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \ldots \;=\; \sum_{i=0}^{\infty} \varepsilon^i\, u_i\left(x, \frac{x}{\varepsilon}\right),$$

where $u_i(x,y)$ are Y-periodic in y, which is an asymptotic expansion based on the macroscopic scale x and the microscopic one $\frac{x}{\varepsilon}$ characterizing the problem (CP_ε). At the end it is shown that u_0, which does not depend on y, is indeed the solution of the homogenized problem (CP_0). For more details see [14].

The energy method of Tartar consists in constructing a family of oscillatory test functions w_ε so that the sequence $\{A\left(\cdot, \frac{\cdot}{\varepsilon}\right) \nabla w_\varepsilon\}$ has a compact divergence[2] in $H^{-1}(\Omega)$ which allows to pass to the limit in (CP_ε) as $\varepsilon \to 0$.

The two-scale convergence method is based on the notion of two-scale convergence whose test functions are of the type $\varphi\left(x, \frac{x}{\varepsilon}\right)$ taking into account both scales of the problem (CP_ε). For more details see [5, 7, 45, 46]. In the next section we focus on this method.

3.2. Reiterated homogenization by the multi-scale convergence method

Consider the multiscale homogenization problem

$$(P_\varepsilon^N) \quad \begin{cases} -\operatorname{div} A\left(x, \frac{x}{l_1(\varepsilon)}, \ldots, \frac{x}{l_N(\varepsilon)}\right) \nabla u_\varepsilon(x) &= f(x) \quad \text{in } \Omega \\ \qquad\qquad\qquad u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

where the matrix function $A \in [L^\infty(\Omega \times Y_1 \times \ldots \times Y_N)]^{n\times n}$ satisfies

i) A is Y_k-periodic, for all $1 \le k \le N$,

ii) $\alpha|\xi|^2 \le A(x, y_1, \ldots, y_N)\xi \cdot \xi \le \beta|\xi|^2$ a.e. in $\Omega \times Y_1 \times \ldots \times Y_N$, for every $\xi \in \mathbb{R}^n$,

[2] Ie, there exists a subsequence of $\left\{\operatorname{div} A\left(\cdot, \frac{\cdot}{\varepsilon}\right) \nabla w_\varepsilon\right\}$ strongly convergent in $H^{-1}(\Omega)$.

$\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$ is a family of separated length scales, defined below, and $f \in L^2(\Omega)$.

Such problem may describe a conductivity problem in a composite material, with a periodic structure, occupying a bounded open set Ω. The multiple separated length scales depend on the parameter ε, and the conductivity tensor is given by $A\left(x, \frac{x}{l_1(\varepsilon)}, \ldots, \frac{x}{l_N(\varepsilon)}\right)$, while f is a fixed source term. The solution of such problem will be the potential u_ε. Bensoussan, Lions and Papanicolaou studied this problem in [14], and called it reiterated homogenization problem.

The length scale separability ensures that each scale is of a different order of magnitude. The trivial case of separated length scales is when they are simply powers, ie $l_k(\varepsilon) = \varepsilon^k$, for each k.

Definition 3.2.1 *A smooth function $l : (0, \varepsilon_0) \to (0, +\infty)$, for some $\varepsilon_0 > 0$, is said to be a length scale if*

$$\lim_{\varepsilon \searrow 0} l(\varepsilon) \;=\; 0.$$

The family of length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$ is said to be separated if

$$\lim_{\varepsilon \searrow 0} \frac{l_{k+1}(\varepsilon)}{l_k(\varepsilon)} \;=\; 0, \qquad \textit{for every} \;\; 1 \le k \le N-1.$$

Here we recall the notion of multiscale convergence, which was introduced by Allaire and Briane in [6] as the generalization of the two-scale convergence introduced by Nguetseng in [45], to treat the homogenization problem (P_ε^N).

Definition 3.2.2 *A sequence $\{u_\varepsilon\} \subset L^2(\Omega)$ is said to multiscale (or $(N+1)$-scale) converge to $u_0 \in L^2(\Omega \times Y_1 \times \ldots \times Y_N)$ if*

$$\lim_{\varepsilon \searrow 0} \int_\Omega u_\varepsilon(x)\, \varphi\left(x, \frac{x}{l_1(\varepsilon)}, \ldots, \frac{x}{l_N(\varepsilon)}\right) dx \;=$$
$$= \;\int_\Omega \int_{Y_1} \cdots \int_{Y_N} u_0(x, y_1, \ldots, y_N)\, \varphi(x, y_1, \ldots, y_N)\; dy_1 \ldots dy_N\; dx$$

for any $\varphi \in L^2[\Omega; C_{per}(Y_1 \times \ldots \times Y_N)]$, and any family of separated length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$.

Theorem 3.2.1 (See [45]) *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then there exists a subsequence $\{u_{\varepsilon_k}\}$ and a function $u_0 \in L^2(\Omega \times Y_1 \times \ldots \times Y_N)$ such that $\{u_{\varepsilon_k}\}$ $(N+1)$-scale converges to u_0.*

In the case of bounded sequences in $H^1(\Omega)$, the following theorem defines the $N+1$-scale limit of bounded sequences of gradients.

Theorem 3.2.2 (See [7]) *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $H^1(\Omega)$. *Then there exists a function* $u \in H^1(\Omega)$ *and functions* $u_k \in L^2[\Omega \times Y_1 \times \ldots \times Y_{k-1}; H^1_{per}(Y_k)]$, *for* $1 \le k \le N$, *such that, up to a subsequence,*

- $\{u_\varepsilon\}$ *is* $(N+1)$*-scale convergent to* u,
- $\{\nabla u_\varepsilon\}$ *is* $(N+1)$*-scale convergent to* $\nabla u + \sum_{k=1}^{N} \nabla_{y_k} u_k$.

Notice that the function u is the weak limit of the sequence $\{u_\varepsilon\}$ in $H^1(\Omega)$, while each function ∇u_k may be considered as the limit at the length scale $l_k(\varepsilon)$ of $\{\nabla u_\varepsilon\}$, for $1 \le k \le N$.

Under the assumption on the $(N+1)$-scale convergence of the sequence of matrix functions

$$A_\varepsilon(x) \;=\; A\left(x, \frac{x}{l_1(\varepsilon)}, \ldots, \frac{x}{l_N(\varepsilon)}\right),$$

and the convergence of its L^2-norm, the following result states the convergence of the solutions u_ε of (P^N_ε) and characterizes its limit.

Theorem 3.2.3 (See [7]) *Let* u_ε *be the solution of* (P^N_ε). *If*

(H1) $\{A_\varepsilon\}$ *is* $(N+1)$*-scale convergent to* A,

(H2) $\lim_{\varepsilon \searrow 0} \|A_\varepsilon\|_{[L^2(\Omega)]^{n\times n}} = \|A\|_{L^2(\Omega\times Y_1\times\ldots\times Y_N)}$,

then

$$u_\varepsilon \;\rightharpoonup\; u \quad in \quad H^1_0(\Omega)$$

where u *is the unique solution of the homogenized problem*

$$(P^N_0) \qquad \begin{cases} -\operatorname{div}\, A_0(x)\nabla u(x) &=& f(x) \quad in \quad \Omega \\ u &\in& H^1_0(\Omega). \end{cases}$$

The homogenized matrix function A_0 *is defined by the inductive homogenization formulae:*

$$A_N(x, y_1, \ldots, y_N) \;=\; A(x, y_1, \ldots, y_N) \;\; a.e. \; in \;\; \Omega \times Y_1 \times \ldots \times Y_N,$$

and for every $0 \le k \le N-1$,

$$A_k(x, y_1, \ldots, y_k)\xi = \int_{Y_{k+1}} A_{k+1}(x, y_1, \ldots, y_{k+1}) \left(\xi + \nabla_{y_{k+1}} w^\xi_{k+1}(x, y_1, \ldots, y_{k+1})\right) dy$$

for any $\xi \in \mathbb{R}^n$, *where* $w^\xi_{k+1} \in L^2[\Omega \times Y_1 \times \ldots \times Y_k; H^1_{per}(Y_{k+1})]$ *is the solution of*

$$\begin{cases} -\operatorname{div}_{y_{k+1}}\, A_{k+1}(x, y_1, \ldots, y_{k+1}) \left(\xi + \nabla_{y_{k+1}} w^\xi_{k+1}(x, y_1, \ldots, y_{k+1})\right) \;=\; 0 \;\; in \; Y_{k+1} \\ \qquad\qquad\qquad\qquad w^\xi_{k+1}(x, y_1, \ldots, y_k, \cdot) \;\in\; H^1_{per}(Y_{k+1}). \end{cases}$$

This means that the matrix function A_k *is obtained by periodic homogenization of* $A_{k+1}(x, y_1, \ldots, y_{k+1})$, *for every* $0 \le k \le N-1$.

Therefore, in the particular case of two oscillatory length scales, ε and ε^2, if u_ε is the solution of

$$(P_\varepsilon^2) \quad \begin{cases} -\operatorname{div} A\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \nabla u_\varepsilon(x) & = & f(x) \quad \text{in } \Omega \\ u_\varepsilon & \in & H_0^1(\Omega), \end{cases}$$

then the sequence $\{u_\varepsilon\}$ is weak convergent in $H_0^1(\Omega)$ to the unique solution u of the homogenized problem

$$(P_0^2) \quad \begin{cases} -\operatorname{div} A_0(x) \nabla u(x) & = & f(x) \quad \text{in } \Omega \\ u & \in & H_0^1(\Omega), \end{cases}$$

with

$$A_0(x)\,\xi \;=\; \int_{Y_1} A_1(x, y_1) \left(\xi + \nabla_{y_1} w_1^\xi(x, y_1)\right) dy_1 \tag{3.1}$$

for any $\xi \in \mathbb{R}^n$. The function $w_1^\xi \in L^2[\Omega; H^1_{per}(Y_1)]$ is the solution of the cell problem

$$\begin{cases} -\operatorname{div}_{y_1} A_1(x, y_1) \left(\xi + \nabla_{y_1} w_1^\xi(x, y_1)\right) & = & 0 \quad \text{in } Y_1 \\ w_1^\xi(x, \cdot) & \in & H^1_{per}(Y_1), \end{cases}$$

with

$$A_1(x, y_1)\,\xi \;=\; \int_{Y_2} A(x, y_1, y_2) \left(\xi + \nabla_{y_2} w_2^\xi(x, y_1, y_2)\right) dy_2$$

for any $\xi \in \mathbb{R}^n$, where the function $w_2^\xi \in L^2[\Omega \times Y_1; H^1_{per}(Y_2)]$ is the solution of

$$\begin{cases} -\operatorname{div}_{y_2} A(x, y_1, y_2) \left(\xi + \nabla_{y_2} w_2^\xi(x, y_1, y_2)\right) & = & 0 \quad \text{in } Y_2 \\ w_2^\xi(x, y_1, \cdot) & \in & H^1_{per}(Y_2). \end{cases}$$

So the matrix A_0 is obtained by iterating 2 times the periodic homogenization problem starting from the faster to the slower length scale.

Notice that assumptions (H_1) and (H_2) hold if the matrix function A satisfies one of the following conditions:

- $A \in [L^\infty[\Omega; C_{per}(Y_1 \times \ldots \times Y_N)]]^{n\times n}$
- $A \in [L^\infty[Y_k; C_{per}(\Omega \times Y_1 \times \ldots \times Y_{k-1} \times Y_{k+1} \times \ldots \times Y_N)]]^{n\times n}$, for any $1 \leq k \leq N$.

3.3. Convergence results for periodic structures whose source term varies with ε

This section focuses on the homogenization of elliptic problems of type

$$(P_{f_\varepsilon}) \quad \begin{cases} -\operatorname{div} A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) & = & f_\varepsilon(x) \quad \text{in } \Omega \\ u_\varepsilon & \in & H_0^1(\Omega), \end{cases}$$

where the matrix function $A = [a_{ij}] \in [L^\infty(Y)]^{n\times n}$ satisfies

i) a_{ij} is Y-periodic, for every $1 \le i, j \le n$,

ii) $A(y)\xi \cdot \xi \ge \alpha|\xi|^2$, for a.e. $y \in Y$ and every $\xi \in \mathbb{R}^n$,

iii) $|A(y)\xi| \le \beta|\xi|$, for a.e. $y \in Y$ and every $\xi \in \mathbb{R}^n$,

for some $0 < \alpha < \beta$, and the sequence $\{f_\varepsilon\}$ is in $H^{-1}(\Omega)$.

Basically there are two results concerning the convergence of solutions u_ε of (P_{f_ε}). One asks for either the strong convergence in $H^{-1}(\Omega)$ or the weak convergence in $L^2(\Omega)$ of the sequence $\{f_\varepsilon\}$, while the other only asks for the weak convergence in $H^{-1}(\Omega)$. However the second one is a partial result in the sense that we do not have a convergence result of the whole sequence of solutions $\{u_\varepsilon\}$.

Theorem 3.3.1 (See [21]) *Let u_ε be the solution of (P_{f_ε}). If the sequence $\{f_\varepsilon\}$ satisfies one of the two following conditions*

i) $f_\varepsilon \to f$ *in* $H^{-1}(\Omega)$,

ii) $f_\varepsilon \rightharpoonup f$ *in* $L^2(\Omega)$,

then

$$\begin{array}{lll} \bullet\ u_\varepsilon \rightharpoonup u_0 & in & H_0^1(\Omega) \\ \bullet\ A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 & in & [L^2(\Omega)]^n, \end{array}$$

where u_0 is the unique solution of the homogenized problem

$$(P_f) \quad \begin{cases} -\operatorname{div} A_0\, \nabla u_0(x) = f(x) & in \quad \Omega \\ u_0 \in H_0^1(\Omega), \end{cases}$$

and the constant matrix $A_0 = [a^0_{ij}] \in \mathbb{R}^{n\times n}$ is given by

$$a^0_{ij} = \frac{1}{|Y|}\int_Y a_{ij}(y)\, dy - \frac{1}{|Y|}\sum_{k=1}^n \int_Y a_{ik}(y)\frac{\partial \overline{w}_j}{\partial y_k}(y)\, dy, \quad \forall\ 1 \le i, j \le n.$$

Here the function $\overline{w}_j$ is solution of

$$\begin{cases} \int_Y A(y)\nabla \overline{w}_j(y)\cdot \nabla v(y)\, dy = \int_Y A(y)e_j \cdot \nabla v(y)\, dy \\ \forall v \in H^1_{per}(Y) : \frac{1}{|Y|}\int_Y v(y)\, dy = 0\,; \quad \overline{w}_j \in H^1_{per}(Y) : \frac{1}{|Y|}\int_Y \overline{w}_j(y)\, dy = 0. \end{cases}$$

Moreover it holds the convergence of energies

$$\lim_{\varepsilon \searrow 0} \int_\Omega A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \cdot \nabla u_\varepsilon(x)\, dx = \int_\Omega A_0 \nabla u_0(x) \cdot \nabla u_0(x)\, dx,$$

and the convergence of $\{A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon\}$ *in the sense of distributions, ie*

$$\lim_{\varepsilon \searrow 0} \int_\Omega A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \cdot \nabla u_\varepsilon(x)\, \varphi(x)\ dx \ = \ \int_\Omega A_0 \nabla u_0(x) \cdot \nabla u_0(x)\, \varphi(x)\ dx,$$

for any $\varphi \in C_0^\infty(\Omega)$.

Notice that the effective coefficient A_0 is different from the weak limit

$$\overline{A} \ = \ \frac{1}{|Y|} \int_Y A(y)\ dy$$

of the sequence $\{A_\varepsilon\}$. Indeed A_0 is the sum of such limit $\overline{A}$ and a corrector term expressed by means of gradients of the functions $\overline{w}_j$. Besides A_0 is exactly the same constant matrix given in Theorem 3.1.1. On the other hand, the right-hand side of the homogenized equation is the limit of $\{f_\varepsilon\}$.

The result is completely different if we assume that $\{f_\varepsilon\}$ converges weakly to f in $H^{-1}(\Omega)$, because in this case we do not have a convergence result for the whole sequence of solutions $\{u_\varepsilon\}$, and the right-hand side of the homogenized equation is not any more the weak limit of $\{f_\varepsilon\}$. The following result is due to Tartar.

Theorem 3.3.2 (See [21]) *Let* u_ε *be the solution of* (P_{f_ε}). *If the sequence* $\{f_\varepsilon\}$ *is weak convergent to* f *in* $H^{-1}(\Omega)$, *then there exists a subsequence* $\{u_{\varepsilon_k}\}$ *of solutions such that*

$$u_{\varepsilon_k} \ \rightharpoonup \ u^\star \ \ in \ \ H_0^1(\Omega),$$

where $u^\star$ *is the unique solution of the homogenized problem*

$$(P_\star) \qquad \begin{cases} -\operatorname{div} A_0 \nabla u^\star(x) \ = \ -\operatorname{div} g^\star(x) & in \quad \Omega \\ \qquad\qquad u^\star \ \in \ H_0^1(\Omega), & \end{cases}$$

with $A_0 = [a_{ij}^0] \in \mathbb{R}^{n\times n}$ *given in the previous theorem, and* $g^\star \in [L^2(\Omega)]^n$ *solution of*

$$\int_\Omega g^\star(x) \nabla\varphi(x)\ dx \ =$$

$$= \ \lim_{k\to\infty} \sum_{i=1}^n \int_\Omega f_{\varepsilon_k}(x) w_i^{\varepsilon_k}(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx \ + \ \int_\Omega f(x)\, (\varphi(x) - x \cdot \nabla\varphi(x))\ dx$$

for any $\varphi \in C_0^\infty(\Omega)$. *Here, for each* $1 \le i \le n$, *the sequence* $\{w_i^{\varepsilon_k}\}_k$ *is given by*

$$w_i^{\varepsilon_k}(x) \ = \ \varepsilon_k\, w_i\left(\frac{x}{\varepsilon_k}\right), \qquad a.e.\ in\ \Omega,$$

and the function w_i *is solution of*

$$\begin{cases} \int_Y A^T(y) \nabla w_i(y) \cdot \nabla v(y)\, dy \ = \ 0, \quad \forall v \in H^1_{per}(Y) : \ \frac{1}{|Y|} \int_Y v(y)\ dy = 0 \\[2ex] w_i - e_i \cdot y \in H^1_{per}(Y), \ \frac{1}{|Y|} \int_Y (w_i(y) - e_i \cdot y)\, dy = 0. \end{cases}$$

Moreover, if the sequence $\{f_\varepsilon\}$ *either converges strongly in* $H^{-1}(\Omega)$ *or weakly in* $L^2(\Omega)$*, then*

$$-\operatorname{div} g^\star = f \qquad \text{and} \qquad u^\star = u_0,$$

where u_0 *is the solution of* (P_f)*.*

3.4. The notion of H-convergence

Though there are many theories in homogenization, possibly the H-convergence is the most general one. Initially, Spagnolo introduced in [67] the notion of G-convergence, ie a notion of convergence of symmetric matrices A_ε coefficients of elliptic problems of the form

$$\begin{cases} -\operatorname{div} A_\varepsilon(x)\,\nabla u_\varepsilon(x) = f(x) & \text{in } \Omega \\ u_\varepsilon \in H_0^1(\Omega), \end{cases}$$

which implies the convergence of solutions u_ε. The notion of H-convergence was introduced by Tartar and Murat in [44] as a convergence of general matrices A_ε, which implies the convergence of $A_\varepsilon \nabla u_\varepsilon$, besides the convergence of solutions u_ε. The H-convergence may be considered as a generalization of G-convergence. For more details see [6, 21, 24, 34, 44, 67], and the references therein.

In order to present the definition of H-convergence, let us introduce the subspace $\mathcal{M}(\alpha,\beta)$ of the space of $n\times n$ real matrices $\mathbb{R}^{n\times n}$.

Definition 3.4.1 *For some constants* $\alpha > 0$ *and* $\beta > 0$*,*

$$\mathcal{M}(\alpha,\beta) = \left\{ M \in \mathbb{R}^{n\times n} \;:\; M\xi\cdot\xi \geq \alpha|\xi|^2, \;\; M^{-1}\xi\cdot\xi \geq \beta|\xi|^2, \;\; \forall\, \xi \in \mathbb{R}^n \right\}$$

is the subspace of coercive matrices with coercive inverses.

Notice that if $M \in \mathcal{M}(\alpha,\beta)$, then $\alpha|\xi|^2 \leq M\xi\cdot\xi \leq \beta|\xi|^2$, for any $\xi \in \mathbb{R}^n$. Moreover $\mathcal{M}(\alpha,\beta)$ is nonempty set if and only if $\alpha\beta \leq 1$.

Definition 3.4.2 (H-convergence) *A sequence of matrices* $A_\varepsilon \in L^\infty(\Omega;\mathcal{M}(\alpha,\beta))$ *is* H*-convergent to the* H*-limit* $A_0 \in L^\infty(\Omega;\mathcal{M}(\alpha,\beta))$ *if, for any* $f \in H^{-1}(\Omega)$*, the sequence of solutions* u_ε *of*

$$(P_\varepsilon) \qquad \begin{cases} -\operatorname{div} A_\varepsilon(x)\,\nabla u_\varepsilon(x) = f(x) & \text{in } \Omega \\ u_\varepsilon \in H_0^1(\Omega) \end{cases}$$

satisfies

- $u_\varepsilon \rightharpoonup u \quad \text{in } H_0^1(\Omega)$
- $A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u \quad \text{in } [L^2(\Omega)]^n,$

where u *is solution of the homogenized equation*

$$(P_0) \qquad \begin{cases} -\operatorname{div} A_0(x)\,\nabla u(x) = f(x) & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

The H-convergence of a sequence of matrices $A_\varepsilon \in L^\infty(\Omega;\mathcal{M}(\alpha,\beta))$ is defined through the weak convergence in $H_0^1(\Omega)$ of solutions of its associated elliptic partial differential equations (P_ε). However the H-limit does not depend on the function f.

The next theorem implies that the set $L^\infty(\Omega;\mathcal{M}(\alpha,\beta))$ is closed under H-convergence.

Theorem 3.4.1 (See [68]) *For any sequence $\{A_\varepsilon\} \subset L^\infty(\Omega;\mathcal{M}(\alpha,\beta))$ there exists a subsequence which H-converges to some matrix $A_0 \in L^\infty(\Omega;\mathcal{M}(\alpha,\beta))$.*

The H-convergence implies the weak convergence of sequences $\{A_\varepsilon \nabla z_\varepsilon\}$ in $[L^\infty_{loc}(\Omega)]^n$, whenever ∇z_ε is the solution of an equation with a varying term f_ε, without any precise boundary condition, as follows.

Proposition 3.4.1 (See [68]) *If $\{A_\varepsilon\} \in L^\infty(\Omega;\mathcal{M}(\alpha,\beta))$ is H-convergent to A_0, and the sequence $\{z_\varepsilon\}$ satisfies*

$$\begin{array}{lrcll} \bullet & \operatorname{div} A_\varepsilon \nabla z_\varepsilon = f_\varepsilon & \to & f & in \quad H^{-1}(\Omega) \\ \bullet & z_\varepsilon & \rightharpoonup & z & in \quad H^1_{loc}(\Omega), \end{array}$$

then

$$A_\varepsilon \nabla z_\varepsilon \rightharpoonup A_0 \nabla z \quad in \quad [L^2_{loc}(\Omega)]^n.$$

The main difficulty in characterizing the weak limit of $\{A_\varepsilon \nabla u_\varepsilon\}$ lies on the fact that the product of two weak convergent sequences does not converge, in general, to the product of the limits. The following compensated compactness result shows that under some additional assumptions such convergence, in the sense of distributions, holds true.

Lemma 3.4.2 (div-curl lemma) (See [68]) *Let $\{U_\varepsilon\}$ and $\{V_\varepsilon\}$ be two sequences in $[L^2(\Omega)]^n$ such that*

$$\begin{array}{lclll} \bullet & U_\varepsilon & \rightharpoonup & U & in \quad [L^2(\Omega)]^n \\ \bullet & V_\varepsilon & \rightharpoonup & V & in \quad [L^2(\Omega)]^n. \end{array}$$

If

$$\begin{array}{lclll} \bullet & \operatorname{div} U_\varepsilon & \to & \operatorname{div} U & in \quad H^{-1}(\Omega) \\ \bullet & \operatorname{curl} V_\varepsilon & \to & \operatorname{curl} V & in \quad [H^{-1}(\Omega)]^{n\times n}, \end{array}$$

then

$$\lim_{\varepsilon \searrow 0} \int_\Omega U_\varepsilon(x)\cdot V_\varepsilon(x)\varphi(x)\,dx = \int_\Omega U(x)\cdot V(x)\varphi(x)\,dx, \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

If we apply the previous result to the case

$$U_\varepsilon = A_\varepsilon \nabla u_\varepsilon \quad \text{and} \quad V_\varepsilon = \nabla u_\varepsilon,$$

with $\operatorname{div} U_\varepsilon = f$ and $\operatorname{curl} V_\varepsilon = 0$, the convergence of energies follows from the H-convergence.

Proposition 3.4.2 (See [68]) *Let $\{A_\varepsilon\}$ be H-convergent to A_0, and u_ε be the solution of (P_ε). Then*

$$\lim_{\varepsilon \searrow 0} \int_\Omega A_\varepsilon(x)\nabla u_\varepsilon(x) \cdot \nabla u_\varepsilon(x)\varphi(x)\, dx = \int_\Omega A_0(x)\nabla u(x) \cdot \nabla u(x)\varphi(x)\, dx,$$

for every $\varphi \in C_0^\infty(\Omega)$, where u is the solution of (P_0).

Chapter 4

Analysis of Young measures

L.C. Young introduced the notion of generalized curves (see [71]) to deal with nonconvex minimizing problems in control theory, which do not have classical solutions. In this type of problems one wishes to study the behaviour of minimizing sequences, and this notion became very useful. Nowadays they are called Young measures and are applied to nonlinear partial differential equations and conservations laws, as well as to the analysis of the microstructure of composite materials.

4.1. Young measures associated with L^p-sequences

Intuitively the Young measure $\nu = \{\nu_x\}_{x\in\Omega}$, associated with a given sequence of functions $u_j : \Omega \to \mathbb{R}^d$, may be thought as a family of probability measures ν_x on $\mathbb{R}^d$ which give the probability distribution of the values of u_j, as $j \to \infty$, near the point x. Namely, for any measurable set $E \subset \mathbb{R}^d$,

$$\nu_x(E) \;=\; \lim_{r\searrow 0}\;\lim_{j\to\infty}\;\frac{|\{y \in B_r(x) : u_j(y) \in E\}|}{|B_r(x)|},$$

where $B_r(x)$ is the ball centred at $x \in \Omega$ with radius $r > 0$.

Theorem 4.1.1 (Fundamental theorem of Young measures) (See [11]) *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable with finite measure, and $\{u_j\}$ be a sequence of Lebesgue measurable functions $u_j : \Omega \to \mathbb{R}^d$. Then there exists a subsequence $\{u_{j_k}\}$ and a family $\{\nu_x\}_{x\in\Omega}$ of positive measures on $\mathbb{R}^d$ such that*

(i) $\|\nu_x\|_{\mathcal{M}} = \int_{\mathbb{R}^d} d\nu_x \leq 1$ for a.e. $x \in \Omega$,

(i′) $\|\nu_x\|_{\mathcal{M}} = 1$, for a.e. $x \in \Omega$, if and only if

$$\lim_{T\to\infty}\;\sup_k\;|\{x \in \Omega \;:\; |u_{j_k}(x)| \geq T\}| \;=\; 0,$$

(II) *if* $K \subset \mathbb{R}^d$ *is a compact subset and*

$$\lim_{k\to\infty} |\{x \in \Omega \ : \ u_{j_k}(x) \notin K\}| \ = \ 0,$$

then $\operatorname{supp} \nu_x \subset K$ *for a.e.* $x \in \Omega$*,*

(III) *for any continuous function* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *so that* $\lim_{|\lambda|\to\infty} \varphi(\lambda) = 0$*, it holds*

$$\varphi(u_{j_k}(\cdot)) \underset{j\to\infty}{\overset{\star}{\rightharpoonup}} \langle \nu_\cdot, \varphi\rangle = \int_{\mathbb{R}^d} \varphi(\lambda)\ d\nu_\cdot(\lambda) \qquad \textit{in} \ \ L^\infty(\Omega),$$

(IV) *if* $\|\nu_x\|_{\mathcal{M}} = 1$*, for a.e.* $x \in \Omega$*, then for any continuous function* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *so that* $\{\varphi(u_{j_k}(\cdot))\}$ *is equi-integrable in* $L^1(E)$*, with* $E \subset \Omega$*, it holds*

$$\varphi(u_{j_k}(\cdot)) \underset{j\to\infty}{\rightharpoonup} \langle \nu_\cdot, \varphi\rangle \qquad \textit{in} \ \ L^1(E).$$

The family $\nu = \{\nu_x\}_{x\in\Omega}$ *is called the Young measure associated with the sequence* $\{u_j\}$*.*

We are interested on the characterization of Young measures associated with sequences $\{u_j\}$ in $L^p(\Omega;\mathbb{R}^d)$.

Definition 4.1.1 *A family* $\nu = \{\nu_x\}_{x\in\Omega}$ *of probability measures is the Young measure associated with a p-equi-integrable sequence of functions* $u_j \in L^p(\Omega;\mathbb{R}^d)$*, for some* $p \in [1,\infty)$*, if and only if, for any* $E \subset \Omega$*,*

$$\lim_{j\to\infty} \int_E \varphi(u_j(x))\ dx \ = \int_E \int_{\mathbb{R}^d} \varphi(\lambda)\ d\nu_x(\lambda)\ dx,$$

for every $\varphi \in E^p = \left\{ \ \varphi \in C(\mathbb{R}^d) \ : \ \lim_{|\lambda|\to\infty} \frac{\varphi(\lambda)}{1+|\lambda|^p} \text{ exists } \right\}$*.*

In order to prove that a family $\nu = \{\nu_x\}_{x\in\Omega}$ of probability measures is the Young measure associated with a sequence $\{u_j\}$, it is enough to check that

$$\lim_{j\to\infty} \int_\Omega \varphi(u_j(x))\,\xi(x)\ dx \ = \ \int_\Omega \int_{\mathbb{R}^d} \varphi(\lambda)\ d\nu_x(\lambda)\ \xi(x)\ dx, \qquad \forall\ \xi \in L^1(\Omega),$$

for every $\varphi \in C_0\left(\mathbb{R}^d\right)$, whenever $\{\varphi(u_j(\cdot))\}$ is weak* convergent in $L^\infty(\Omega)$. It is even enough to take φ and ξ in dense countable subsets of $C_0(\mathbb{R}^d)$ and $L^1(\Omega)$, respectively.

The representation of weak limits is an important application of Young measures, provided the barycenter of such measure is the weak limit of the sequence of functions associated with. Indeed, if $\nu = \{\nu_x\}_{x\in\Omega}$ is the Young measure associated with the bounded sequence $\{u_j\} \subset L^p(\Omega;\mathbb{R}^d)$, then, for every Carathéodory function $\psi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ so that $\{\psi(\cdot, u_j(\cdot))\}$ is weak convergent in $L^1(\Omega)$, we have

$$\psi(\cdot, u_j(\cdot)) \underset{j\to\infty}{\rightharpoonup} \overline{\psi}(\cdot) \ = \ \int_{\mathbb{R}^d} \psi(\cdot,\lambda)\ d\nu_\cdot(\lambda),$$

as a consequence of the following theorem, taking $g(t) = t^p$, for some $p \geq 1$.

Theorem 4.1.2 (See [49]) *Let $\{u_j\}$ be a sequence of measurable functions defined on Ω with values in $\mathbb{R}^d$, such that*

$$\sup_{j\in\mathbb{N}} \int_\Omega g(|u_j(x)|)\, dx \;<\; \infty, \tag{4.1}$$

where $g : [0,+\infty) \to [0,+\infty]$ is a continuous, nondecreasing function so that $\lim_{t\to\infty} g(t) = +\infty$. Then there exists a Young measure $\nu = \{\nu_x\}_{x\in\Omega}$ associated with a subsequence $\{u_{j_k}\}$ such that, whenever $\psi : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a Carathéodory function and the sequence $\{\psi(\cdot, u_{j_k}(\cdot))\}$ is weak convergent in $L^1(\Omega)$, its weak limit is the function $\overline{\psi} : \Omega \to \mathbb{R}$ defined by

$$\overline{\psi}(x) \;=\; \int_{\mathbb{R}^d} \psi(x,\lambda)\, d\nu_x(\lambda).$$

Obviously, if we want to represent the weak limit in $L^1(\Omega)$ of our sequences, it is indispensable their weak convergence in $L^1(\Omega)$. However bounded sequences in L^1 need not to be weak convergent in $L^1(\Omega)$, because this Banach space is not reflexive. So, besides boundedness, the equi-integrability is necessary and sufficient to ensure the weak convergence in $L^1(\Omega)$.

Theorem 4.1.3 (See [49]) *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a family of probability measures supported on $\mathbb{R}^d$ and depending measurably on $x \in \Omega$. There exists a sequence of functions $\{u_j\}$ such that $\{g(|u_j|)\}$ is weak convergent in $L^1(\Omega)$, and its associated Young measure is ν, if and only if*

$$\int_\Omega \int_{\mathbb{R}^d} g(|\lambda|)\, d\nu_x(\lambda)\, dx \;<\; \infty.$$

Particularly, the measure $\nu = \{\nu_x\}_{x\in\Omega}$ is associated with a p-equi-integrable sequence $\{u_j\}$ if and only if its p-moment is finite, ie

$$\int_\Omega \int_{\mathbb{R}^d} |\lambda|^p\, d\nu_x(\lambda)\, dx \;<\; \infty.$$

Theorem 4.1.4 (See [49]) *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a family of probability measures associated with $\{u_j\}$. Then*

$$\liminf_{j\to\infty} \int_E \psi(x, u_j(x))\, dx \;\geq\; \int_E \int_{\mathbb{R}^d} \psi(x,\lambda)\, d\nu_x(\lambda)\, dx,$$

for every Carathéodory function $\psi : E \times \mathbb{R}^d \to \mathbb{R}$ bounded from below, and every measurable subset $E \subset \Omega$.

4.2. Homogenization and localization

In several situations, it is useful to deal with Young measures ν which do not depend on $x \in \Omega$, ie homogeneous Young measures. There are two different processes to obtain a homogeneous Young measure from one which is not: homogenization and localization. The homogenization process consists in collecting the relevant information concerning single elements ν_x, $x \in \Omega$, into a unique homogeneous Young measure, while the localization procedure concentrates on a particular element ν_a, with $a \in \Omega$.

Theorem 4.2.1 (See [49]) *Let Ω and D be two regular domains in $\mathbb{R}^n$ with $|\partial\Omega| = |\partial D| = 0$. Let $\{u_j\}$ be a sequence of measurable functions $u_j : \Omega \to \mathbb{R}^d$, such that*

$$\sup_{j\in\mathbb{N}} \int_\Omega g\left(|u_j(x)|\right)\, dx \;<\; \infty,$$

where $g : \mathbb{R} \to [0, +\infty)$ is a continuous, nondecreasing function with $\lim_{t\to\infty} g(t) = \infty$, and let $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure associated with some subsequence $\{u_{j_k}\}$. Then there exists a sequence of measurable functions $v_j : D \to \mathbb{R}^d$, such that

$$\sup_{j\in\mathbb{N}} \int_D g\left(|v_j(y)|\right)\, dy \;<\; \infty,$$

whose associated Young measure is the homogeneous measure $\overline{\nu}$ given by

$$\langle \overline{\nu}, \varphi\rangle \;=\; \frac{1}{|\Omega|}\int_\Omega \int_{\mathbb{R}^d} \varphi(\lambda)\, d\nu_x(\lambda)\, dx \;=\; \frac{1}{|\Omega|}\int_\Omega \langle \nu_x, \varphi\rangle\, dx, \quad \forall\, \varphi \in C_0(\mathbb{R}^d).$$

The homogenous measure $\overline{\nu}$, defined through the average of $\nu = \{\nu_x\}_{x\in\Omega}$, is the Young measure associated with the sequence of functions $v_j : D \to \mathbb{R}^d$ given by

$$v_j(y) \;=\; u_j\left(\frac{y - x_k^{(j)}}{\varepsilon_k^{(j)}}\right) \qquad \text{in} \quad x_k^{(j)} + \varepsilon_k^{(j)}\Omega,$$

where, for each $j \in \mathbb{N}$, $\{x_k^{(j)} + \varepsilon_k^{(j)}\Omega\}_k$ is a family of pairwise disjoint sets such that

$$D \;=\; \bigcup_k\, x_k^{(j)} + \varepsilon_k^{(j)}\Omega \;\cup\; N_j \qquad |N_j| = 0, \quad \varepsilon_k^{(j)} < \frac{1}{j}.$$

In particular, if $\{u_j\} = \{f\}$ is a constant sequence, then the sequence of functions f_j defined by

$$f_j(y) \;=\; f\left(\frac{y - x_k^{(j)}}{\varepsilon_k^{(j)}}\right) \qquad \text{in} \quad x_k^{(j)} + \varepsilon_k^{(j)}\Omega,$$

generates the homogenous Young measure $\overline{\nu}$ given by

$$\langle \overline{\nu}, \varphi \rangle \; = \; \frac{1}{|\Omega|} \int_{\Omega} \varphi(f(x)) \; dx,$$

so that the Riemann-Lebesgue lemma holds true.

Lemma 4.2.2 (Riemann-Lebesgue) *Let Ω and D be regular domains in $\mathbb{R}^n$ with $|\partial\Omega| = |\partial D| = 0$ and let f be a function in $L^p\left(\Omega; \mathbb{R}^d\right)$. Then there exists a sequence of functions $f_j : D \to \mathbb{R}^d$ whose associated Young measure is homogeneous and defined by*

$$\langle \overline{\nu}, \varphi \rangle \; = \; \frac{1}{|\Omega|} \int_{\Omega} \varphi(f(x)) \; dx, \qquad \forall \; \varphi \in C_0(\mathbb{R}^d).$$

Notice that Proposition 4.3.3 below follows from the previous lemma.

Now let us recall the localization principle of Young measures based on a blow-up argument around each $x \in \Omega$.

Theorem 4.2.3 (See [49]) *Let Ω and D be two regular domains in $\mathbb{R}^n$ with $|\partial\Omega| = |\partial D| = 0$. Let $u_j : \Omega \to \mathbb{R}^d$ be such that*

$$\sup_j \int_{\Omega} g(|u_j(x)|) \; dx \; < \; \infty,$$

where $g : \mathbb{R} \to [0, +\infty)$ is a continuous, nondecreasing function with $\lim_{t\to\infty} g(t) = \infty$, and let $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure associated with some subsequence $\{u_{j_k}\}$. Then, for a.e. $x \in \Omega$, there exists a sequence $\{u_j(x + r_j\cdot)\}$ defined in D, with $r_j \searrow 0$, such that

$$\sup_j \int_{D} g(|u_j(x + r_j y)|) \; dy \; < \; \infty,$$

and it generates the homogenous Young measure ν_x.

4.3. Oscillations and concentrations

The strong convergence of a sequence of functions in a Lebesgue space is a necessary and sufficient condition for its associated Young measure be a Dirac measure concentrated on the limit.

Proposition 4.3.1 (See [49]) *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure associated with a sequence $\{u_j\} \subset L^p\left(\Omega; \mathbb{R}^d\right)$, such that $\{|u_j|^p\}$ is weak convergent in $L^1(\Omega)$, for $p < \infty$. Then $\nu_x = \delta_{u(x)}$, for a.e. $x \in \Omega$, if and only if the sequence $\{u_j\}$ is strong convergent to u in $L^p\left(\Omega; \mathbb{R}^d\right)$.*

In particular cases, we may have strong convergence only for some components of the sequence. Also for this type of sequences it is possibly to characterize their associated Young measures.

Proposition 4.3.2 (See [49]) *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure associated with a bounded sequence of functions $u_j = (w_j, z_j) : \Omega \to \mathbb{R}^d \times \mathbb{R}^e$ in $L^p\left(\Omega; \mathbb{R}^{d+e}\right)$, for which the sequence $\{w_j\}$ is strongly convergent to w in $L^p\left(\Omega; \mathbb{R}^d\right)$, and let $\mu = \{\mu_x\}_{x\in\Omega}$ be the Young measure associated with the sequence $\{z_j\}$. Then $\nu_x = \delta_{w(x)} \otimes \mu_x$, for a.e. $x \in \Omega$.*

Lemma 4.3.1 (See [49]) *Let $\{u_j\}$ and $\{v_j\}$ be two bounded sequences in $L^p(\Omega)$. If one of the following conditions holds true:*

1. $| \{ x \in \Omega \ : \ u_j(x) \neq v_j(x) \} | \underset{j\to\infty}{\longrightarrow} 0,$

2. $\| u_j - v_j \|_{L^p(\Omega)} \underset{j\to\infty}{\longrightarrow} 0,$

then the Young measure associated with both sequences is the same.

The lost of strong convergence, for a bounded sequence $\{u_j\} \subset L^p(\Omega; \mathbb{R}^d)$, may be due to rapid oscillations in u_j or by concentration effects. Though Young measures associated with rapid oscillating sequences capture some information on the oscillations, as follows in the proposition below, they do not capture concentration effects.

Proposition 4.3.3 (See [49]) *Let $u \in L^p(Q; \mathbb{R}^d)$, with $p \geq 1$, be extended by Q-periodicity to $\mathbb{R}^n$, and let the sequence of functions $u_j(x) = u(jx)$ be defined in Ω. Then the homogenous Young measure $\overline{\nu}$ associated with the sequence $\{u_j\}$ is given by*

$$\langle \overline{\nu}, \varphi \rangle \ = \ \int_Q \varphi(u(y)) \ dy, \qquad \qquad \forall \ \varphi \in C_0(\mathbb{R}^n),$$

ie

$$\lim_{j\to\infty} \int_\Omega \varphi(x, u_j(x)) \ dx \ = \ \int_\Omega \int_Q \varphi(x, u(y)) \ dy \ dx$$

for any Carathéodory function φ bounded from below.

The concentration effects occur when, given a weak convergent sequence $\{u_j\}$ to u in $L^p(\Omega; \mathbb{R}^d)$, the sequence $\{u_j - u\}$ converges in measure[1] to 0 and the total

[1]The sequence $\{u_j - u\}$ converges in measure to 0 iff, for any $\varepsilon > 0$, $\lim_{j\to\infty} |\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\}|=0$. For example, if $\{\Omega_j\}$ is a sequence of subsets $\Omega_j \subset \Omega$ such that $\lim_{j\to\infty} |\Omega_j| = 0$ and each point of Ω belongs to infinitely many Ω_j, the sequence $u_j = \chi_{\Omega_j}$ converges in measure to 0.

mass $\int_\Omega |u_j(x) - u(x)|^p dx$, as j goes to 0, is concentrated in a set of zero Lebesgue measure. The last situation may occur when the sequence of measures $\{|u_j - u|^p \mathcal{L}^n\}$ is weak convergent, in the sense of measures, to $\nu = m\delta_A$, for some $A \in \Omega$, so that the total mass is concentrated at the point A.

Clearly, the Young measure associated with a subsequence of $\{u_j\}$ converging in measure to u, which is $\{\delta_{u(x)}\}_{x\in\Omega}$, does not capture possible concentration effects.

Proposition 4.3.4 (See [49]) *If $\{u_j\}$ converges in measure to u, and $\{z_j\}$ generates the Young measure $\mu = \{\mu_x\}_{x\in\Omega}$, then the sequence $\{u_j + z_j\}$ generates the Young measure ν given by*

$$\langle \nu_x, \varphi \rangle \;=\; \int_{\mathbb{R}^d} \varphi(\lambda + u(x))\; d\mu_x(\lambda) \qquad \forall \varphi \in C_0(\mathbb{R}^d).$$

In particular, if $u = 0$, then $\{u_j + z_j\}$ generates the same Young measure as $\{z_j\}$.

It follows that the sequences $\{(u_j - u) + z_j\}$ and $\{z_j\}$ generate the same Young measure ν, and thus the perturbation $u_j - u$, for which a concentration phenomenon may occur, has no effect on ν.

4.4. Gradient Young measures

The characterization of Young measures generated by sequences of gradients is due to Kinderlehrer and Pedregal, see [36, 37]. An important application of gradient Young measures is the characterization itself of minimizing sequences of variational problems.

Definition 4.4.1 *A family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ supported on $\mathbb{R}^{d\times n}$ is called a gradient Young measure if it is generated by a sequence of gradients $\{\nabla u_j\}$, for some bounded sequence $\{u_j\} \subset W^{1,p}\left(\Omega; \mathbb{R}^d\right)$, with $1 \le p \le \infty$.*

Let us focus on the scalar case ($d = 1$).

Theorem 4.4.1 (See [36]) *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a family of probability measures supported on $\mathbb{R}^n$. There exists a bounded sequence $\{u_j\}$ in $W^{1,p}(\Omega)$ such that the sequence $\{|\nabla u_j|^p\}$, for $1 < p < \infty$, is weak convergent in $L^1(\Omega)$, and ν is the Young measure associated with $\{\nabla u_j\}$, if and only if*

i) for some $u \in W^{1,p}(\Omega)$,

$$\nabla u(x) \;=\; \int_{\mathbb{R}^n} \lambda\; d\nu_x(\lambda) \qquad a.e.\ in\ \Omega,$$

ii)

$$\int_\Omega \int_{\mathbb{R}^n} |\lambda|^p \; d\nu_x(\lambda) \; dx \; < \; \infty, \qquad \textit{for } 1 < p < \infty,$$

or,

$$\operatorname{supp} \nu_x \subset K, \quad \textit{for a.e. } x \in \Omega, \quad \textit{and} \quad p = \infty,$$

where $K \subset \mathbb{R}^n$ is a fixed compact set.

Notice that this theorem is not valid for $p = 1$, because a bounded sequence in $W^{1,1}(\Omega)$ may not be weak convergent in $W^{1,1}(\Omega)$, and therefore we do not know if the first moment $\int_{\mathbb{R}^n} \lambda \; d\nu_x(\lambda)$ is the gradient of a function in $W^{1,1}(\Omega)$. But if we assume this fact, then the theorem is valid.

Let us introduce two homogenization results concerning gradient Young measures, which follow the ideas in the homogenization theorem and the Riemann-Lebesgue Lemma, presented before.

Theorem 4.4.2 (See [36]) *Let $\{u_j\}$ be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^d)$, with affine boundary values $u_Y(x) = Yx$, for some $Y \in \mathbb{R}^{d\times n}$. Let $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure associated with the sequence of gradients $\{\nabla u_j\}$. Then there exists a bounded sequence $\{v_j\}$ in $W^{1,p}(\Omega;\mathbb{R}^d)$, with the same boundary values as $\{u_j\}$, such that the Young measure associated with its sequence of gradients $\{\nabla v_j\}$ is homogeneous and defined by*

$$\langle \overline{\nu}, \varphi \rangle \; = \; \frac{1}{|\Omega|} \int_\Omega \int_{\mathbb{R}^{d\times n}} \varphi(\lambda) \; d\nu_x(\lambda) \; dx.$$

for all $\varphi \in C_0(\mathbb{R}^{d\times n})$

Indeed, given a bounded sequence $\{u_j\}$ in $W^{1,p}(\Omega;\mathbb{R}^d)$, we may define the sequence of functions v_j on Ω by putting

$$v_j(x) \; = \; \begin{cases} \varepsilon_k^{(j)} \, u_j \left(\frac{x - a_k^{(j)}}{\varepsilon_k^{(j)}} \right) + u_Y(a_k^{(j)}) & \text{if} \quad x \in a_k^{(j)} + \varepsilon_k^{(j)}\Omega, \\ u_Y(x) & \text{otherwise,} \end{cases}$$

where $\{a_k^{(j)} + \varepsilon_k^{(j)}\Omega\}$ is a family of pairwise disjoint sets such that, for each $j \in \mathbb{N}$,

$$\Omega \; = \; \bigcup_k \left(a_k^{(j)} + \varepsilon_k^{(j)} \overline{\Omega} \right) \cup N_j, \quad |N_j| = 0.$$

Thus

$$\nabla v_j(x) \; = \; \nabla u_j \left(\frac{x - a_k^{(j)}}{\varepsilon_k^{(j)}} \right) \qquad \text{in} \quad a_k^{(j)} + \varepsilon_k^{(j)}\Omega,$$

and $v_j - u_Y \in W_0^{1,p}(\Omega)$. Notice that we must bear in mind the boundary values when we deal with sequences in $W^{1,p}(\Omega;\mathbb{R}^d)$, moreover we have to enforce affine boundary values for u_j.

The same basic idea is taken in the following lemma where, instead of considering a sequence $\{u_j\}$, we use only one function u.

Lemma 4.4.3 (Riemann-Lebesgue) (See [49]) *Let Ω and D be open, bounded, regular subsets in $\mathbb{R}^n$ and let $u \in W^{1,p}(D;\mathbb{R}^d)$, with affine boundary values $u_Y(x) = Yx$, for some $Y \in \mathbb{R}^{d\times n}$. Then there exists a bounded sequence $\{v_j\}$ in $W^{1,p}(\Omega;\mathbb{R}^d)$, with the same boundary values as u, for which the Young measure associated with the sequence of gradients $\{\nabla v_j\}$ is homogeneous and defined by*

$$\langle \overline{\nu}, \varphi \rangle \ = \ \frac{1}{|D|} \int_D \varphi(\nabla u(x))\ dx,$$

for every $\varphi \in X^p = \left\{ \varphi : \mathbb{R}^{d\times n} \to \mathbb{R} \text{ continuous } : \ |\varphi(A)| \le C(1+|A|^p),\ C \in \mathbb{R} \right\}$.

Theorem 4.4.4 (See [49]) *Let $\{u_j\}$ be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^d)$, and $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure associated with $\{\nabla u_j\}$. Let, for a.e. $a \in \Omega$,*

$$F(a) \ = \ \int_{\mathbb{R}^{d\times n}} \lambda \ d\nu_a(\lambda) \qquad \text{and} \qquad u_a(x) \ = \ F(a)\, x.$$

Then there exists a bounded sequence $\{w_j^a\}$ in $W^{1,p}(\Omega;\mathbb{R}^d)$ such that $w_j^a - u_a \in W_0^{1,p}(\Omega;\mathbb{R}^d)$, and the Young measure associated with $\{\nabla w_j^a\}$ is the homogenous measure ν_a.

For each $a \in \Omega$, the function w_j^a may be defined in Ω by putting

$$w_j^a(x) \ = \ \frac{1}{r_j}\left(u_j(a + r_j x) - M_a^{(j)}\right),$$

for some sequence $r_j \searrow 0$, and constant $M_a^{(j)}$ such that

$$\int_\Omega w_j^a(x)\ dx \ = \ \frac{1}{|\Omega|}\int_\Omega u_a(x)\ dx, \qquad \text{for every } \ j \in \mathbb{N}.$$

4.5. Laminates

An interesting example of gradient Young measures are the so called laminates, ie gradient Young measures which are the convex combination of Dirac measures. For instances, if we take two rank-one connected matrices A and B in $\mathbb{R}^{d\times n}$, ie $B - A = a \otimes \overrightarrow{n}$, then the probability measure

$$\sigma \ = \ t\,\delta_A + (1-t)\,\delta_B$$

is a laminate, for any $t \in (0,1)$. Indeed, consider the sequence of functions $u_j : \Omega \to \mathbb{R}^d$ defined by

$$u_j(x) \;=\; Bx \;+\; \frac{1}{j}\int_0^{jx\cdot\vec{n}} \chi_{(0,t)}(s)\, ds\; a$$

such that

$$\nabla u_j(x) \;=\; B + \chi_{(0,t)}(jx\cdot\vec{n})\, a\otimes\vec{n} \;=\; A\,\chi_{(0,t)}(jx\cdot\vec{n}) + B\,(1-\chi_{(0,t)}(jx\cdot\vec{n})).$$

The function ∇u_j takes different constant values in alternating bands normal to $\vec{n}$, so that u_j is continuous on the interfaces if $A\vec{m} = B\vec{m}$ for vectors $\vec{m}$ perpendicular to $\vec{n}$, ie $B - A = a\otimes\vec{n}$. The sequence $\{\nabla u_j\}$ generates the first order laminate σ.

Proposition 4.5.1 (See [12]) *Let $A, B, Y \in \mathbb{R}^{d\times n}$ such that $B - A = a\otimes\vec{n}$ and $Y = tA + (1-t)B$, for some $t \in (0,1)$. Then there exists a bounded sequence $\{u_j\} \subset W^{1,\infty}(\Omega;\mathbb{R}^d)$ such that $u_j(x) = Yx$ on $\partial\Omega$, and $\sigma = t\,\delta_A + (1-t)\,\delta_B$ is the homogenous Young measure associated with $\{\nabla u_j\}$.*

Moreover, we may consider three matrices A, B, C in $\mathbb{R}^{d\times n}$ such that

$$A - (\lambda B + (1-\lambda)C) \;=\; a\otimes\vec{n} \qquad \text{and} \qquad C - B = b\otimes\vec{m}, \qquad \lambda\in(0,1).$$

Then there exists also a sequence $\{u_j\} \subset W^{1,\infty}(\Omega;\mathbb{R}^d)$ so that $\{\nabla u_j\}$ generates the Young measure

$$\sigma \;=\; t\,\delta_A + (1-t)\;(\lambda\,\delta_B + (1-\lambda)\,\delta_C)\,.$$

In this way was introduced the (H_j) condition on the pairs $\{(t_k, A_k)\}_{1\le k\le j}$ in order to

$$\sigma \;=\; \sum_{k=1}^{j} t_k^j\;\delta_{A_k^j}$$

be a gradient Young measure. See, for instance [12, 23, 47, 43].

Definition 4.5.1 *A set of pairs $\{(t_k, A_k)\}_{1\le k\le j} \subset (0,1)\times\mathbb{R}^{d\times n}$, with $\sum_k^j t_k = 1$, is said to satisfy the (H_j) condition if:*

1. *for $j = 2$, rank $(A_1 - A_2) \le 1$,*
2. *for $j > 2$, rank $(A_1 - A_2) \le 1$ and, if (after permutation of indices) we set*

$$s_1 = t_1 + t_2, \quad B_1 = \frac{t_1}{s_1}A_1 + \frac{t_2}{s_1}A_2, \quad s_k = t_{k+1}, \quad B_k = A_{k+1}, \quad 2\le k\le j-1,$$

the set of pairs $\{(s_k, B_k)\}_{1\le k\le j-1}$ satisfies the (H_{j-1}) condition.

Thus it follows the definition of a laminate.

Definition 4.5.2 *A probability measure σ, with compact support in $\mathbb{R}^{d\times n}$, is called a laminate if there exists a sequence of sets of pairs $\{(t_k^j, A_k^j)\}_{1\leq k\leq j} \subset (0,1)\times\mathbb{R}^{d\times n}$, satisfying the (H_j) condition, such that σ is the limit, in the sense of measures, of the sequence $\left\{\sum_k^j t_k^j \delta_{A_k^j}\right\}$, ie*

$$\langle \sigma, \varphi\rangle \;=\; \lim_{j\to\infty} \sum_{k=1}^{j} \; t_k^j \;\varphi(A_k^j), \qquad \forall\; \varphi \in C(\mathbb{R}^{d\times n}).$$

Therefore, if σ is a laminate with barycenter Y in $\mathbb{R}^{d\times n}$, then there exists a sequence $\{u_j\} \subset W^{1,\infty}(\Omega;\mathbb{R}^d)$ such that

1. $u_j(x) = Yx$ on $\partial\Omega$,
2. u_j is weak* convergent to Yx in $W^{1,\infty}(\Omega;\mathbb{R}^d)$,
3. $\lim_j \int_\Omega \varphi(\nabla u_j(x))\,dx \;=\; |\Omega| \int_{\mathbb{R}^{d\times n}} \varphi(\lambda)\,d\sigma(\lambda)$, for every $\varphi \in C(\mathbb{R}^{d\times n})$, ie σ is a gradient Young measure,

provided the support of σ is a compact set.

However, if the probability measure σ is a gradient Young measure, we cannot ensure that it is a laminate. Indeed, there are examples of gradient Young measures which fail the compatibility condition, ie (H_j) condition. See [49] and the references therein for more details.

4.6. Decomposition of sequences of gradient

Since Young measures do not capture concentration effects, as discussed in Section 2.3, it is important, when dealing with gradient Young measures, to separate the sequence of gradients into the oscillating part and a remainder carrying the concentration effects. The following lemma states that any bounded sequence of gradients in $L^p(\Omega;\mathbb{R}^{n\times d})$, with $1 < p < \infty$, may be written as the sum of a p-equi-integrable sequence of gradients and a remainder which converges to 0 in measure.

Lemma 4.6.1 (See [31]) *Let $\{u_j\}$ be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^d)$ with $1 < p < \infty$. Then there exists a subsequence $\{u_{j_k}\}$ and a sequence $\{v_j\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ such that*

i) $\{|\nabla v_j|^p\}$ *is equi-integrable,*

ii) $\lim_{j\to\infty} |\{x \in \Omega : u_{j_k}(x) \neq v_j(x)$ or $\nabla u_{j_k}(x) \neq \nabla v_j(x)\}| = 0$.

In particular, both sequences $\{\nabla u_{j_k}\}$ and $\{\nabla v_j\}$ generate the same Young measure. If Ω has Lipschitz boundary, then $\{v_j\} \subset W^{1,\infty}(\Omega;\mathbb{R}^d)$.

In the scalar case, $d = 1$, and when $p = 1$, if some weak* convergent sequence in $L^\infty(\Omega;\mathbb{R}^n)$ may be decomposed as the sum of a strong convergent sequence of gradients and a weak convergent sequence, both in $L^1(\Omega;\mathbb{R}^n)$, we may not ensure, in general, that it is a gradient sequence itself in the sense that the sequence $\{\operatorname{curl} V_j\}$ does not converge strongly to 0 in $W^{-1,p}(\Omega)$, as follows from the next lemma.

Lemma 4.6.2 (See [38]) *Let $\{V_j\}$ be a weak* convergent sequence to V in $L^\infty(\Omega;\mathbb{R}^n)$. If $V_j = \nabla v_j + E_j$ with*

i) $\{v_j\}$ weak convergent in $W^{1,1}(\Omega)$,

ii) $\{E_j\}$ strong convergent in $L^1(\Omega;\mathbb{R}^n)$,

then $\operatorname{curl} V_j \to \operatorname{curl} V$ in $W^{-1,p}(\Omega)$, for all $p < +\infty$.

The following lemmas are a particular situation of Lemmas 2.15, 2.16, 2.17 in [32], respectively, where we take the linear operator $\mathcal{A} = \operatorname{curl}$. See also Proposition 2.3 in [19].

Lemma 4.6.3 *Let $\{V_j\}$ be a bounded sequence in $L^p(\Omega;\mathbb{R}^n)$, with $1 < p < +\infty$, such that*

i) $V_j \rightharpoonup V$ in $L^p(\Omega;\mathbb{R}^n)$,

ii) $\operatorname{curl} V_j \to 0$ in $W^{-1,p}(\Omega)$,

iii) $\{V_j\}$ generates the Young measure $\nu = \{\nu_x\}_{x\in\Omega}$.

Then there exists a bounded sequence $\{v_j\} \subset W^{1,p}(\Omega)$ so that $\{\nabla v_j\}$ is q-equi-integrable,

$$\| V_j - \nabla v_j \|_{L^q(\Omega;\mathbb{R}^n)} \underset{j}{\longrightarrow} 0 \quad \forall\, 1 \le q < p, \qquad \int_\Omega \nabla v_j(x)\, dx = \int_\Omega V(x)\, dx,$$

and, in particular, ν is the gradient Young measure generated by $\{\nabla v_j\}$. Moreover, if $\Omega = Q$ then $\{\nabla v_j - V\} \subset L^p_{per}(Q;\mathbb{R}^n)$.

Lemma 4.6.4 *Let $\{V_j\}$ be a bounded sequence in $L^1(\Omega;\mathbb{R}^n)$ such that*

i) $V_j \rightharpoonup V$ in $L^1(\Omega;\mathbb{R}^n)$,

ii) $\operatorname{curl} V_j \to 0$ in $W^{-1,r}(\Omega)$, for some $r \in \left(1, \frac{n}{n-1}\right)$

iii) $\{V_j\}$ generates the Young measure $\nu = \{\nu_x\}_{x\in\Omega}$.

Then there exists a bounded sequence $\{v_j\} \subset W^{1,1}(\Omega)$ *so that* $\{\nabla v_j\}$ *is equi-integrable,*

$$\| V_j - \nabla v_j \|_{L^1(\Omega;\mathbb{R}^n)} \underset{j}{\longrightarrow} 0, \qquad \int_\Omega \nabla v_j(x)\, dx = \int_\Omega V(x)\, dx,$$

and, in particular, ν *is the gradient Young measure generated by* $\{\nabla v_j\}$. *Moreover, if* $\Omega = Q$ *then* $\{\nabla v_j - V\} \subset L^1_{per}(Q;\mathbb{R}^n)$.

Lemma 4.6.5 *Let* $\{V_j\}$ *be a bounded sequence in* $L^\infty(\Omega;\mathbb{R}^n)$ *such that*

i) $V_j \rightharpoonup^\star V$ *in* $L^\infty(\Omega;\mathbb{R}^n)$,

ii) $\operatorname{curl} V_j \rightharpoonup 0$ *in* $L^p(\Omega)$, *for some* $p > n$,

iii) $\{V_j\}$ *generates the Young measure* $\nu = \{\nu_x\}_{x\in\Omega}$.

Then there exists a bounded sequence $\{v_j\} \subset W^{1,\infty}(\Omega)$ *so that*

$$\| V_j - \nabla v_j \|_{L^\infty(\Omega;\mathbb{R}^n)} \underset{j}{\longrightarrow} 0, \qquad \int_\Omega \nabla v_j(x)\, dx = \int_\Omega V(x)\, dx,$$

and, in particular, ν *is the gradient Young measure generated by* $\{\nabla v_j\}$. *Moreover, if* $\Omega = Q$ *then* $\{\nabla v_j - V\} \subset L^\infty_{per}(Q;\mathbb{R}^n)$.

4.7. Multi-scale Young measures

A Young measure $\mu = \{\mu_x\}_{x\in\Omega}$ associated with a sequence $\{u_\varepsilon\}$ keeps the information how the values of the sequence are distributed in a neighbourhood of x, when ε goes to 0. However the Young measure loses all information about the oscillatory behaviour of its associated sequence, namely the number of oscillating length scales, the directions of oscillation, ...

In order to study the multi-scale oscillatory behaviour of a sequence $\{u_\varepsilon\}$, Pedregal introduced in [52] the notion of multi-scale Young measure. This notion comes from the study of the joint Young measure $\theta = \{\theta_x\}_{x\in\Omega}$ associated with the sequence

$$\left\{ \left(u_\varepsilon(\cdot), \left\langle \frac{\cdot}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{\cdot}{l_N(\varepsilon)} \right\rangle \right) \right\} \tag{4.2}$$

where $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$ is a family of separated length scales. In this way, several oscillatory test-functions $\left\langle \frac{x}{l_i(\varepsilon)} \right\rangle$ are considered, jointly with the sequence $\{u_\varepsilon\}$.

Proposition 4.7.1 (See [52]) *Let* $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$ *be a family of separated length scales. Then the Young measure associated with the sequence*

$$\left\{\left(\left\langle\frac{\cdot}{l_1(\varepsilon)}\right\rangle, \ldots, \left\langle\frac{\cdot}{l_N(\varepsilon)}\right\rangle\right)\right\},$$

defined in Ω*, is the Lebesgue measure over* Q^N *:*

$$\underbrace{\mathcal{L}^n_Q \otimes \ldots \otimes \mathcal{L}^n_Q}_{N \text{ times}}.$$

The joint Young measure $\theta = \{\theta_x\}_{x\in\Omega}$, associated with the sequence (4.2), gives more information about the oscillations of $\{u_\varepsilon\}$ than its associated Young measure. From the slicing decomposition, for a.e. $x \in \Omega$, we may decompose each probability measure θ_x as

$$\theta_x \;=\; \mu_{x,y_1,\ldots,y_N} \otimes \underbrace{\mathcal{L}^n_Q \otimes \ldots \otimes \mathcal{L}^n_Q}_{N \text{ times}},$$

for some family of probability measures $\{\mu_{x,y_1,\ldots,y_N}\}_{x\in\Omega,(y_1,\ldots,y_N)\in Q^N}$.

Definition 4.7.1 *A family of probability measures* $\{\mu_{x,y}\}_{x\in\Omega, y\in Q^N}$*, supported on* $\mathbb{R}^d$*, is said to be the multi-scale Young measure associated with the sequence of functions* $u_\varepsilon : \Omega \to \mathbb{R}^d$*, at the separated length scales* $l_1(\varepsilon), \ldots, l_N(\varepsilon)$*, if the joint Young measure* $\theta = \{\theta_x\}_{x\in\Omega}$*, supported on* $\mathbb{R}^d \times Q^N$*, associated with the sequence*

$$\left\{\left(u_\varepsilon(\cdot), \left\langle\frac{\cdot}{l_1(\varepsilon)}\right\rangle, \ldots, \left\langle\frac{\cdot}{l_N(\varepsilon)}\right\rangle\right)\right\},$$

may be decomposed, for a.e. $x \in \Omega$ *and* $y = (y_1, \ldots, y_N) \in Q^N$*, as*

$$\theta_x \;=\; \mu_{x,y_1,\ldots,y_N} \otimes \mathcal{L}^n_Q \otimes \ldots \otimes \mathcal{L}^n_Q.$$

The multi-scale Young measure $\{\mu_{x,y}\}_{x\in\Omega,y\in Q^N}$ associated with a sequence $\{u_\varepsilon\}$, at the family of separated length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$, satisfies the equality

$$\lim_{\varepsilon\searrow 0} \int_\Omega \psi\left(x, u_\varepsilon(x), \left\langle\frac{x}{l_1(\varepsilon)}\right\rangle, \ldots, \left\langle\frac{x}{l_N(\varepsilon)}\right\rangle\right) dx \;= \tag{4.3}$$

$$= \int_\Omega \int_Q \cdots \int_Q \int_{\mathbb{R}^d} \psi(x, \lambda, y_1, \ldots, y_N)\; d\mu_{x,y_1,\ldots,y_N}(\lambda)\; dy_1 \ldots dy_N\; dx,$$

for every Charathéodory function $\psi : \Omega \times \mathbb{R}^d \times Q^N \to \mathbb{R}$ such that $\left\{\psi\left(\cdot, u_\varepsilon(\cdot), \left\langle\frac{\cdot}{l_1(\varepsilon)}\right\rangle, \ldots, \left\langle\frac{\cdot}{l_N(\varepsilon)}\right\rangle\right)\right\}$ is weak convergent in $L^1(\Omega)$. Moreover, it gives also information about the usual Young measure associated with $\{u_\varepsilon\}$. Indeed, the family of probability measures $\eta = \{\eta_x\}_{x\in\Omega}$, supported on $\mathbb{R}^d$, defined by

$$d\eta_x(\lambda) \;=\; d\mu_{x,y_1,\ldots,y_N}(\lambda) \otimes dy_1 \otimes \ldots \otimes dy_N, \qquad \text{for a.e. } x \in \Omega,$$

is the Young measure associated with the sequence $\{u_\varepsilon\}$, provided

$$\lim_{\varepsilon \searrow 0} \int_\Omega \psi\left(x, u_\varepsilon(x)\right) dx \;=\; \int_\Omega \int_Q \cdots \int_Q \int_{\mathbb{R}^d} \psi(x,\lambda)\; d\mu_{x,y_1,\ldots,y_N}(\lambda)\; dy_1 \ldots dy_N\; dx,$$

for every Charathéodory function $\psi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that $\{\psi(\cdot, u_\varepsilon(\cdot))\}$ is weak convergent in $L^1(\Omega)$.

Theorem 4.7.1 (See [52]) *Let $\{\mu_{x,y}\}_{x\in\Omega, y\in Q^N}$ be the multi-scale Young measure, supported on $\mathbb{R}^d$, associated with the sequence $\{u_\varepsilon\}$ at the family of separated length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$. Then, for a.e. $x \in \Omega$, there exists a length scale $r(\varepsilon)$ such that $r_i(\varepsilon) = \frac{r(\varepsilon)}{l_i(\varepsilon)} \nearrow \infty$, for every $1 \le i \le N$, and*

$$\langle \varphi, \mu_{x,y_1,\ldots,y_N} \rangle = \lim_{\varepsilon \searrow 0} \frac{1}{r_i(\varepsilon)^n} \int_{r_i(\varepsilon)Q} \varphi(u_\varepsilon(x + l_i(\varepsilon)[z] + l_1(\varepsilon)y_1 + \ldots + l_N(\varepsilon)y_N))\; dz$$

for a.e. $(y_1, \ldots, y_N) \in Q^N$ and every $\varphi \in C_0(\mathbb{R}^d)$.

The notion of multi-scale convergence may be rewritten using multi-scale Young measures. Namely, if $\{\mu_{x,y}\}_{x\in\Omega, y\in Q^N}$ is the multi-scale Young measure associated with $\{u_\varepsilon\}$, and $\{u_\varepsilon\}$ multi-scale converges to u_0, then

$$\lim_{\varepsilon \searrow 0} \int_\Omega u_\varepsilon(x)\; \varphi\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_N(\varepsilon)} \right\rangle\right) dx \;=$$

$$= \int_\Omega \int_Q \cdots \int_Q \left(\int_{\mathbb{R}^d} \lambda\; d\mu_{x,y_1,\ldots,y_N}(\lambda) \right)\; \varphi(x, y_1, \ldots, y_N)\; dy_1 \ldots dy_N\; dx,$$

for any Charathéodory function $\varphi : \Omega \times Q^N \to \mathbb{R}$, and it holds

$$u_0(x, y_1, \ldots, y_N) \;=\; \int_{\mathbb{R}^d} \lambda\; d\mu_{x,y_1,\ldots,y_N}(\lambda), \qquad \text{for a.e. } (y_1, \ldots, y_N) \in Q^N.$$

It follows that the multi-scale limit u_0 is the first moment of the multi-scale Young measure $\{\mu_{x,y}\}_{x\in\Omega, y\in Q^N}$.

Moreover, the homogenization of multiple integrals, with multi-scale periodic densities, may be analyzed through the multi-scale Young measures, as follows.

Theorem 4.7.2 (See [52]) *Let $f(x, y_1, \ldots, y_N, \rho) : \Omega \times Q^N \times \mathbb{R}^n \to \mathbb{R}$ be such that*

i) f is measurable in Ω, continuous in $Q^N \times \mathbb{R}^n$, and convex in $\mathbb{R}^n$,

ii) f is Q-periodic in y_k, for every $1 \le k \le N$,

iii) there exist $C \ge c > 0$ for which

$$c\,|\rho|^p \;\le\; f(x, y_1, \ldots, y_N, \rho) \;\le\; C\,(1 + |\rho|^p), \qquad \textit{with } p > 1,$$

for every $(x, y_1, \ldots, y_N, \rho) \in \Omega \times Q^N \times \mathbb{R}^n$.

Let $\{l_1(\varepsilon), \dots, l_N(\varepsilon)\}$ be a family of separated length scales. Then the Γ-limit, with respect to the weak topology of $W^{1,p}(\Omega)$, of the sequence of functionals

$$I_\varepsilon(u) \;=\; \int_\Omega f\left(x, \frac{x}{l_1(\varepsilon)}, \dots, \frac{x}{l_N(\varepsilon)}, \nabla u(x)\right)\, dx \tag{4.4}$$

defined in $W^{1,p}(\Omega)$, is the functional

$$I(u) \;=\; \int_\Omega f_{hom}(x, \nabla u(x))\, dx,$$

where $f_{hom} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$f_{hom}(x,\rho) = \inf_{v_i \in \Psi_i} \int_{Q^N} f\left(x, y_1, \dots, y_N, \rho + \sum_{i=1}^N \nabla_{y_i} v_i(x, y_1, \dots, y_i)\right) dy_1 \dots dy_N,$$

with

$$\Psi_i = \left\{ v_i : \Omega \times Q^i \to \mathbb{R} \,:\, v_i(\cdot, y_1, \dots, y_i) \in L^p(\Omega),\; v_i(x, y_1, \dots, y_{i-1}, \cdot) \in W^{1,p}_{per}(Q) \right\},$$

for every $1 \le i \le N$.

When the density f is non-convex in the last variable, it is known that the homogenized density f_{hom} is defined in a different way.

Theorem 4.7.3 (See [51]) *Let $f(x, y_1, \dots, y_N, \rho) : \Omega \times Q^N \times \mathbb{R}^n \to \mathbb{R}$ be such that*

i') f is measurable in Ω, and continuous in $Q^N \times \mathbb{R}^n$,

ii) f is Q-periodic in y_k, for every $1 \le k \le N$,

iii) there exist $C \ge c > 0$ for which

$$c\,|\rho|^p \;\le\; f(x, y_1, \dots, y_N, \rho) \;\le\; C\,(1 + |\rho|^p), \qquad \textit{with } p > 1,$$

for every $(x, y_1, \dots, y_N, \rho) \in \Omega \times Q^N \times \mathbb{R}^n$.

Then the limit energy density of the sequence of functionals I_ε in (4.4) is the function $f_{hom} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_{hom}(x,\rho) \;=$$
$$=\; \lim_{T\to\infty} \inf_{v_i \in \Psi_i} \frac{1}{|TQ^N|} \int_{TQ^N} f\left(x, y_1, \dots, y_N, \rho + \sum_{i=1}^N \nabla_{y_i} v_i(x, y_1, \dots, y_i)\right) dy_1 \cdot dy_N.$$

Part II

Main Results

Chapter 5

Γ-convergence of non-periodic integral functionals

5.1. Introduction

A main issue in variational convergence is to determine explicitly the Γ-limit of sequences of functionals I_j defined in an appropriate space. Namely, it is important to study sufficient conditions under which sequences of functionals are Γ-convergent to integral functionals, whose densities are Charathéodory functions. By now, in the case of sequences of abstract functionals, some sufficient conditions are well known, as it was discussed in Section 2.2. In the case of sequences of integral functionals the explicit characterization of the limit energy density is known in the periodic setting, when the functionals I_j are defined by

$$I_j(u) \;=\; \int_\Omega W(jx, \nabla u(x))\; dx,$$

where $W : \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ is Q-periodic in the first variable, and satisfies a standard growth condition with respect to the second one. For more details see Section 2.3.

In this chapter, we are interested in understanding the structure of general sequences $\{a_j\}$, for which the density of the Γ-limit of sequences of functionals of the form

$$I_j(u) \;=\; \int_\Omega W(a_j(x), \nabla u(x))\; dx \tag{5.1}$$

may be explicitly characterized through the integrand W, and the sequence $\{a_j\}$ itself. This problem was firstly studied in [50], where a sufficient condition on the sequence $\{a_j\}$, called the Average Gradient Property (AGP), was introduced so that the Γ-limit of the sequence $\{I_j\}$ above could be effectively determined. Even though in some situations this concept is tractable (for instance in the periodic setting), the concept itself turns out to be a bit complicated to grasp.

Therefore, the main point of this chapter is to introduce and explore a much more tangible condition, which we called the Composition Gradient Property (CGP), for reasons to be understood soon. The CGP leads to a rather clear way of understanding the structure on the sequence $\{a_j\}$, for which the Γ-limit can be effectively computed. One main advantage of it is the easy way that we may check whether a sequence $\{a_j\}$ verifies the CGP. Besides, it is sufficient for the AGP in a general non-periodic setting.

In order to understand how the CGP comes out, let us first understand the significance of the AGP, indicating briefly the process of finding the Γ-limit of the sequence of functionals given by (5.1) through Young measures. Let $\{u_j\}$ be a weak convergent sequence to u in $W^{1,p}(\Omega)$, and let $\sigma = \{\sigma_x\}_{x\in\Omega}$ be the Young measure associated with the sequence $\{a_j\}$, supported on $\mathbb{R}^m$. Moreover, let $\eta = \{\eta_x\}_{x\in\Omega}$ be the joint Young measure associated with the sequence of pairs $\{(a_j, \nabla u_j)\}$, which may be decomposed as

$$\eta_x(\lambda,\rho) \;=\; \mu_{x,\lambda}(\rho)\otimes\sigma_x(\lambda) \qquad \text{for a.e. } x\in\Omega.$$

Then we may obtain the following estimates on the lower limit:

$$\begin{aligned}
\liminf_{j\to\infty} \int_\Omega W(a_j(x),\nabla u_j(x))\,dx \;&\geq\; \int_\Omega\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} W(\lambda,\rho)\;d\mu_{x,\lambda}(\rho)\;d\sigma_x(\lambda)\;dx\\
&\geq\; \int_\Omega\int_{\mathbb{R}^m} CW\left(\lambda,\int_{\mathbb{R}^n}\rho\;d\mu_{x,\lambda}(\rho)\right)d\sigma_x(\lambda)\;dx\\
&=\; \lim_{j\to\infty}\int_\Omega CW(a_j(x),\varphi(x,a_j(x)))\;dx,
\end{aligned}$$

where $CW(\lambda,\cdot)$ represents the convexification of $W(\lambda,\cdot)$ in $\mathbb{R}^n$, for any $\lambda\in\mathbb{R}^m$, and we have defined the field $\varphi:\Omega\times\mathbb{R}^m\to\mathbb{R}^n$ by putting

$$\varphi(x,\lambda) \;=\; \int_{\mathbb{R}^n}\rho\;d\mu_{x,\lambda}(\rho). \tag{5.2}$$

Notice that, the weak limit ∇u of $\{\nabla u_j\}$ in $L^p(\Omega;\mathbb{R}^n)$ is given by

$$\nabla u(x) \;=\; \int_{\mathbb{R}^m}\varphi(x,\lambda)\,d\sigma_x(\lambda) \qquad \text{for a.e. } x\in\Omega.$$

The AGP condition is tailored to ensure that the sequence of compositions $\{\varphi(\cdot,a_j(\cdot))\}$ is "essentially a sequence of gradients", as commented in Section 5.2. Indeed, the composition $\varphi(\cdot,a_j(\cdot))$ consists in a reorganization, through averaging, of the initial sequence of gradients ∇u_j over "level sets" of a_j. If such a reorganization does not furnish a sequence of gradients, ie $\{a_j\}$ does not verify the AGP, then there is not much that can be done in determining the integrand for the Γ-limit, because we cannot recover a gradient sequence for which the inequalities above are indeed equalities. But if it does, then the Γ-limit can be effectively determined through a minimization process in all fields φ for which the composition $\{\varphi(\cdot, a_j(\cdot))\}$ is "essentially a sequence of gradients".

The difficulty with the AGP is that its formal and rigorous definition is rather involved. This is somehow not surprising as it is supposed to ensure, as indicated above, that the process going from a sequence of gradients $\{\nabla u_j\}$ to the sequence $\{\varphi(\cdot, a_j(\cdot))\}$ through (5.2) produces again a sequence of gradients. Because of this we introduce a new definition, which is explored in Section 5.3.

Definition 5.1.1 *A sequence $\{a_j\} \subset L^q(\Omega; \mathbb{R}^m)$, with associated Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$, satisfies the Composition Gradient Property (CGP) (with respect to the exponent $q > 1$) if there exists a Carathéodory map $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ such that, for a.e. $x \in \Omega$,*

1. *$\varphi(x, \cdot)$ is one-to-one over the support of σ_x, for a.e. $x \in \Omega$;*
2. *$\{\varphi(x, a_j(x + r_j\cdot))\}$ is "essentially a sequence of gradients" in the sense*

$$\| \operatorname{curl} \varphi(x, a_j(x + r_j\cdot)) \|_{W^{-1,q}(B)} \underset{j}{\longrightarrow} 0,$$

for some sequence $r_j \searrow 0$.

The main result in this chapter is to show the sufficiency of the previous condition, as follows.

Theorem 5.1.1 *Let $\{a_j\}$ be a sequence, with associated Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$, such that the inverse image of any ball by a_j is always a set of finite perimeter. If $\{a_j\}$ satisfies the CGP, then it also satisfies the AGP.*

This result provides an easy and practical way of checking whether a sequence $\{a_j\}$ satisfies the AGP. See Section 5.4. Namely, for each $x \in \Omega$, one has to find an one-to-one continuous map $\varphi_x : \mathbb{R}^m \to \mathbb{R}^n$ for which the sequence $\{\varphi_x(a_j(x + r_j\cdot))\}$ may be approximated by a sequence of gradients in the unit ball B, when r_j is chosen so that the sequence $\{a_j(x + r_j\cdot)\}$ generates the homogeneous Young measure σ_x.

Once it is known whether a sequence satisfies the CGP, let us consider a sequence of functionals defined in $W^{1,p}(\Omega)$ by

$$I_j(u) = \int_\Omega W(a_j(x), \nabla u(x))\, dx, \tag{5.3}$$

where $W(\lambda, \rho)$ is continuous in $\mathbb{R}^m \times \mathbb{R}^n$, and satisfies

i) $c_1(|\rho|^p - 1) \le W(a_j(x), \rho) \le c_2(|\rho|^p + 1)$, for a.e. $x \in \Omega$ and every $j \in \mathbb{N}$,

ii) $|W(\lambda_1,\rho)-W(\lambda_2,\rho)|\le w(|\lambda_1-\lambda_2|)|\rho|^p$, for some continuous function w with $w(0)=0$.

Then finding the Γ-limit of $\{I_j\}$, as a main application of this condition, requires to find, for each $x\in\Omega$, a sequence of radii r_j so that $\{a_j(x+r_j\cdot)\}$ generates the homogeneous Young measure σ_x, in the unit ball B, and to find one-to-one continuous maps φ_x such that $\{\varphi_x(a_j(x+r_j\cdot))\}$ is "essentially a sequence of gradients". Specifically, and to stress the scope of the CGP in the computation of Γ-limits, we rewrite Theorem 1.2 in [50] replacing the AGP by the CGP as follows, so that its proof remains intact due to Theorem 5.1.1 above.

Theorem 5.1.2 *Let $\{a_j\}$ be a uniformly bounded sequence, with Young measure $\sigma=\{\sigma_x\}_{x\in\Omega}$, verifying the CGP. For each $x\in\Omega$, let $\{r_j\}$ be a sequence of radii such that the sequence $\{a_j(x+r_j\cdot)\}$ generates the homogeneous Young measures σ_x in the unit ball B, and put*

$$\mathcal{A}_x=\left\{\varphi:\mathbb{R}^m\to\mathbb{R}^n\, continuous,\, one\,to\,one\;:\;\|\operatorname{curl}\varphi(a_j(x+r_j\cdot))\|_{W^{-1,q}(B)}\to 0\right\},$$

for some $q>p>1$. Then the Γ-limit (in the weak topology of $W^{1,p}(\Omega)$) of the sequence of functionals in (5.3) is given by

$$I(u)=\int_\Omega \overline{W}(x,\nabla u(x))\,dx,$$

where the density $\overline{W}:\Omega\times\mathbb{R}^n\to\mathbb{R}$ is defined by

$$\overline{W}(x,\rho)\;=\;\inf_{\varphi\in\mathcal{A}_x}\left\{\int_{\mathbb{R}^m} CW(\lambda,\varphi(\lambda))\;d\sigma_x(\lambda)\;:\;\rho=\int_{\mathbb{R}^m}\varphi(\lambda)\;d\sigma_x(\lambda)\right\}.$$

Notice how functional I can never provide the Γ-limit for a sequence of functionals determined by a sequence of functions a_j not satisfying the CGP condition, as in this situation the class $\mathcal{A}_x$ of admissible fields φ would be empty. This is discussed in Section 5.5.

An interesting corollary of the (reinforced) CGP condition can be deduced identifying sequences of functionals having the same Γ-limit.

Corollary 5.1.3 *Let $\{a_j\}\subset L^\infty(\Omega;\mathbb{R}^m)$ be a sequence with Young measure $\sigma=\{\sigma_x\}_{x\in\Omega}$, for which there exists a Carathéodory map $\varphi:\Omega\times\mathbb{R}^m\to\mathbb{R}^n$ such that, for a.e. $x\in\Omega$,*

i) $\varphi(x,\cdot)$ *is one-to-one over* $\operatorname{supp}\sigma_x$*;*

ii) $\{\varphi(x,a_j(x+r_j\cdot))\}$ *is weak** *convergent in* $L^\infty(B;\mathbb{R}^n)$*, for some* $r_j\searrow 0$*;*

iii) $\operatorname{curl}\varphi(x,a_j(x+r_j\cdot))\rightharpoonup 0$ *in* $L^p(B)$*, for some* $p>n$*.*

Then there is a sequence of gradients $\{\nabla u_j\}$ *bounded in* $L^\infty(\Omega;\mathbb{R}^n)$*, and a Carathéodory map* $\phi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$*, so that the* Γ*-limits of the two sequences of functionals*

$$I_j(u) \;=\; \int_\Omega W(a_j(x), \nabla u(x))\, dx \quad \text{and} \quad J_j(u) \;=\; \int_\Omega W(\phi(x, \nabla u_j(x)), \nabla u(x))\, dx$$

coincide for every integrand $W : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ *as in the previous theorem.*

The relevance of this corollary is clear. In a sense, we can restrict ourselves to computing Γ-limits associated with sequences of gradients. Γ-limits of functionals generated by oscillating sequences of functions which are not gradients may be hard to compute in the non-periodic setting.

In Section 5.6 we explore some examples of sequences $\{a_j\}$ satisfying the CGP, and compute explicitly the Γ-limit of their associated sequences of functionals.

5.2. The Average Gradient Property

In this section we discuss the definition of Average Gradient Property (AGP) in order to explore some sufficient and necessary conditions for it. In plain words, one would say that the sequence $\{a_j\}$ satisfies the AGP if averages of gradients over "level sets" of a_j are gradients themselves.

Definition 5.2.1 *A sequence* $\{a_j\} \subset L^q(\Omega;\mathbb{R}^m)$*, with associated Young measure* $\sigma = \{\sigma_x\}_{x\in\Omega}$*, is said to satisfy the Average Gradient Property (AGP) (with respect to the exponent* $q > 1$*) if, for a.e.* $x \in \Omega$*, whenever:*

1. *the sequence of functions* $a_j^x(\cdot) = a_j(x + r_j \cdot) : B \subset \mathbb{R}^n \to \mathbb{R}^m$*, for some* $r_j \searrow 0$*, generates the homogeneous measure* σ_x*;*

2. *for each* $j \in \mathbb{N}$*, there exists a family of pairwise disjoint balls in* $\mathbb{R}^m$*,*

$$\left\{ B(\lambda_k^{(j)}, r_k^{(j)}) \right\}_k ,$$

   with radii $r_k^{(j)} < r_j$ *and center* $\lambda_k^{(j)} \in \mathbb{R}^m$*, for every* $k \in \mathbb{N}$*, such that*

$$\sigma_x \left(\mathbb{R}^m \setminus \bigcup_k B(\lambda_k^{(j)}, r_k^{(j)}) \right) \;=\; 0,$$

3. $v \in W_0^{1,q}(B)$*,*

then the sequence of fields $V_j^x : B \subset \mathbb{R}^n \to \mathbb{R}^n$ *defined by*

$$V_j^x(y) \;=\; \sum_k \chi_{\Omega_{j,k}^x}(y)\,\frac{1}{|\Omega_{j,k}^x|}\int_{\Omega_{j,k}^x} \nabla v(z)\; dz,$$

where

$$\Omega_{j,k}^x \;=\; (a_j^x)^{-1}\left(B(\lambda_k^{(j)}, r_k^{(j)})\right) \;=\; \left\{\; y \in B \;:\; a_j(x + r_j y) \in B(\lambda_k^{(j)}, r_k^{(j)}) \;\right\},$$

is "essentially a sequence of gradients" in the precise sense

$$\|\,\mathrm{curl}\; V_j^x\,\|_{W^{-1,q}(B)} \;\underset{j}{\longrightarrow}\; 0. \tag{5.4}$$

This means that, to verify whether a sequence $\{a_j\}$, with associated Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$, satisfies the AGP, we should fix any point $x \in \Omega$, and follow the three steps:

1st. Take a positive real sequence $r_j \searrow 0$ for which the rescaled sequence $\{a_j(x + r_j\cdot)\}$, defined in the unit ball $B \subset \mathbb{R}^n$, generates the homogeneous Young measure σ_x. Since the sequence $\{a_j\}$ generates σ, this is a standard procedure called localization of Young measures, as referred to in Section 4.2, so that there is always such a positive real sequence.

2nd. A covering by pairwise disjoint balls of the support of σ_x in $\mathbb{R}^m$ should be built, for each $j \in \mathbb{N}$, so that the family of inverse images of such a covering by $a_j(x+r_j\cdot)$ is a covering by pairwise disjoint sets of the unit ball $B \subset \mathbb{R}^n$, as it is exemplified below. Therefore any sequence of piecewise constant fields V_j^x defined in B by the average of any gradient field ∇v over each inverse image is well-defined.

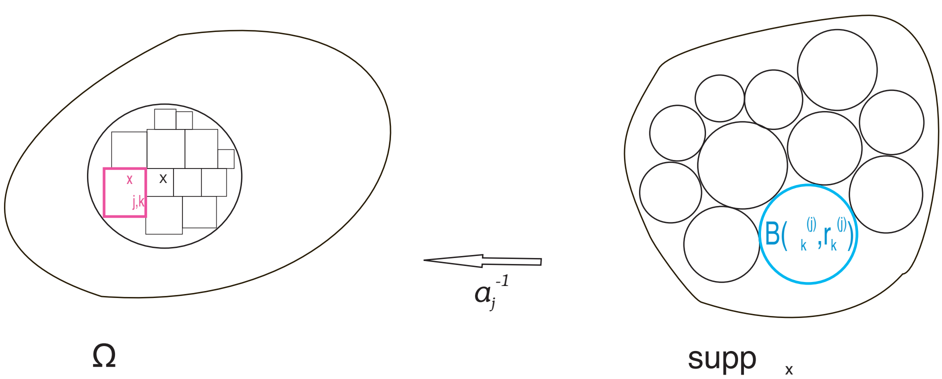

Figure 5.1: Coverings of $B(x, r_j)$ and $\mathrm{supp}\,\sigma_x$.

3rd. It remains to check whether the sequence $\{V_j^x\}$, defined previously for any arbitrary gradient field ∇v, is "essentially a sequence of gradients", in the sense that the sequence $\{\operatorname{curl} V_j^x\}$ converges strongly to 0 in $W^{-1,q}(B)$.

If $\{V_j^x\}$ converges weakly in $L^q(B;\mathbb{R}^n)$ and $\{\operatorname{curl} V_j^x\}$ converges strongly to 0 in $W^{-1,q}(B)$, then it follows from Lemma 4.6.3 that there exists a bounded sequence $\{v_j\} \subset W^{1,q}(B)$ such that

$$\| V_j^x - \nabla v_j \|_{L^p(B;\mathbb{R}^n)} \underset{j}{\longrightarrow} 0, \qquad \text{for any } \; p < q.$$

Notice that there exists a gap between the exponents. So we will say that the sequence $\{V_j^x\}$ is "essentially a sequence of gradients" whenever

$$\| \operatorname{curl} V_j^x \|_{W^{-1,q}(B)} \underset{j}{\longrightarrow} 0,$$

provided it may be approximated by a sequence of gradients.

On the other hand, if there exists a bounded sequence $\{v_j\} \subset W^{1,q}(B)$ such that

$$\| V_j^x - \nabla v_j \|_{L^q(B;\mathbb{R}^n)} \underset{j}{\longrightarrow} 0,$$

we may conclude

$$\| \operatorname{curl} V_j^x \|_{W^{-1,q}(B)} \underset{j}{\longrightarrow} 0,$$

because the first strong convergence implies the strong convergence of $\{\operatorname{curl}\,(V_j^x - \nabla v_j)\}$ to 0 in $W^{-1,q}(B)$, and we have, due to the linearity of curl,

$$\operatorname{curl}\,(V_j^x - \nabla v_j) \;=\; \operatorname{curl} V_j^x - \operatorname{curl} \nabla v_j \;=\; \operatorname{curl} V_j^x.$$

A laminate is a simple example of a sequence satisfying the AGP condition. Namely, consider the sequence $\{a_j\}$ defined in Q by

$$a_j(x) \;=\; A_1 \chi_{(0,t)}\left(\left\langle jx \cdot \vec{n} \right\rangle\right) + A_2 \left(1 - \chi_{(0,t)}\left(\left\langle jx \cdot \vec{n} \right\rangle\right)\right)$$

for some fixed unit vector $\vec{n} \in \mathbb{R}^n$ and $t \in (0,1)$, with associated Young measure

$$\sigma \;=\; t\,\delta_{A_1} + (1-t)\,\delta_{A_2}$$

supported on $\{A_1, A_2\} \subset \mathbb{R}^m$. In the definition of V_j^x, we may drop from now the parameter x, due to the homogeneity of σ. Thus, the sequence of fields $V_j : Q \to \mathbb{R}^n$ defined by

$$V_j(y) \;=\; \left(\frac{1}{|\Omega_{j,1}|}\int_{\Omega_{j,1}} \nabla v(z)\; dz\right) \chi_{\Omega_{j,1}}(y) \;+\; \left(\frac{1}{|\Omega_{j,2}|}\int_{\Omega_{j,2}} \nabla v(z)\; dz\right) \chi_{\Omega_{j,2}}(y),$$

for any $v \in W_0^{1,q}(Q)$, with

$$\Omega_{j,1} \;=\; \{\ y \in Q \ :\ a_j(y) = A_1\ \} \quad \text{and} \quad \Omega_{j,2} \;=\; \{\ y \in Q \ :\ a_j(y) = A_2\ \}$$

such that $Q = \Omega_{j,1} \cup \Omega_{j,2}$, satisfies the condition

$$\|\,\mathrm{curl}\ V_j\,\|_{W^{-1,q}(Q)} \;\underset{j}{\longrightarrow}\; 0.$$

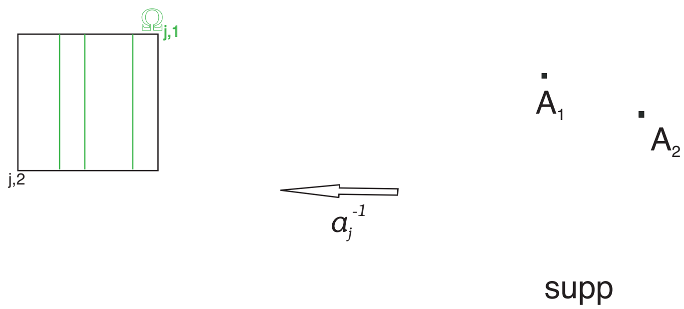

Figure 5.2: Laminate satisfies the AGP.

Indeed, if we put

$$V_j^1 \;=\; \frac{1}{|\Omega_{j,1}|}\int_{\Omega_{j,1}} \nabla v(z)\ dz \quad \text{and} \quad V_j^2 \;=\; \frac{1}{|\Omega_{j,2}|}\int_{\Omega_{j,2}} \nabla v(z)\ dz,$$

then it comes from Green's formula

$$\begin{aligned} V_j^1 - V_j^2 \;&=\; \frac{1}{|\Omega_{j,1}|}\int_{\Omega_{j,1}} \nabla v(z)\ dz - \frac{1}{|\Omega_{j,2}|}\int_{\Omega_{j,2}} \nabla v(z)\ dz \\ &=\; \left(\frac{1}{|\Omega_{j,1}|} + \frac{1}{|\Omega_{j,2}|}\right)\int_{\partial\Omega_{j,1}} v(z)\ d\mathcal{H}^{n-1}(z)\,\vec{n}, \end{aligned}$$

provided $-\vec{n}$ is the outerward normal vector to $\partial\Omega_{j,2}$, while $\vec{n}$ is to $\partial\Omega_{j,1}$. Thus we conclude

$$\begin{aligned} &\lim_{j\to\infty} \|\,\mathrm{curl}\ V_j\,\|_{W^{-1,q}(Q)} \;= \\ &= \lim_{j\to\infty}\ \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \int_Q [\,V_j(y)\otimes\nabla w(y) - \nabla w(y)\otimes V_j(y)\,]\ dy \right| \;= \\ &= \lim_{j\to\infty}\ \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \int_{\Omega_{j,1}} \left[\,V_j^1\otimes\nabla w(y) - \nabla w(y)\otimes V_j^1\,\right]\ dy \;+ \right. \\ &\qquad\qquad \left. + \int_{\Omega_{j,2}} \left[\,V_j^2\otimes\nabla w(y) - \nabla w(y)\otimes V_j^2\,\right]\ dy \right| \;= \end{aligned}$$

$$= \lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \int_{\partial\Omega_{j,1}} \left[(V_j^1 - V_j^2)\otimes \vec{n} - \vec{n}\otimes(V_j^1 - V_j^2)\right] w(y)\, d\mathcal{H}^{n-1}(y) \right| =$$

$$= \lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \int_{\partial\Omega_{j,1}} \left[\left(\frac{1}{|\Omega_{j,1}|} + \frac{1}{|\Omega_{j,2}|}\right)\left(\int_{\partial\Omega_{j,1}} v(z)\ d\mathcal{H}^{n-1}(z)\right) \right.\right.$$

$$\left.\left. \left(\vec{n}\otimes\vec{n} - \vec{n}\otimes\vec{n}\right)\right] w(y)\, d\mathcal{H}^{n-1}(y) \right| = 0.$$

Remark 5.2.1 *The sequence of functions $a_j : Q \to Q$ defined by*

$$a_j(x) = \langle j\,x\rangle \qquad\qquad for\ every\ j\in\mathbb{N},$$

which generates the (homogenous) Lebesgue measure over Q, satisfies the AGP. Indeed, if we consider, for each $j\in\mathbb{N}$, a family of pairwise disjoint cubes $\{Q_k^{(j)}\}$, with side length $h_k^{(j)} \searrow 0$, as $j\to\infty$, such that

$$\left| Q \setminus \bigcup_k Q_k^{(j)} \right| = 0, \qquad and \qquad \left| Q \setminus \bigcup_k a_j\left(Q_k^{(j)}\right) \right| = 0,$$

then the sequence of fields $V_j : Q \to \mathbb{R}^n$ defined by

$$V_j(y) = \sum_k \chi_{Q_k^{(j)}}(y)\, \frac{1}{|Q_k^{(j)}|} \int_{Q_k^{(j)}} \nabla v(z)\ dz,$$

for any $v\in W_0^{1,p}(Q)$, converges strongly to ∇v in $L^p(Q;\mathbb{R}^n)$. Notice that, for a.e. $y\in Q$,

$$\lim_{j\to\infty} |V_j(y) - \nabla v(y)| = \lim_{j\to\infty} \left| \frac{1}{|Q_{k(j)}^{(j)}|} \int_{Q_k^{(j)}} \nabla v(z)\ dz - \nabla v(y) \right|$$

$$\le \lim_{j\to\infty} \frac{1}{|Q_{k(j)}^{(j)}|} \int_{Q_{k(j)}^{(j)}} |\nabla v(z) - \nabla v(y)|\ dz = 0,$$

where $\{Q_{k(j)}^{(j)}\}$ is a diagonal sequence of cubes shrinking to the Lebesgue point y. Applying the Lebegue Dominated Convergence Theorem, it is clear the strong convergence in $L^p(Q;\mathbb{R}^n)$. Since $\operatorname{curl}\nabla v = 0$, we conclude that $\{\operatorname{curl} V_j\}$ converges strongly to 0 in $W^{-1,p}(Q)$.

5.3. The Composition Gradient Property

A main contribution of this chapter is the definition of a new sufficient structural condition, called the Composition Gradient Property (CGP), on the sequence

$\{a_j\}$, to the explicit characterization of the limit energy density of sequences $\{I_j\}$ determined by $\{a_j\}$. In this section we explore the CGP condition having in mind the proof of the main result, Theorem 5.1.1, and a completely general way of identifying, or constructing, sequences satisfying it. Basically, a sequence $\{a_j\}$ satisfies the CGP if composed with an one-to-one, continuous map is "essentially a sequence of gradients".

Definition 5.1.1 *A sequence* $\{a_j\} \subset L^q(\Omega; \mathbb{R}^m)$*, with associated Young measure* $\sigma = \{\sigma_x\}_{x\in\Omega}$*, satisfies the Composition Gradient Property (CGP) (with respect to the exponent* $q > 1$*) if there exists a Carathéodory map* $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ *such that, for a.e.* $x \in \Omega$*,*

1. $\varphi(x, \cdot)$ *is one-to-one over the support of* σ_x*;*
2. $\{\varphi(x, a_j(x + r_j\cdot))\}$ *is "essentially a sequence of gradients", in the sense*

$$\| \operatorname{curl} \varphi(x, a_j(x + r_j\cdot)) \|_{W^{-1,q}(B)} \underset{j}{\longrightarrow} 0,$$

for some sequence $r_j \searrow 0$*.*

When a sequence $\{a_j\}$, with associated Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$, satisfies the CGP, there exists a Carathéodory map φ such that the push-forward measure of σ through φ, given by

$$\{\varphi(x, \cdot)_\sharp \sigma_x\}_{x\in\Omega}\,,$$

becomes a gradient Young measure. Indeed, it follows from Lemma 4.6.3, there exists a bounded sequence $\{u_j\} \subset W^{1,q}(\Omega)$ for which

$$\| \nabla u_j - \varphi(\cdot, a_j(\cdot)) \|_{L^p(\Omega;\mathbb{R}^n)} \underset{j}{\longrightarrow} 0, \qquad p < q.$$

This clearly implies that the sequence $\{\varphi(\cdot, a_j(\cdot))\}$ generates a gradient Young measure. Since σ is the Young measure associated with $\{a_j\}$, we conclude that $\{\varphi(x, \cdot)_\sharp \sigma_x\}_{x\in\Omega}$ is the one associated with $\{\varphi(\cdot, a_j(\cdot))\}$. So, to recover a gradient sequence, from the reorganization of a given sequence $\{a_j\}$, is the main idea beyond the CGP condition, as the name referred to.

Notice that condition 2. in the definition of CGP is not equivalent to say that the family of probability measures $\{\varphi(x, \cdot)_\sharp \sigma_x\}_{x\in\Omega}$ is a gradient Young measure. Namely, the sequence $\{\varphi(\cdot, a_j(\cdot))\}$ may not be "essentially a sequence of gradients", even in the case $\{\varphi(x, \cdot)_\sharp \sigma_x\}_{x\in\Omega}$ is a gradient Young measure. For example, consider the homogenous measure

$$\varphi_\sharp \sigma \;=\; t\, \delta_{\varphi(A_1)} \;+\; (1-t)\, \delta_{\varphi(A_2)}$$

with $\varphi(A_1) - \varphi(A_2) \parallel \vec{n}$, and $\vec{n} \neq \vec{m}$, which is generated by the sequences

$$\varphi(a_j(x)) \;=\; \varphi(A_1)\,\chi_{(0,t)}\left(\left\langle jx\cdot\vec{n}\right\rangle\right) \;+\; \varphi(A_2)\left(1-\chi_{(0,t)}\left(\left\langle jx\cdot\vec{n}\right\rangle\right)\right)$$

and

$$\varphi(\tilde{a}_j(x)) \;=\; \varphi(A_1)\,\chi_{(0,t)}\left(\left\langle jx\cdot\vec{m}\right\rangle\right) \;+\; \varphi(A_2)\left(1-\chi_{(0,t)}\left(\left\langle jx\cdot\vec{m}\right\rangle\right)\right),$$

so that $\{\varphi(a_j(\cdot))\}$ is a gradient sequence while $\{\varphi(\tilde{a}_j(\cdot))\}$ is not.

Remark 5.3.1 *Notice that the sequence of periodic functions $a_j : Q \to Q$ defined by $a_j(x) = \langle jx \rangle$, whose associated Young measure is the Lebesgue measure supported on Q, does not satisfy the CGP condition. Indeed, the existence of a map $\varphi : Q \to \mathbb{R}^n$ such that $\{\operatorname{curl}\varphi(\langle j\cdot\rangle)\}$ converges strongly to 0 in $W^{-1,q}(Q)$ requires constant values on the boundary, which is incompatible with the one-to-one condition on the map φ.*

In order to understand the structure of sequences $\{a_j\}$ satisfying the CGP, we provide some necessary and sufficient conditions, as follows.

Lemma 5.3.1 *Let $\{a_j\} \subset L^q(\Omega;\mathbb{R}^m)$ be a sequence, with associated Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$, and $\{\nabla u_j\} \subset L^\infty(\Omega;\mathbb{R}^n)$, with associated Young measure $\nu = \{\nu_x\}_{x\in\Omega}$. Assume there exists a Carathéodory map $\phi : \Omega\times\mathbb{R}^n \to \mathbb{R}^m$ such that, for a.e. $x \in \Omega$,*

1. *$\phi(x,\cdot)$ is one-to-one over $\operatorname{supp}\nu_x$;*
2. *$\| a_j(x+r_j\cdot) - \phi(x,\nabla u_j(x+s_j\cdot)) \|_{L^\infty(B;\mathbb{R}^m)} \to 0$, for some sequences $r_j,\, s_j \searrow 0$ so that $\{a_j(x+r_j\cdot)\}$ generates the homogenous measure σ_x, and $\{\nabla u_j(x+s_j\cdot)\}$ generates ν_x.*

Then there exists a Carathéodory map $\varphi : \Omega\times\mathbb{R}^m \to \mathbb{R}^n$ for which, for a.e. $x\in\Omega$,

- *i) $\varphi(x,\cdot)$ is one-to-one over $\operatorname{supp}\sigma_x$;*
- *ii) $\|\operatorname{curl}\,\varphi(x, a_j(x+r_j\cdot))\|_{W^{-1,q}(B)} \to 0$.*

Proof. For a.e. $x\in\Omega$, $\phi(x,\cdot)$ is continuous and one-to-one in the compact set $\operatorname{supp}\nu_x$, so that we can consider the inverse map $\phi^{-1}(x,\cdot) : \operatorname{Im}\phi(x,\cdot) \subset \mathbb{R}^m \to \operatorname{supp}\nu_x \subset \mathbb{R}^n$, which is continuous and one-to-one in $\operatorname{Im}\phi(x,\cdot)$. Thus

$$\begin{aligned}
&\lim_{j\to\infty}\int_B \left|\,\phi^{-1}(x,a_j(x+r_jy)) - \nabla u_j(x+s_jy)\,\right|^q\,dy \;= \\
=\;&\lim_{j\to\infty}\int_B \left|\,\phi^{-1}(x,a_j(x+r_jy)) - \phi^{-1}(x,\phi(x,\nabla u_j(x+s_jy)))\,\right|^q\,dy \;= \\
\le\;&\lim_{j\to\infty}\int_B c\,|\,a_j(x+r_jy) - \phi(x,\nabla u_j(x+s_jy))\,|^q\,dy \;=\; 0,
\end{aligned}$$

for some constant $c > 0$. This implies that the sequence $\{\operatorname{curl} \phi^{-1}(x, a_j(x + r_j\cdot))\}$ converges strongly to 0 in $W^{-1,q}(B)$, ie

$$\| \operatorname{curl} \phi^{-1}(x, a_j(x + r_j\cdot)) \|_{W^{-1,q}(B)} \underset{j}{\longrightarrow} 0.$$

Notice that, the sequence $\{\phi^{-1}(x, a_j(x + r_j\cdot))\}$ generates the homogenous Young measure ν_x, which is the Young measure associated with $\{\nabla u_j(x + s_j\cdot)\}$. Thus

$$\nu_x = \phi^{-1}(x,\cdot)_\sharp \sigma_x,$$

and σ_x may be characterized by the push-forward of ν_x through $\phi(x,\cdot)$, ie

$$\sigma_x = \phi(x,\cdot)_\sharp \nu_x,$$

so that

$$\operatorname{Im} \phi(x,\cdot) = \{ \phi(x,\rho) \in \mathbb{R}^m : \rho \in \operatorname{supp} \nu_x \} = \operatorname{supp} \sigma_x.$$

□

In this way we can find many sequences $\{a_j\}$ satisfying the CGP. It is enough to consider a sequence of gradients $\{\nabla u_j\}$, with associated Young measure ν, and a continuous and one-to-one map $\phi(x,\cdot)$ over $\operatorname{supp} \nu_x$, and put

$$a_j(x) = \phi(x, \nabla u_j(x)).$$

Particularly, we may simply consider sequences of functionals determined by sequences of gradients $a_j = \nabla u_j$. Many examples of sequences $\{a_j\}$ satisfying the CGP condition are explored in the last section of this chapter.

However the reverse implication, in the previous lemma, is not exactly true, as it follows from next lemma.

Lemma 5.3.2 *Let $\{a_j\} \subset L^\infty(\Omega;\mathbb{R}^m)$ generate the Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$, and $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ be a Carathéodory map such that, for a.e. $x \in \Omega$,*

i) $\varphi(x,\cdot)$ *is one-to-one over* $\operatorname{supp} \sigma_x$*;*

ii) $\{\varphi(x, a_j(x + r_j\cdot))\}$ *is weak* convergent in* $L^\infty(B;\mathbb{R}^n)$*, for some* $r_j \searrow 0$*;*

iii) $\operatorname{curl} \varphi(x,\cdot, a_j(x + r_j\cdot)) \rightharpoonup 0$ *in* $L^p(B)$*, for some* $p > n$*.*

Then there exist a Carathéodory map $\phi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ so that, for a.e. $x \in \Omega$, there exists a sequence $\{\nabla v_j\} \subset L^\infty(B;\mathbb{R}^n)$ with associated Young measure ν_x, and

1. $\phi(x,\cdot)$ *is one-to-one over* $\operatorname{supp} \nu_x$*;*

2. $\| a_j(x + r_j\cdot) - \phi(x, \nabla v_j(\cdot)) \|_{L^\infty(B;\mathbb{R}^m)} \to 0.$

Proof. For a.e. $x \in \Omega$, since $\{\varphi(x, a_j(x+r_j\cdot))\}$ is weak* convergent in $L^\infty(B;\mathbb{R}^n)$ and $\{\operatorname{curl}\varphi(x,\cdot,a_j(x+r_j\cdot))\}$ converges weakly to 0 in $L^p(B)$, for some $p > n$, it follows from Lemma 4.6.5, there exists a sequence of gradients $\{\nabla v_j\} \subset L^\infty(B;\mathbb{R}^n)$ for which

$$\| \nabla v_j - \varphi(x, a_j(x + r_j\cdot)) \|_{L^\infty(B;\mathbb{R}^n)} \underset{j}{\longrightarrow} 0.$$

Particularly, the sequences $\{\nabla v_j\}$ and $\{\varphi(x, a_j(x + r_j\cdot))\}$ generate the same homogenous gradient Young measure. On the other hand, since $\varphi(x,\cdot)$ is continuous and one-to-one over $\operatorname{supp}\sigma_x$, there exists the inverse map $\varphi^{-1}(x,\cdot) : \operatorname{Im}\varphi(x,\cdot) \subset \mathbb{R}^n \to \operatorname{supp}\sigma_x \subset \mathbb{R}^m$ such that

$$\| \varphi^{-1}(x, \nabla v_j(\cdot)) - a_j(x + r_j\cdot) \|_{L^\infty(B;\mathbb{R}^m)} \underset{j}{\longrightarrow} 0.$$

□

5.4. Sufficiency of the CGP

This section is dedicated to the proof of Theorem 5.1.1, which is based on the following lemma. This lemma states that, if there exists a sequence of piecewise constant fields, in a given partition of the domain Ω, which is "essentially a sequence of gradients", then the sequence of fields with constant value, equal to the average of gradients, in each set of such partition, is "essentially a sequence of gradients", too.

Lemma 5.4.1 *For each $j \in \mathbb{N}$, let $\{\Omega_k^{(j)}\}_{k\in K(j)} \subset \mathbb{R}^n$ be a countable family of pairwise disjoint sets, with finite perimeter, such that $\overline{\Omega} = \bigcup_{k\in K(j)} \overline{\Omega}_k^{(j)}$. If the sequence of functions $V_j : \Omega \to \mathbb{R}^n$ given by*

$$V_j(x) = \sum_{k\in K(j)} F_k^{(j)} \chi_{\Omega_k^{(j)}}(x),$$

with $|F_k^{(j)} - F_i^{(j)}| \geq c > 0$, whenever $k \neq i$ and for all j, is "essentially a sequence of gradients" in the sense $\| \operatorname{curl} V_j \|_{W^{-1,q}(\Omega)} \underset{j}{\to} 0$, then the sequence of functions $U_j : \Omega \to \mathbb{R}^n$ defined by

$$U_j(x) = \sum_{k\in K(j)} \frac{1}{|\Omega_k^{(j)}|} \int_{\Omega_k^{(j)}} \nabla v(y)\, dy\, \chi_{\Omega_k^{(j)}}(x),$$

for any $v \in W_0^{1,q}(\Omega)$, is "essentially a sequence of gradients", too.

Proof. Assume that, for some $q > 1$,

$$\lim_{j\to\infty} \| \operatorname{curl} V_j \|_{W^{-1,q}(\Omega)} = \lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} |\langle \operatorname{curl} V_j, w\rangle| =$$

$$= \lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \int_\Omega [\, V_j(x) \otimes \nabla w(x) - \nabla w(x) \otimes V_j(x)\,]\, dx \right| = 0,$$

with

$$\int_\Omega [\, V_j(x) \otimes \nabla w(x) - \nabla w(x) \otimes V_j(x)\,]\, dx =$$

$$= \sum_{k\in K(j)} \left(F_k^{(j)} \otimes \int_{\Omega_k^{(j)}} \nabla w(x)\, dx - \int_{\Omega_k^{(j)}} \nabla w(x)\, dx \otimes F_k^{(j)} \right).$$

It follows from the Green formula

$$\int_{\Omega_k^{(j)}} \nabla w(x)\; dx = \int_{\partial\Omega_k^{(j)}} w(x)\, \vec{n}_k^{(j)}(x)\; d\mathcal{H}^{n-1}(x),$$

where $\vec{n}_k^{(j)}(x)$ is the outer unit normal vector at $x \in \partial\Omega_k^{(j)}$.

Then we realize that, for each k, there exists i for which

$$\partial\Omega_k^{(j)} \bigcap \partial\Omega_i^{(j)} \neq \emptyset.$$

Let $\Gamma_{k,i}^{(j)}$ be such intersection of boundaries, ie

$$\Gamma_{k,i}^{(j)} = \partial\Omega_k^{(j)} \bigcap \partial\Omega_i^{(j)} \neq \emptyset,$$

so that $\vec{n}_k^{(j)}(x) = -\vec{n}_i^{(j)}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_{k,i}^{(j)}$. Therefore

$$\sum_{k\in K(j)} \left(F_k^{(j)} \otimes \int_{\Omega_k^{(j)}} \nabla w(x)\, dx - \int_{\Omega_k^{(j)}} \nabla w(x)\, dx \otimes F_k^{(j)} \right) =$$

$$= \sum_{i,k\in K(j)} \left(\left(F_k^{(j)} - F_i^{(j)} \right) \otimes \int_{\Gamma_{k,i}^{(j)}} w(x)\, \vec{n}_k^{(j)}(x)\; d\mathcal{H}^{n-1}(x) - \right.$$

$$\left. \int_{\Gamma_{k,i}^{(j)}} w(x)\, \vec{n}_k^{(j)}(x)\; d\mathcal{H}^{n-1}(x) \otimes \left(F_k^{(j)} - F_i^{(j)} \right) \right),$$

and

$$\lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \sum_{k\in K(j)} \int_{\Gamma_{k,i}^{(j)}} w(x) \left[\left(F_k^{(j)} - F_i^{(j)} \right) \otimes \vec{n}_k^{(j)}(x) - \right.\right.$$

$$\left.\left. \vec{n}_k^{(j)}(x) \otimes \left(F_k^{(j)} - F_i^{(j)} \right) \right] d\mathcal{H}^{n-1}(x) \right| = 0. \qquad (5.5)$$

Provided $|F_k^{(j)} - F_i^{(j)}| \geq c > 0$, no intersection of boundaries is missing in the previous sum. From (5.5) and the arbitrariness of w, we conclude that

$$\lim_{j\to\infty} \sup_{i,k\in K(j)} \sup_{x\in\Gamma(j)_{k,i}} \left| \left(F_k^{(j)} - F_i^{(j)}\right) \otimes \vec{n}_k^{(j)}(x) - \vec{n}_k^{(j)}(x) \otimes \left(F_k^{(j)} - F_i^{(j)}\right) \right| = 0 \quad (5.6)$$

This means that there exist constants $c_k^{(j)}$ such that the differences $\vec{n}_k^{(j)}(x) - c_k^{(j)}\left(F_k^{(j)} - F_i^{(j)}\right)$ converge to zero uniformly along the interfaces $\Gamma_{k,i}^{(j)}$, ie at the limit, the jumps of the vector field V_j across interfaces $\Gamma_{k,i}^{(j)}$ are parallel to the normals to those same interfaces.

Take now any $v \in W_0^{1,q}(\Omega)$, and put

$$G_k^{(j)} = \frac{1}{|\Omega_k^{(j)}|} \int_{\Omega_k^{(j)}} \nabla v(y)\, dy = \frac{1}{|\Omega_k^{(j)}|} \int_{\partial\Omega_k^{(j)}} v(y)\, \vec{n}_k^{(j)}(y)\, d\mathcal{H}^{n-1}(y),$$

so that

$$U_j(x) = \sum_{k\in K(j)} \chi_{\Omega_k^{(j)}}(x)\, G_k^{(j)}.$$

Then, as before, $\|\operatorname{curl} U_j\|_{W^{-1,q}(\Omega)} \to 0$, as $j \to \infty$, if

$$\lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}} \leq 1} \left| \sum_{k\in K(j)} \int_{\Gamma_{k,i}^{(j)}} w(x) \left[\left(G_k^{(j)} - G_i^{(j)}\right) \otimes \vec{n}_k^{(j)}(x) + \right.\right. \quad (5.7)$$
$$\left.\left. - \vec{n}_k^{(j)}(x) \otimes \left(G_k^{(j)} - G_i^{(j)}\right)\right] d\mathcal{H}^{n-1}(x) \right| = 0.$$

Notice that, since $\Gamma_{k,i}^{(j)} = \partial\Omega_k^{(j)} \bigcap \partial\Omega_i^{(j)}$, we may write

$$\begin{aligned}
& G_k^{(j)} - G_i^{(j)} = \\
= \; & \frac{1}{|\Omega_k^{(j)}|} \int_{\partial\Omega_k^{(j)}} v(y)\, \vec{n}_k^{(j)}(y)\, d\mathcal{H}^{n-1}(y) - \frac{1}{|\Omega_i^{(j)}|} \int_{\partial\Omega_i^{(j)}} v(y)\, \vec{n}_i^{(j)}(y)\, d\mathcal{H}^{n-1}(y) \\
= \; & \frac{1}{|\Omega_k^{(j)}|} \int_{\Gamma_{k,i}^{(j)}} v(y)\, \vec{n}_k^{(j)}(y)\, d\mathcal{H}^{n-1}(y) + \frac{1}{|\Omega_i^{(j)}|} \int_{\Gamma_{k,i}^{(j)}} v(y)\, \vec{n}_k^{(j)}(y)\, d\mathcal{H}^{n-1}(y) \\
= \; & \left(\frac{1}{|\Omega_k^{(j)}|} + \frac{1}{|\Omega_i^{(j)}|}\right) \int_{\Gamma_{k,i}^{(j)}} v(y)\, \vec{n}_k^{(j)}(y)\, d\mathcal{H}^{n-1}(y) \\
= \; & C_{k,i}^j \int_{\Gamma_{k,i}^{(j)}} v(y)\, \vec{n}_k^{(j)}(y)\, d\mathcal{H}^{n-1}(y),
\end{aligned}$$

so that the limit in (5.7) is equal to

$$\lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \sum_{k\in K(j)} C^j_{k,i} \int_{\Gamma^{(j)}_{k,i}} w(x) \left[\int_{\Gamma^{(j)}_{k,i}} v(y)\, \vec{n}^{(j)}_k(y)\, d\mathcal{H}^{n-1}(y) \otimes \vec{n}^{(j)}_k(x) \right.\right.$$
$$\left.\left. - \vec{n}^{(j)}_k(x) \otimes \int_{\Gamma^{(j)}_{k,i}} v(y)\, \vec{n}^{(j)}_k(y)\; d\mathcal{H}^{n-1}(y) \right] d\mathcal{H}^{n-1}(x) \right| .$$

Provided, the difference $\vec{n}^{(j)}_k(y) - c^{(j)}_k \left(F^{(j)}_k - F^{(j)}_i \right)$ converges to zero uniformly in $\Gamma^{(j)}_{k,i}$, we may replace $\vec{n}^{(j)}_k(y)$ by $c^{(j)}_k \left(F^{(j)}_k - F^{(j)}_i \right)$ in the previous limit, which implies

$$\lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \sum_{k\in K(j)} \left(\frac{1}{|\Omega^{(j)}_k|} + \frac{1}{|\Omega^{(j)}_i|} \right) \int_{\Gamma^{(j)}_{k,i}} \int_{\Gamma^{(j)}_{k,i}} w(x) v(y)\, d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x) \right.$$
$$\left. \left[\left(F^{(j)}_k - F^{(j)}_i \right) \otimes \left(F^{(j)}_k - F^{(j)}_i \right) - \left(F^{(j)}_k - F^{(j)}_i \right) \otimes \left(F^{(j)}_k - F^{(j)}_i \right) \right] \right| \;=\; 0.$$

□

From the previous lemma we may deduce the structure of a partition of Ω where a gradient sequence may be defined. Namely the normals to the interfaces should be determined by the jumps of the sequence through such interfaces. In the next remark we focus on an example of a partition where a gradient sequence cannot be defined.

Remark 5.4.1 *For each $j \in \mathbb{N}$, consider the family of pairwise disjoint sets*

$$\left\{ B_{\frac{k}{j}} \setminus B_{\frac{k-1}{j}} \right\}_{k=1}^{j},$$

where B_i stands for the ball centred at the origin and with radius i, such that

$$\overline{B} \;=\; \bigcup_{k=1}^{j} \overline{B}_{\frac{k}{j}} \setminus \overline{B}_{\frac{k-1}{j}}.$$

The sequence of functions $V_j : B \to \mathbb{R}^2$ defined by

$$V_j(x) \;=\; \sum_{k=1}^{j} F^{(j)}_k \; \chi_{B_{\frac{k}{j}} \setminus B_{\frac{k-1}{j}}}(x)$$

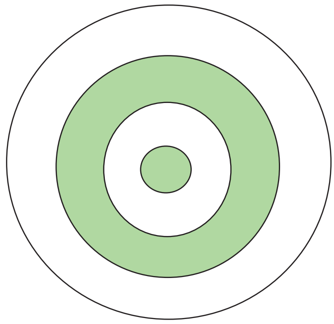

Figure 5.3: Curved layers centred at the origin.

is not "essentially a gradient sequence", in the sense that the sequence $\{\text{curl } V_j\}$ *does not converge strongly to* 0 *in* $W^{-1,q}(B)$. *Indeed*

$$\begin{aligned}
&\lim_{j\to\infty} \|\,\text{curl } V_j\,\|_{W^{-1,q}(B)} = \\
&= \lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \int_B \left[\, V_j(x)\otimes\nabla w(x) - \nabla w(x)\otimes V_j(x)\,\right]\, dx \right| = \\
&= \lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \sum_{k=1}^{j} \int_{B_{\frac{k}{j}}\setminus B_{\frac{k-1}{j}}} \left[F_k^{(j)}\otimes\nabla w(x) - \nabla w(x)\otimes F_k^{(j)}\right]\, dx \right| = \\
&= \lim_{j\to\infty} \sup_{\|w\|_{W_0^{1,q'}}\le 1} \left| \sum_{k=1}^{j-1} \int_{\Sigma_{\frac{k}{j}}} \left[\left(F_k^{(j)+} - F_k^{(j)-}\right)\otimes \vec{n}_{\frac{k}{j}}(x)\, w(x) \,- \right.\right. \\
&\qquad\qquad \left.\left. w(x)\,\vec{n}_{\frac{k}{j}}(x)\otimes\left(F_k^{(j)+} - F_k^{(j)-}\right)\, dS \right] \right| \neq 0,
\end{aligned}$$

where $\Sigma_{\frac{k}{j}} = \partial B_{\frac{k}{j}}$, $\vec{n}_{\frac{k}{j}}(x)$ *is the outer unit normal for* $x\in\Sigma_{\frac{k}{j}}$, *and* $F_k^{(j)+} - F_k^{(j)-} = [V_j]_{\Sigma_{\frac{k}{j}}}$ *is the jump of* V_j *through* $\Sigma_{\frac{k}{j}}$. *The last limit is not equal to* 0 *because the normal* $\vec{n}_{\frac{k}{j}}(x)$ *depends on the point* $x\in\Sigma_{\frac{k}{j}}$, *and it cannot be approximated by the constant jumps* $F_k^{(j)+} - F_k^{(j)-}$ *along the interface* $\Sigma_{\frac{k}{j}}$.

Now, we are able to prove Theorem 5.1.1, ie if $\{a_j\}$ satisfies the CGP condition, then it satisfies the AGP. Namely, if the composition $\{\varphi(\cdot, a_j(\cdot))\}$ is "essentially a gradient sequence", we have to show that there exists a partition Ω such that any sequence of piecewise constant fields, equal to the average of gradients, in each set of the partition, is "essentially a gradient sequence", too.

Proof. (of Theorem 5.1.1) Let $\sigma = \{\sigma_x\}_{x\in\Omega}$ be the Young measure associated with the sequence $\{a_j\}$, so that $\operatorname{supp}\sigma_x$ is a bounded open set in $\mathbb{R}^m$, for a.e.

$x \in \Omega$. Let $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ be a Carthéodory map such that, for a.e. $x \in \Omega$, $\varphi(x, \cdot)$ is continuous and one-to-one in the support of σ_x, and the sequence $\{\operatorname{curl} \varphi(x, a_j(x+r_j \cdot))\}$ converges strongly to 0 in $W^{-1,q}(B)$. For a.e. $x \in \Omega$, let $\{r_j\}$ be a sequence of positive real values, tending to 0, for which the rescaled sequence $\{a_j(x + r_j \cdot)\}$, defined in the unit ball $B \subset \mathbb{R}^n$, generates the homogeneous Young measure σ_x. Thus

$$\lim_{j\to\infty} \frac{1}{|B|} \int_B \varphi(x, a_j(x + r_j y))\, dy \;=\; \int_{\mathbb{R}^m} \varphi(x, \lambda)\, d\sigma_x(\lambda).$$

From Lemma A.5.2, for each $j \in \mathbb{N}$, there exists a set of points $\{\lambda_k^{(j)}\} \subset \operatorname{supp} \sigma_x$ and positive numbers $r_k^{(j)} < r_j$ such that

$$\{\, B(\lambda_k^{(j)}, r_k^{(j)})\,\}_k$$

is a family of pairwise disjoint balls, centred at $\lambda_k^{(j)}$ with radius $r_k^{(j)}$, for which

$$\sigma_x \left(\mathbb{R}^m \setminus \bigcup_k B(\lambda_k^{(j)}, r_k^{(j)}) \right) \;=\; 0,$$

and

$$\int_{\mathbb{R}^m} \varphi(x, \lambda)\, d\sigma_x(\lambda) \;=\; \lim_{j\to\infty} \sum_k \sigma_x(B(\lambda_k^{(j)}, r_k^{(j)}))\, \varphi(x, \lambda_k^{(j)}).$$

Notice that

$$\lim_{j\to\infty} \sup_k \left| \sigma_x(B(\lambda_k^{(j)}, r_k^{(j)})) \;-\; \frac{|a_j^{-1}(x + r_j B(\lambda_k^{(j)}, r_k^{(j)}))|}{|B|} \right| \;=\; 0.$$

So, let us define the subset $\Omega_{j,k}^x$ by

$$\Omega_{j,k}^x \;=\; a_j^{-1}(x + r_j B(\lambda_k^{(j)}, r_k^{(j)}))$$

such that it has finite perimeter, and for each $j \in \mathbb{N}$, we may consider a countable family of pairwise disjoint sets $\{\Omega_{j,k}^x\}_k$ satisfying

$$\left| B \setminus \bigcup_k \Omega_{j,k}^x \right| \;=\; 0.$$

We define the fields $V_j^x : B \subset \mathbb{R}^n \to \mathbb{R}^n$ by putting

$$\begin{aligned} V_j^x(y) \;&=\; \sum_k \chi_{\Omega_{j,k}^x}(y)\, \frac{1}{|\Omega_{j,k}^x|} \int_{\Omega_{j,k}^x} \varphi(x, a_j(x + r_j z))\, dz \;= \\ &=\; \sum_k \chi_{\Omega_{j,k}^x}(y)\, F_{j,k}^x, \end{aligned}$$

so that $|F^x_{j,k} - F^x_{j,i}| \geq c > 0$, whenever $k \neq i$, provided the right choice of $\{\Omega^x_{j,k}\}_k$, and $\varphi(x,\cdot)$ is one-to-one over $\operatorname{supp}\sigma_x$. Since

$$\begin{aligned}
0 &= \lim_{j\to\infty}\left|\frac{1}{|B|}\int_B \varphi(x,a_j(x+r_jy))\,dy \;-\; \sum_k \sigma_x(B(\lambda_k^{(j)},r_k^{(j)}))\;\varphi(x,\lambda_k^{(j)})\right| \\
&= \lim_{j\to\infty}\left|\sum_k\left(\frac{1}{|B|}\int_{\Omega^x_{j,k}} \varphi(x,a_j(x+r_jy))\,dy \;-\; \sigma_x(B(\lambda_k^{(j)},r_k^{(j)}))\;\varphi(x,\lambda_k^{(j)})\right)\right| \\
&= \lim_{j\to\infty}\left|\sum_k\left(\int_{\Omega^x_{j,k}} \varphi(x,a_j(x+r_jy))\,dy \;-\; |\Omega^x_{j,k}|\,\varphi(x,\lambda_k^{(j)})\right)\right| \\
&= \lim_{j\to\infty}\left|\sum_k\left(|\Omega^x_{j,k}|F^x_{j,k} \;-\; |\Omega^x_{j,k}|\,\varphi(x,\lambda_k^{(j)})\right)\right|,
\end{aligned}$$

we conclude

$$\lim_{j\to\infty}\int_B \left|V_j^x(y)-\varphi(x,a_j(x+r_jy))\right|^q dy \;=\; \lim_{j\to\infty}\sum_k |\Omega_k^{(j)}|\left|F^x_{j,k}-\varphi(x,\lambda_k^{(j)})\right|^q \;=\; 0.$$

Thus it follows

$$\|\operatorname{curl} V_j^x\|_{W^{-1,q}(B)} \;\underset{j}{\longrightarrow}\; 0, \tag{5.8}$$

because, by hypotheses, $\{\operatorname{curl}\varphi(x,a_j(x+r_j\cdot))\}$ converges strongly to 0 in $W^{-1,q}(B)$. Then we may apply Lemma 5.4.1 to the sequence $\{V_j^x\}$, ie we may replace, in the definition of V_j^x, the field $\varphi(x,a_j(x+r_j\cdot))$ by any gradient field ∇v, with $v \in W_0^{1,q}(B)$, and the sequence $\{V_j^x\}$ is again "essentially a gradient sequence". In this way, we have proved that $\{a_j\}$ satisfies the AGP condition.

□

5.5. Explicit characterization of the density of the Γ-limit

The explicit characterization of the limit energy density of sequences of functionals of the type

$$I_j(u) \;=\; \int_\Omega W(a_j(x),\nabla u(x))\,dx \tag{5.9}$$

defined in $W^{1,p}(\Omega)$, with $W(\lambda,\rho)$ continuous in $\mathbb{R}^m\times\mathbb{R}^n$ and satisfying

i) $c\,(|\rho|^p-1) \;\leq\; W(a_j(x),\rho) \;\leq\; C\,(|\rho|^p+1)$, for $(x,\rho)\in\Omega\times\mathbb{R}^n$,

ii) $|W(\lambda_1,\rho) - W(\lambda_2,\rho)| \;\leq\; w(|\lambda_1-\lambda_2|)|\rho|^p$, for every $\lambda_1, \lambda_2 \in \mathbb{R}^m$,

where $w : \mathbb{R}^+ \to \mathbb{R}$ is continuous and $w(0) = 0$, was already studied in [50], for sequences $\{a_j\}$ satisfying the AGP condition. Though the AGP is a general condition, it is not so tractable, as explained in the Introduction. In the previous section, a much more understandable condition was introduced, which allows to handle the procedure of computing and characterizing the limit energy density.

Therefore, according with Theorem 5.1.2, if the sequence $\{a_j\}$ satisfies the CGP, the sequence of functionals given by (5.9) is Γ-convergent to

$$I(u) \;=\; \int_\Omega \overline{W}(x, \nabla u(x))\; dx,$$

where the density $\overline{W} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\overline{W}(x,\rho) \;=\; \inf_{\varphi \in \mathcal{A}_x} \left\{ \int_{\mathbb{R}^m} CW(\lambda, \varphi(\lambda))\; d\sigma_x(\lambda) \;:\; \rho = \int_{\mathbb{R}^m} \varphi(\lambda)\; d\sigma_x(\lambda) \right\}$$

with

$$\mathcal{A}_x = \left\{ \varphi : \mathbb{R}^m \to \mathbb{R}^n \text{ continuous, one to one } : \|\operatorname{curl}\varphi(a_j(x + r_j\cdot))\|_{W^{-1,q}(B)} \to 0 \right\},$$

for some $q > p > 1$, whenever the sequence $\{a_j(x + r_j\cdot)\}$ generates the homogenous Young measure σ_x. Clearly, this characterization of $\overline{W}$, through the minimization of the functional

$$F(\varphi) \;=\; \int_{\mathbb{R}^m} CW(\lambda, \varphi(\lambda))\; d\sigma_x(\lambda)$$

under the constraint

$$\rho = \int_{\mathbb{R}^m} \varphi(\lambda)\; d\sigma_x(\lambda),$$

in the admissible set $\mathcal{A}_x$, is well-defined whenever $\mathcal{A}_x$ is non-empty. The CGP condition ensures that $\mathcal{A}_x$ is non-empty.

Proposition 5.5.1 *Let $\{a_j\}$ generate the Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$ so that, for a.e. $x \in \Omega$, the sequence $\{a_j(x+r_j\cdot)\}$ generates the homogenous Young measure σ_x. If $\{a_j\}$ satisfies the CGP (with respect to exponent $q > 1$) then, for a.e. $x \in \Omega$, the admissible set*

$$\mathcal{A}_x = \left\{ \varphi : \mathbb{R}^m \to \mathbb{R}^n \textit{ continuous, one to one } : \|\operatorname{curl}\varphi(a_j(x + r_j\cdot))\|_{W^{-1,q}(B)} \to 0 \right\}$$

is non-empty.

Proof. It follows immediately from the definition of CGP. □

Thus we may compute explicitly the limit energy density of the sequence $\{I_j\}$ whenever the sequence $\{a_j\}$ may be transformed into a sequence of gradients

$\{\varphi(\cdot, a_j(\cdot))\}$, in the sense $\{\text{curl } \varphi(\cdot, a_j(\cdot))\}$ converges strongly to 0 in $W^{-1,q}(\Omega)$. Moreover, if we consider a reinforced CGP condition, ie assuming $\{\text{curl } \varphi(\cdot, a_j(\cdot))\}$ converges weakly to 0 in $L^p(\Omega)$, stated in Corollary 5.1.3, it is possible to identify different sequences of functionals with the same limit energy.

Proof. (of Corollary 5.1.3) Let $\{a_j\}$ be a sequence in $L^\infty(\Omega;\mathbb{R}^m)$ satisfying the hypotheses in Corollary 5.1.3. Then, from Lemma 5.3.2, there exists a Carathéodory map $\phi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$, and a sequence of gradients $\{\nabla u_j\} \subset L^\infty(\Omega;\mathbb{R}^n)$, such that

$$\| \, a_j - \phi(\cdot, \nabla u_j(\cdot)) \, \|_{L^\infty(\Omega;\mathbb{R}^m)} \underset{j}{\longrightarrow} 0.$$

Since W is continuous and satisfies $ii)$ we have

$$\lim_{j\to\infty} \int_\Omega W(a_j(x), \nabla v_j(x)) \, dx \; = \; \lim_{j\to\infty} \int_\Omega W(\phi(x, \nabla u_j(x)), \nabla v_j(x)) \, dx,$$

for any bounded sequence $\{v_j\} \subset W^{1,p}(\Omega)$. Obviously, the sequence $\{\phi(\cdot, \nabla u_j(\cdot))\}$ also satisfies the CGP, and

$$\Gamma - \lim_{j\to\infty} \int_\Omega W(a_j(x), \nabla u(x)) \, dx \; = \; \Gamma - \lim_{j\to\infty} \int_\Omega W(\phi(x, \nabla u_j(x)), \nabla u(x)) \, dx,$$

for every $u \in W^{1,p}(\Omega)$. □

5.6. Examples

Applying Theorem 5.1.2 to some examples of sequences $\{a_j\}$ satisfying the CGP, we obtain interesting explicit formulae of the limit energy density.

5.6.1. Laminates

i) Let us consider the function $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$a(x,y) \; = \; A_1 \; \chi_{(0,t(x))} \left(\left\langle y \cdot \vec{n} \right\rangle \right) + A_2 \left(1 - \chi_{(0,t(x))} \left(\left\langle y \cdot \vec{n} \right\rangle \right) \right),$$

where $A_1, A_2 \in \mathbb{R}^m$, the volume fraction $t : \Omega \to (0,1)$, and the unit normal vector $\vec{n} \in \mathbb{R}^n$. For a.e. $x \in \Omega$, $a(x,\cdot)$ is a periodic function at the direction of the vector $\vec{n}$, ie

$$a(x,y) \; = \; a(x, y + k\vec{n}), \qquad \forall \, y \in \mathbb{R}^n, \; \forall \, k \in \mathbb{Z}.$$

We define the sequence of functions $a_j : \Omega \to \mathbb{R}^m$ by putting

$$a_j(x) \; = \; a(x, jx) \; = \; A_1 \; \chi_{(0,t(x))} \left(\left\langle jx \cdot \vec{n} \right\rangle \right) + A_2 \left(1 - \chi_{(0,t(x))} \left(\left\langle jx \cdot \vec{n} \right\rangle \right) \right),$$

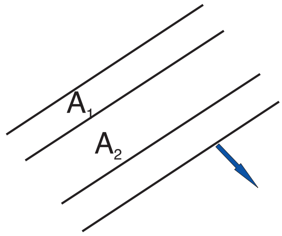

Figure 5.4: First order laminate.

which generates the non-homogenous Young measure $\sigma = \{\sigma_x\}_{x\in\Omega}$ given by

$$\sigma_x \;=\; t(x)\;\delta_{A_1} + (1-t(x))\;\delta_{A_2}, \qquad \text{for a.e. } x\in\Omega.$$

In this case, to prove that $\{a_j\}$ satisfies the CGP, it is enough to consider a function $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\varphi(A_1) - \varphi(A_2) \;\parallel\; \vec{n},$$

ie the difference $\varphi(A_1) - \varphi(A_2)$ is proportional to the normal vector $\vec{n}$. Thus the sequence $\{\varphi(a_j(\cdot))\}$ is "essentially a gradient sequence".

The set $\mathcal{A}_x$, of admissible functions φ for which the push-forward measure

$$\varphi_\sharp\sigma_x \;=\; t(x)\,\delta_{B_1} + (1-t(x))\,\delta_{B_2},$$

with

$$B_1 \;=\; \varphi(A_1) \qquad \text{and} \qquad B_2 \;=\; \varphi(A_2),$$

is a first order laminate, may be defined by

$$\mathcal{A}_x \;=\; \Big\{\; B_1, B_2 \in \mathbb{R}^n \;:\; B_1 - B_2 \;\parallel\; \vec{n} \;\Big\}.$$

Therefore we may compute explicitly the density of the Γ-limit of the sequence of functionals of the form

$$I_j(u) \;=$$
$$= \int_\Omega \Big[\, W_1(\nabla u(x))\chi_{(0,t(x))}\left(\left\langle jx\cdot\vec{n}\right\rangle\right) + W_2(\nabla u(x))\left(1-\chi_{(0,t(x))}\left(\left\langle jx\cdot\vec{n}\right\rangle\right)\right)\Big]\,dx,$$

where

$$W_1(\rho) \;=\; W(A_1,\rho) \qquad \text{and} \qquad W_2(\rho) \;=\; W(A_2,\rho)$$

for every $\rho\in\mathbb{R}^n$. Namely, the sequence $\{I_j\}$ is Γ-convergent to the functional

$$I(u) \;=\; \int_\Omega \overline{W}(x,\nabla u(x))\;dx,$$

where $\overline{W} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\begin{aligned}\overline{W}(x,\rho) &= \min_{\varphi\in\mathcal{A}_x} \left\{ \int_{\mathbb{R}^m} CW(\lambda,\varphi(\lambda))\ d\sigma_x(\lambda)\ :\ \rho = \int_{\mathbb{R}^m} \varphi(\lambda)\ d\sigma_x(\lambda) \right\} = \\ &= \min_{B_1,B_2\in\mathbb{R}^n} \Big\{ t(x)CW_1(B_1) + (1-t(x))CW_2(B_2)\ :\ (B_1 - B_2) \parallel \vec{n}, \\ &\qquad\qquad \rho = t(x)B_1 + (1-t(x))B_2 \ \Big\}.\end{aligned}$$

Here $CW_i(\cdot)$ is the convex envelope of $W_i(\cdot)$ in $\mathbb{R}^n$, for $i = 1, 2$.

Remark 5.6.1 *The sequence of functions* $a_j : B \to \mathbb{R}^m$ *defined by*

$$\begin{aligned}a_j(x) &= A_1\ \chi_{(0,t(x))}(\langle|jx|\rangle) + A_2\left(1 - \chi_{(0,t(x))}(\langle|jx|\rangle)\right) = \\ &= \sum_{k=1}^{j} A_1\ \chi_{B_{\frac{k-1}{j}+\frac{t(x)}{j}}\setminus B_{\frac{k-1}{j}}}(x) + A_2\left(1 - \chi_{B_{\frac{k-1}{j}+\frac{t(x)}{j}}\setminus B_{\frac{k-1}{j}}}(x)\right),\end{aligned}$$

where $t : B \to (0,1)$ *and* B_i *is the ball centred at the origin with radius* i*, generates the same non-homogenous Young measure*

$$\sigma_x = t(x)\ \delta_{A_1} + (1-t(x))\ \delta_{A_2}, \qquad \textit{for a.e. } x \in B.$$

Indeed

$$\begin{aligned}&\lim_{j\to\infty} \int_B f(a_j(x))\xi(x)\ dx = \\ &= \lim_{j\to\infty} \sum_{k=1}^{j} \int_{B_{\frac{k-1}{j}+\frac{t(x)}{j}}\setminus B_{\frac{k-1}{j}}} f(A_1)\xi(x)\ dx + \int_{B_{\frac{k}{j}}\setminus B_{\frac{k-1}{j}+\frac{t(x)}{j}}} f(A_2)\xi(x)\ dx = \\ &= \int_B [t(x)f(A_1) + (1-t(x))f(A_2)]\,\xi(x)\ dx,\end{aligned}$$

for every $f \in C_0(\mathbb{R}^m)$ *and* $\xi \in L^1(B)$*. However, it does not satisfy the CGP, because the jumps of any composition* $\varphi(a_j(\cdot))$ *across the interfaces* $\partial B_{\frac{k-1}{j}+\frac{t(x)}{j}}$ *cannot effectively determine the normal vector to such interfaces, as previously commented in Remark 5.4.1.*

ii) We may also consider a sequence of functions $a_j : \Omega \to \mathbb{R}^m$ oscillating between three values, for instance

$$\begin{aligned}a_j(x) &= A_1\ \chi_{(0,t)}\left(\left\langle jx\cdot\vec{n}\right\rangle\right) + \\ &+ \left(1 - \chi_{(0,t)}\left(\left\langle jx\cdot\vec{n}\right\rangle\right)\right)\left(A_2\ \chi_{(0,s)}\left(\left\langle jx\cdot\vec{m}\right\rangle\right) + A_3\left(1 - \chi_{(0,s)}\left(\left\langle jx\cdot\vec{m}\right\rangle\right)\right)\right),\end{aligned}$$

where $A_1, A_2, A_3 \in \mathbb{R}^m$, $\vec{n}$, $\vec{m} \in \mathbb{R}^n$ and $s, t \in (0,1)$. The sequence $\{a_j\}$ generates the homogenous Young measure σ given by

$$\sigma \;=\; t\,\delta_{A_1} + (1-t)\,\left(s\,\delta_{A_2} + (1-s)\,\delta_{A_3}\right).$$

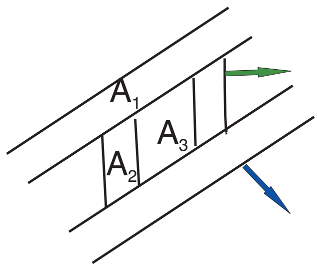

Figure 5.5: Second order laminate.

Moreover, it satisfies the CGP condition, because, for any field $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that

- $\varphi(A_1) \;\neq\; \varphi(A_2) \;\neq\; \varphi(A_3)$,
- $\varphi(A_2) - \varphi(A_3) \;\parallel\; \vec{m}, \qquad \varphi(A_1) - (s\,\varphi(A_2) + (1-s)\,\varphi(A_3)) \;\parallel\; \vec{n}$,

the composition sequence $\{\varphi(a_j(\cdot))\}$ is " essentially a sequence of gradients". Thus the admissible set $\mathcal{A}_x$ may be defined by

$$\mathcal{A}_x \;=\; \left\{\; B_1, B_2, B_3 \in \mathbb{R}^n \;:\; B_1 - (s\,B_2 + (1-s)\,B_3) \parallel \vec{n}, \quad B_2 - B_3 \parallel \vec{m} \;\right\}.$$

In this way, we conclude that the limit energy density, of the sequence

$$I_j(u) \;=\; \int_\Omega W(a_j(x), \nabla u(x))\; dx,$$

is the homogenous function $\overline{W} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\overline{W}(\rho) = \inf_{B_i \in \mathbb{R}^n} \Big\{\; t\,W_1(B_1) + (1-t)\,(s\,W_2(B_2) + (1-s)\,W_3(B_3)) \;:\; (B_2 - B_3) \parallel \vec{m},$$
$$\rho = t\,B_1 + (1-t)(s\,B_2 + (1-s)\,B_3),\; (B_1 - (s\,B_2 + (1-s)\,B_3)) \parallel \vec{n} \;\Big\}.$$

5.6.2. Non-periodic sequences

i) More generally, we may consider a sequence of functions $a_j : \Omega \to \mathbb{R}^m$ of the form

$$a_j(x) \;=\; \sum_{k=1}^{j} A_k^j\; \chi_{\Omega_k^j}(x) \tag{5.10}$$

where $\{\Omega_k^j\}$ is a family of pairwise disjoint sets covering Ω, so that $\vec{n}_k^j \in \mathbb{R}^n$ is the unit normal vector to Ω_k^j, and there is a field $\phi : \mathbb{R}^m \to \mathbb{R}^n$ in such a way that, for every $j \in \mathbb{N}$, the pairs

$$\left\{ \left(\frac{|\Omega_k^j|}{|\Omega|}, \varphi(A_k^j) \right) \right\}_k^j$$

satisfy the (H_j) condition. See Section 4.5 for more details. The Young measure associated with $\{a_j\}$ is

$$\sigma = \lim_{j\to\infty} \sum_{k=1}^{j} \frac{|\Omega_k^j|}{|\Omega|} \delta_{A_k^j}.$$

Then, clearly, the composition sequence $\varphi(a_j(\cdot))$, given by

$$\varphi(a_j(x)) = \sum_{k=1}^{j} \varphi(A_k^j) \chi_{\Omega_k^j}(x),$$

is "essentially a sequence of gradients". In this situation, the admissible set $\mathcal{A}_x$ is given by

$$\mathcal{A}_x = \left\{ B_k^j \in \mathbb{R}^n : \left\{ \left(\frac{|\Omega_k^j|}{|\Omega|}, B_k^j \right) \right\}_k \text{satisfies}\,(H_j)\,\text{condition, with normals} \left\{ \vec{n}_k^j \right\}_k \right\}.$$

The limit energy density, of the family of functionals

$$I_j(u) = \sum_{k=1}^{j} \int_\Omega W_k^j(\nabla u(x)) \chi_{\Omega_k^j}(x)\, dx$$

where

$$W_k^j(\rho) = W(A_k^j, \rho),$$

is given by

$$\overline{W}(\rho) = \lim_{j\to\infty} \inf_{B_k^j \in \mathbb{R}^n} \left\{ \sum_{k=1}^{j} \frac{|\Omega_k^j|}{|\Omega|} W_k^j(B_k^j) \ : \ \rho = \sum_{k=1}^{j} \frac{|\Omega_k^j|}{|\Omega|} B_k^j, \right.$$
$$\left. \left\{ \left(\frac{|\Omega_k^j|}{|\Omega|}, B_k^j \right) \right\}_k \text{satisfies } (H_j) \text{ condition, with normals } \left\{ \vec{n}_k^j \right\}_k \right\}.$$

ii) Another typical situation is concerned with Vitali coverings of a certain domain Ω by small copies of another domain D. Namely, assume that, for each $j \in \mathbb{N}$, $\{x_k^{(j)} + r_k^{(j)} D\}_k$ is a Vitali covering of Ω by pairwise disjoint sets $x_k^{(j)} + r_k^{(j)} D$, where

$\{x_k^{(j)}\} \subset \Omega$ and $r_k^{(j)} \leq \frac{1}{j}$. For any function $v \in W_0^{1,p}(D)$, let us define the sequence of functions $v_j : \Omega \to \mathbb{R}$ by putting

$$v_j(x) \;=\; r_j^{(j)}\, v\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) \qquad \text{if} \;\; x \in x_k^{(j)} + r_k^{(j)} D.$$

Then consider the sequence of functions $a_j : \Omega \to \mathbb{R}^n$ defined by

$$a_j(x) \;=\; \nabla v\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) \qquad \text{if} \;\; x \in x_k^{(j)} + r_k^{(j)} D,$$

such that it generates the homogenous Young measure $\overline{\delta}_{\nabla v}$ given by

$$\langle \overline{\delta}_{\nabla v}, \varphi \rangle \;=\; \frac{1}{|D|} \int_D \varphi(\nabla v(y))\; dy, \qquad \forall\, \varphi \in C_0(\mathbb{R}^n).$$

This corresponds to the homogenization procedure for gradient Young measures, as referred to in Section 4.2. It follows by construction that the sequence $\{a_j\}$ is "essentially a sequence of gradients".

Therefore, the sequence of functionals defined by

$$I_j(u) \;=\; \sum_k \int_{x_k^{(j)} + r_k^{(j)} D} W\left(\nabla v\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right), \nabla u(x)\right)\; dx$$

is Γ-convergent to the functional

$$I(u) \;=\; \int_\Omega \overline{W}(\nabla u(x))\; dx,$$

where

$$\overline{W}(\rho) \;=\; \inf_{z \in W^{1,p}(D)} \left\{ \frac{1}{|D|} \int_D CW(\nabla v(y), \rho + \nabla z(y))\; dy \;:\; \frac{1}{|D|} \int_D \nabla z(y)\; dy = 0 \right\},$$

for every $\rho \in \mathbb{R}^n$. Notice that, the admissible set $\mathcal{A}$ is defined by

$$\mathcal{A} \;=\; \{\; \varphi : \mathbb{R}^n \to \mathbb{R}^n \text{ continuous } : \; \operatorname{curl} \varphi(\nabla v(\cdot)) = 0 \;\},$$

and, after changing the variables $\varphi(\nabla v) = \rho + \nabla z$, we reach the previous representation.

In the particular case when the function v is defined in the unit ball B by

$$v(y) \;=\; \frac{|y|^2}{2} - \frac{1}{2},$$

so that $\nabla v(y) = y$, it follows

$$\overline{W}(\rho) \;=\; \inf_{z \in W^{1,p}(B)} \left\{ \frac{1}{|B|} \int_B CW(y, \rho + \nabla z(y))\; dy \;:\; \frac{1}{|B|} \int_B \nabla z(y)\; dy \;=\; 0 \right\}.$$

Chapter 6

Γ-convergence of quadratic functionals with oscillating linear perturbations

6.1. Introduction

In this chapter, we are interested in studying, from a variational point of view, the homogenization of second-order elliptic equations of the form

$$(P_\varepsilon)\quad \begin{cases} -\operatorname{div} A_\varepsilon(x)\,\nabla u_\varepsilon(x) &= \operatorname{div} b_\varepsilon(x) \quad \text{in } \Omega \\ u_\varepsilon &\in H^1_0(\Omega), \end{cases}$$

where the leading coefficient A_ε and the source term $\operatorname{div} b_\varepsilon$ are rapidly oscillating as ε goes to 0. More precisely, we are looking for an explicit characterization of the homogenized leading coefficient and source term, in the unusual situation where the source term also oscillates, depending on ε. Notice that, the homogenization of problem (P_ε) with constant source term, ie $\operatorname{div} b_\varepsilon = f$, has been widely studied in the last decades. Moreover, problem (P_ε) has been already studied in the case of non-constant source term, where the leading coefficient is of the form $A_\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$. See Chapter 3 for more details. Indeed, it is known that, if $A_\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$, for some Q-periodic matrix function $A = [a_{ij}] \in [L^\infty(Y)]^{n\times n}$ satisfying $\alpha|\rho|^2 \le A(y)\rho\cdot\rho \le \beta|\rho|^2$, for a.e. $y \in Q$ and every $\rho \in \mathbb{R}^n$, and the sequence $\{\operatorname{div}\, b_\varepsilon\}$ converges weakly in $H^{-1}(\Omega)$, then there exists a subsequence of solutions u_ε of (P_ε) weakly converging in $H^1_0(\Omega)$ to the solution u_0 of the homogenized problem

$$(P_\star)\quad \begin{cases} -\operatorname{div} A_0\,\nabla u_0(x) &= \operatorname{div} g^\star(x) \quad \text{in } \Omega \\ u_0 &\in H^1_0(\Omega), \end{cases}$$

where A_0 is the effective matrix and $g^\star \in [L^2(\Omega)]^n$. The function $g^\star$ is determined in a rather elaborate way. Our aim is to give a more explicit characterization, and a better understanding, of the leading coefficient and the source term in the homogenized equation, by means of the Young measures associated with the sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$, considering periodic and non-periodic sequences.

Namely, we study the Γ-convergence of quadratic functionals, with oscillatory linear perturbations, of the type

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\, \nabla u(x)^T \, \frac{A_\varepsilon(x)}{2} \, \nabla u(x) \;+\; b_\varepsilon(x) \cdot \nabla u(x) \,\right] dx, \tag{6.1}$$

in order to understand the interaction between the oscillations of A_ε and b_ε, and how it affects homogenization. In order to obtain the Γ-convergence of such energies, we apply the results obtained in the previous chapter, and those in [52]. Any time we can compute explicitly the Γ-limit, we will have precise information on the term $g^\star$. A main tool in this endeavour will be the joint Young measure associated with pairs $\{(A_\varepsilon, b_\varepsilon)\}$, which is not necessarily the product of Young measures associated with each sequence separately.

We treat separately the case where $n = 1$ and the one where $n > 1$, because the first one is simpler to handle, and a more explicit characterization of the function $g^\star$ is obtained in the general non-periodic setting. When $n = 1$, in Section 6.2 we consider the sequence of energies I_ε defined in $H_0^1(\Omega)$ by

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\, \frac{a_\varepsilon(t)}{2} \, u'(t)^2 \;+\; b_\varepsilon(t)\, u'(t) \,\right] dt,$$

where $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$ are weak* convergent sequences in $L^\infty(\Omega)$, and the first one is uniformly bounded away from zero. We conclude, by Theorem 6.2.1 below, that such sequence of energies is Γ-convergent to the functional I defined by

$$I(u) \;=\; \int_\Omega \psi(t, u'(t)) \; dt,$$

where the limit energy density ψ is a quadratic function in the second variable, given explicitly by

$$\psi(t,\rho) \;=\; \frac{a_0(t)}{2}\,\rho^2 + a_0(t)k(t)\rho + \frac{a_0(t)k(t)^2}{2} - \int_{\mathbb{R}^2} \frac{\beta^2}{2\alpha} \; d\eta_t(\alpha,\beta),$$

for every $(t,\rho) \in \Omega \times \mathbb{R}^n$. Here the functions $a : \Omega \to (0,\infty)$ and $k : \Omega \to \mathbb{R}$ are defined by putting

$$a_0(t) \;=\; \left(\int_{\mathbb{R}} \frac{1}{\alpha} \; d\sigma_t(\alpha) \right)^{-1} \qquad \text{and} \qquad k(t) \;=\; \int_{\mathbb{R}^2} \frac{\beta}{\alpha} \; d\eta_t(\alpha,\beta),$$

where $\eta = \{\eta_t\}_{t\in\Omega}$ is the joint Young measure associated with the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$, and $\sigma = \{\sigma_t\}_{t\in\Omega}$ is the one associated with $\{a_\varepsilon\}$. The interaction between the two sequences $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$ enters into this density ψ through their joint Young measure. Then we deduce, from Corollary 6.2.3, that the homogenized problem associated with it is

$$(P_\star) \quad \begin{cases} -\frac{d}{dt}\, a_0(t)\, u_0'(t) & = & \frac{d}{dt}\, g^\star(t) \qquad \text{in } \Omega \\ u_0 & \in & H_0^1(\Omega), \end{cases}$$

where the function $g^\star$ is the linear coefficient of the density ψ given by

$$g^\star(t) \;=\; a_0(t)\, k(t) \qquad \text{in } \; \Omega.$$

In the periodic setting, if $a_\varepsilon(t) = a\left(\left\langle \frac{t}{\varepsilon}\right\rangle\right)$ and $b_\varepsilon = b\left(t, \left\langle \frac{t}{\varepsilon}\right\rangle\right)$ oscillate at the same length scale ε, we conclude that

$$a_0 \;\equiv\; \int_0^1 a(y)\; dy \qquad \text{and} \qquad g^\star(t) \;=\; a_0 \int_0^1 \frac{b(t,y)}{a(y)}\; dy,$$

see Proposition 6.2.1. However, when the leading and source terms oscillate at distinct length scales, the homogenized source term is indeed the weak* limit of $\{b_\varepsilon\}$. Namely if $a_\varepsilon(t) = a\left(\left\langle \frac{t}{\varepsilon}\right\rangle\right)$ and $b_\varepsilon = b\left(t, \left\langle \frac{t}{\varepsilon^2}\right\rangle\right)$, then

$$g^\star(t) \;=\; \int_0^1 b(t,y)\; dy \qquad \text{in } \; \Omega.$$

The joint Young measure η plays an important role in the characterization of $g^\star$ and in the understanding of how the oscillatory behaviour affect it. If η is the product of the Young measures associated with $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$, then the function $g^\star$ is completely defined through the Young measure associated with $\{b_\varepsilon\}$, in such a way that the sequence $\{a_\varepsilon\}$ does not interfere at all in such a coefficient. In this situation $g^\star$ is the weak* limit of the sequence $\{b_\varepsilon\}$. When the joint Young measure is not the product of the measures associated with each sequence, the function $g^\star$ is not the weak* limit of $\{b_\varepsilon\}$. This homogenized coefficient also depends on the Young measure associated with the sequence $\{a_\varepsilon\}$.

When $n > 1$, we treat separately the periodic multi-scale situation and the non-periodic one. More precisely, in Section 6.3 the Γ-convergence, of the sequence of energies I_ε defined by (6.1), is treated in the case where the sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ are defined by

$$A_\varepsilon(x) = A\left(x, \left\langle \frac{x}{l_1(\varepsilon)}\right\rangle, \ldots, \left\langle \frac{x}{l_N(\varepsilon)}\right\rangle\right)$$

and

$$b_\varepsilon(x) = b\left(x, \left\langle \frac{x}{l_1(\varepsilon)}\right\rangle, \ldots, \left\langle \frac{x}{l_N(\varepsilon)}\right\rangle\right),$$

for some matrix function $A \in [L^\infty(\Omega \times Q \times \ldots \times Q)]^{n\times n}$ and $b \in [L^\infty(\Omega \times Q \times \ldots \times Q)]^{n}$. Here $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$ is a family of separated length scales. Notice that we are assuming that both sequences oscillate at the same length scales. For the sake of simplicity, let us present an example illustrating our result. If we consider the sequences $A_\varepsilon(x) = A\left(x, \left\langle \frac{x}{l(\varepsilon)} \right\rangle\right)$ and $b_\varepsilon(x) = b\left(x, \left\langle \frac{x}{l(\varepsilon)} \right\rangle\right)$, oscillating at the same length scale $l(\varepsilon)$, we conclude that the sequence of solutions of

$$\begin{cases} -\operatorname{div} A\left(x, \left\langle \frac{x}{l(\varepsilon)} \right\rangle\right) \nabla u_\varepsilon(x) &= \operatorname{div} b\left(x, \left\langle \frac{x}{l(\varepsilon)} \right\rangle\right) \quad \text{in } \Omega \\ u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

is weak convergent to the solution of the homogenized problem

$$\begin{cases} -\operatorname{div} A_0(x) \nabla u_0(x) &= \operatorname{div} g^\star(x) \quad \text{in } \Omega \\ u_0 &\in H_0^1(\Omega), \end{cases}$$

where the matrix function $A_0 : \Omega \to \mathbb{R}^{n\times n}$ is defined by

$$A_0(x) \;=\; \int_Q (I_n + [\nabla_y w_j(x,y)])^T A(x,y)\, (I_n + [\nabla_y w_j(x,y)])\; dy,$$

and $g^\star : \Omega \to \mathbb{R}^n$ by

$$g^\star(x) = \int_Q (I_n + [\nabla_y w_j(x,y)])^T \,[\; A(x,y)\nabla_y z(x,y) \;+\; b(x,y)\;]\; dy.$$

Here I_n is the $n \times n$-identity matrix, $[\nabla_y w_j(x,y)]$ is the $n \times n$-matrix whose columns are the vectors $\nabla_y w_j(x,y)$, and the function $w_j(x,\cdot)$ is the solution of the cell problem

$$\begin{cases} -\operatorname{div}_y A(x,y)\,(e_j + \nabla_y w_j(x,y)) &= 0 \qquad \text{in } Q \\ w_j(x,\cdot) &\in H^1_{per}(Q), \end{cases}$$

for every $1 \le j \le n$, and $z(x,\cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_y A(x,y)\,\nabla_y z(x,y) &= \operatorname{div} b(x,y) \qquad \text{in } Q \\ z(x,\cdot) &\in H^1_{per}(Q). \end{cases}$$

Clearly the homogenized coefficient $g^\star$ depends on the sequence $\{A_\varepsilon\}$, provided both sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillate at the same length scale.

In Section 6.4, we also treat the case where the sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillate at distinct separated length scales, ie

$$A_\varepsilon(x) \;=\; A\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_N(\varepsilon)} \right\rangle\right)$$

and

$$b_\varepsilon(x) \;=\; b\left(x, \left\langle \frac{x}{l_{N+1}(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_{N+M}(\varepsilon)} \right\rangle\right),$$

where $\{l_1(\varepsilon), \dots, l_{N+M}(\varepsilon)\}$ is a family of $N+M$ separated length scales. For instance, when the sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillate at distinct separated length scales $\{l_1(\varepsilon), l_2(\varepsilon)\}$, so that

$$A_\varepsilon(x) \;=\; A\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle\right) \qquad \text{and} \qquad b_\varepsilon(x) \;=\; b\left(x, \left\langle \frac{x}{l_2(\varepsilon)} \right\rangle\right),$$

we conclude that the homogenized source term $g^\star$ is indeed the weak* limit of the sequence $\{b_\varepsilon\}$, ie

$$g^\star(x) \;=\; \int_Q b(x,y)\, dy,$$

so that it does not depend on the oscillatory behaviour of $\{A_\varepsilon\}$. See Corollary 6.4.4.

Finally, in the general non-periodic setting, we focus on the homogenization of problem (P_ε) when the sequence of pairs $\{(A_\varepsilon, b_\varepsilon)\}$ is assumed to satisfy the Composition Gradient Property (CGP). This structural assumption was introduced in the previous chapter, so that Γ-limits can be computed explicitly through the Young measure associated with relevant sequences. As such, it is a sufficient condition which allows to furnish an explicit form of the density of the Γ-limit. We apply these ideas to general quadratic functionals. See Section 6.5. An interesting example, which may be considered also in the periodic setting, is the following Dirichlet problem for a laminate composite material

$$\begin{cases} -\operatorname{div}\, a_\varepsilon(x)\nabla u_\varepsilon(x) &=& \operatorname{div}\, b_\varepsilon(x) \qquad \text{in } \Omega \\ u_\varepsilon &\in& H^1_0(\Omega), \end{cases}$$

where the sequence of pairs $(a_\varepsilon, b_\varepsilon) : \Omega \to (1,+\infty) \times \mathbb{R}^n$ is given by

$$(a_\varepsilon(x), b_\varepsilon(x)) \;=\; (a_1,b_1)\chi_{(0,t(x))}\left(\frac{x}{\varepsilon}\cdot \vec{n}\right) \;+\; (a_2,b_2)\left(1-\chi_{(0,t(x))}\left(\frac{x}{\varepsilon}\cdot\vec{n}\right)\right).$$

Here, for a.e. $x \in \Omega$, $\chi_{(0,t(x))}(s)$ is the characteristic function of the interval $(0,t(x))$ over $(0,1)$, extended by periodicity to $\mathbb{R}$. In this situation, whenever we assume that there exists a continuous and one-to-one map $\phi : (1,+\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ such that its composition $\phi(a_\varepsilon(\cdot), b_\varepsilon(\cdot))$ satisfies the continuity condition on the interface, ie

$$\phi(a_1,b_1) - \phi(a_2,b_2) \;\parallel\; \vec{n},$$

so that the CGP holds, we may define explicitly the effective coefficients. Indeed, we conclude that the associated homogenized equation is

$$\begin{cases} -\operatorname{div}\, a_0(x)\,\nabla u_0(x) &=& \operatorname{div}\, g^\star(x) \qquad \text{in } \Omega \\ u_0 &\in& H^1_0(\Omega). \end{cases}$$

Here the effective coefficient $a_0 : \Omega \to \mathbb{R}$ is defined by

$$a_0(x) \;=\; \frac{a_1\,a_2}{(1-t(x))\,a_1 + t(x)\,a_2}$$

and $g^\star : \Omega \to \mathbb{R}^n$ by

$$g^\star(x) \;=\; \frac{t(x)\,a_2}{(1-t(x))\,a_1 + t(x)\,a_2}\,b_1 \;+\; \frac{(1-t(x))\,a_1}{(1-t(x))\,a_1 + t(x)\,a_2}\,b_2.$$

6.2. The general one-dimensional case

In this section we are interested on the homogenization of ordinary equations of type

$$\begin{cases} -\frac{d}{dt}\,a_\varepsilon(t)\,u_\varepsilon'(t) &=& \frac{d}{dt}\,b_\varepsilon(t) \qquad \text{in } \Omega \\ \qquad\qquad u_\varepsilon &\in& H_0^1(\Omega), \end{cases}$$

where $\{a_\varepsilon\} \subset L^\infty(\Omega)$ satisfies $a_\varepsilon(t) \geq c > 0$ a.e. in Ω, $\{b_\varepsilon\} \subset L^\infty(\Omega)$, and Ω is a bounded open subset of $\mathbb{R}$. The homogenized equation may be defined whenever the explicit characterization of the density of the Γ-limit of functionals

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\, \frac{a_\varepsilon(t)}{2}\,u'(t)^2 \;+\; b_\varepsilon(t)\,u'(t) \,\right] dt \tag{6.2}$$

is known.

Theorem 6.2.1 *Let $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$ be weak* convergent sequences in $L^\infty(\Omega)$ such that $a_\varepsilon(t) \geq c > 0$ a.e. in Ω, and $\eta = \{\eta_t\}_{t\in\Omega}$ is the Young measure generated by the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$. Then the sequence of functionals in (6.2) is Γ-convergent, in the weak topology of $H_0^1(\Omega)$, to the functional I defined in $H_0^1(\Omega)$ by*

$$I(u) \;=\; \int_\Omega \psi(t, u'(t))\; dt,$$

where $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$\psi(t,\rho) \;=\; \frac{a_0(t)}{2}\,\rho^2 \;+\; a_0(t)k(t)\,\rho \;+\; \left[\frac{a_0(t)k(t)^2}{2} \;-\; \int_{\mathbb{R}^2} \frac{\beta^2}{2\alpha}\; d\eta_t(\alpha,\beta)\right], \tag{6.3}$$

with

$$a_0(t) = \left(\int_{\mathbb{R}^2} \frac{1}{\alpha}\; d\eta_t(\alpha,\beta)\right)^{-1} = \left(\int_{\mathbb{R}} \frac{1}{\alpha}\; d\sigma_t(\alpha)\right)^{-1} \quad \text{and} \quad k(t) \;=\; \int_{\mathbb{R}^2} \frac{\beta}{\alpha}\; d\eta_t(\alpha,\beta),$$

for a.e. $t \in \Omega$. Here $\sigma = \{\sigma_t\}_{t\in\Omega}$ is the Young measure associated with $\{a_\varepsilon\}$.

Proof. Consider the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$, with associated Young measure $\eta = \{\eta_t\}_{t\in\Omega}$ supported on $\mathbb{R}^2$, which satisfies the CGP condition, see Definition 5.1.1. Indeed, for any map $\varphi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, continuous and injective in the second variable, if we define $F_\varepsilon : \Omega \to \mathbb{R}$ by putting

$$F_\varepsilon(t) \;=\; \int_0^t \varphi(s, a_\varepsilon(s), b_\varepsilon(s))\; ds,$$

it holds $F'_\varepsilon(t) = \varphi(t, a_\varepsilon(t), b_\varepsilon(t))$ in Ω. Thus, it follows from Theorem 5.1.2,

$$\Gamma - \lim_{\varepsilon \searrow 0} \int_\Omega \left[\frac{a_\varepsilon(t)}{2}\, u'(t)^2 \;+\; b_\varepsilon(t)\, u'(t) \right] dt \;=\; \int_\Omega \psi(t, u'(t))\; dt,$$

for every $u \in H^1_0(\Omega)$, where $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\psi(t,\rho) = \min_\varphi \left\{ \int_{\mathbb{R}^2} \left[\frac{\alpha}{2}\, \varphi(\alpha,\beta)^2 + \beta\, \varphi(\alpha,\beta) \right] d\eta_t(\alpha,\beta) \;:\; \rho = \int_{\mathbb{R}^2} \varphi(\alpha,\beta)\; d\eta_t(\alpha,\beta) \right\}.$$

For fixed $(t,\rho) \in \Omega \times \mathbb{R}$, let us find the minimizer of the functional

$$F(\varphi) \;=\; \int_{\mathbb{R}^2} \left[\frac{\alpha}{2}\, \varphi(\alpha,\beta)^2 \;+\; \beta\, \varphi(\alpha,\beta) \right] d\eta_t(\alpha,\beta)$$

under the constraint

$$\rho \;=\; \int_{\mathbb{R}^2} \varphi(\alpha,\beta)\; d\eta_t(\alpha,\beta).$$

First, consider a new strict convex functional

$$G(\varphi) \;=\; \int_{\mathbb{R}^2} \left[\frac{\alpha}{2}\, \varphi(\alpha,\beta)^2 \;+\; \beta\, \varphi(\alpha,\beta) \;+ \lambda\, \varphi(\alpha,\beta) \right] d\eta_t(\alpha,\beta),$$

where λ is a parameter. Its minimizer φ has to satisfy the equation

$$\frac{d}{dt} G(\varphi + sv)\lfloor_{s=0} \;=\; 0, \qquad \text{for every } v \in C^\infty_c(\mathbb{R}^2).$$

Notice that

$$G(\varphi + sv) \;=\; \int_{R^2} \left[\frac{\alpha}{2}\, \varphi(\alpha,\beta)^2 + (\beta+\lambda)\, \varphi(\alpha,\beta) \right] d\eta_t(\alpha,\beta) \;+$$
$$+\; s^2 \int_{\mathbb{R}^2} \frac{\alpha}{2}\, v(\alpha,\beta)^2 d\eta_t(\alpha,\beta) \;+\; s \int_{\mathbb{R}^2} [\, \alpha\, \varphi(\alpha,\beta) + (\beta+\lambda)\,]\, v(\alpha,\beta)\, d\eta_t(\alpha,\beta),$$

so that

$$\frac{d}{dt} G(\varphi + sv)\lfloor_{s=0} = \int_{\mathbb{R}^2} [\, \alpha\, \varphi(\alpha,\beta) + (\beta+\lambda)\,]\, v(\alpha,\beta)\, d\eta_t(\alpha,\beta) \;=\; 0.$$

The arbitrariness of v implies that

$$\varphi(\alpha,\beta) \;=\; -\frac{\beta+\lambda}{\alpha}.$$

Since φ should satisfy the previous constraint, we get the value of the parameter

$$\lambda \;=\; -a_0(t)\, (\rho + k(t)),$$

with

$$a_0(t) = \left(\int_{\mathbb{R}^2} \frac{1}{\alpha}\, d\eta_t(\alpha,\beta) \right)^{-1} = \left(\int_{\mathbb{R}} \frac{1}{\alpha}\, d\sigma_t(\alpha) \right)^{-1} \quad \text{and} \quad k(t) \ = \ \int_{\mathbb{R}^2} \frac{\beta}{\alpha}\, d\eta_t(\alpha,\beta).$$

In this way, we conclude that the minimizer φ is given by

$$\varphi(\alpha,\beta) \ = \ \frac{a_0(t)}{\alpha}\rho \ + \ \frac{a_0(t)k(t)}{\alpha} \ - \ \frac{\beta}{\alpha}.$$

The proof is finished when we replace this expression in $\psi(t,\rho)$. □

The density of the Γ-limit of $\{I_\varepsilon\}$ is a quadratic function, whose quadratic and linear coefficients, $a_0(t)$ and $a_0(t)k(t)$, respectively, are defined through the Young measure associated with $\{a_\varepsilon\}$, and the joint Young measure associated with $\{(a_\varepsilon, b_\varepsilon)\}$. Notice that a_0 is not the weak* limit of $\{a_\varepsilon\}$, and k is the weak* limit of $\{b_\varepsilon/a_\varepsilon\}$.

Whenever the joint Young measure η is a homogeneous measure (ie, does not depend on t), the functions a and k defined in Ω will be constants, and the limit energy density ψ will be homogeneous, too. Since the joint Young measure is not necessarily the product of the Young measures associated with each sequence, the following Corollary remarks the special case when it holds true. Notice that this is the situation when the oscillations of both terms a_ε and b_ε take place at different scales, as it is commented at the end of this section.

Corollary 6.2.2 *Under the hypothesis of previous theorem, if the joint Young measure $\eta = \{\eta_t\}_{t\in\Omega}$ is the product of the Young measures associated with $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$, namely*

$$\eta(\alpha,\beta) \ = \ \theta(\beta)\otimes\sigma(\alpha),$$

where $\sigma = \{\sigma_t\}_{t\in\Omega}$ is associated with $\{a_\varepsilon\}$ and $\theta = \{\theta_t\}_{t\in\Omega}$ with $\{b_\varepsilon\}$, then

$$\psi(t,\rho) \ = \ \frac{a_0(t)}{2}\rho^2 + b(t)\rho + \frac{1}{2a_0(t)}\left(b(t)^2 - \int_{\mathbb{R}} \beta^2 d\theta_t(\beta) \right),$$

with

$$a_0(t) \ = \ \left(\int_{\mathbb{R}} \frac{1}{\alpha}\, d\sigma_t(\alpha) \right)^{-1} \quad \text{and} \quad b(t) \ = \ \int_{\mathbb{R}} \beta\ \theta_t(\beta),$$

for a.e. $t\in\Omega$ and every $\rho\in\mathbb{R}$.

Thus the linear coefficient b is the weak* limit of $\{b_\varepsilon\}$ when the joint Young measure associated with $\{(a_\varepsilon, b_\varepsilon)\}$ is the product of Young measures. The proof of the corollary is just to recompute the formulae in the previous theorem, when $\eta_t = \theta_t \otimes \sigma_t$ a.e. in Ω.

From Theorem 6.2.1 it follows the result on the convergence of solutions u_ε, of the second order ordinary equations associated with functionals I_ε.

Corollary 6.2.3 *Let $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$ be weak* convergent sequences in $L^\infty(\Omega)$ such that $a_\varepsilon(t) \geq c > 0$ a.e. in Ω, and $\eta = \{\eta_t\}_{t\in\Omega}$ is the Young measure generated by the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$. Let u_ε be the solution of*

$$(P_\varepsilon)\ \begin{cases} -\frac{d}{dt}\, a_\varepsilon(t)\, u'_\varepsilon(t) &= \frac{d}{dt}\, b_\varepsilon(t) \quad \text{in } \Omega \\ u_\varepsilon &\in H^1_0(\Omega). \end{cases}$$

Then the sequence $\{u_\varepsilon\}$ converges weakly to the solution u_0 in $H^1_0(\Omega)$ of the homogenized problem

$$(P_\star)\ \begin{cases} -\frac{d}{dt}\, a_0(t)\, u'_0(t) &= \frac{d}{dt}\, g^\star(t) \quad \text{in } \Omega \\ u_0 &\in H^1_0(\Omega), \end{cases}$$

with

$$g^\star(t) \;=\; a_0(t)\, k(t)$$

a.e. in Ω, where the functions a and k are defined in Theorem 6.2.1. If the joint Young measure η is the product of Young measures, then

$$g^\star(t) \;=\; b(t)$$

a.e. in Ω, where b is the weak* limit of $\{b_\varepsilon\}$.

The leading coefficient a_0 and the source term $g^\star$ are defined through the Young measures σ and η, so that $g^\star$ is not necessarily the weak* limit of the sequence $\{b_\varepsilon\}$, because it depends on the oscillatory behaviour of the sequence $\{a_\varepsilon\}$ as well through its associated Young measure. Whenever η is the product of each measure, $g^\star$ is indeed the weak* limit of $\{b_\varepsilon\}$.

Let us see how the homogenized coefficients look like when the sequences $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$ are periodic and oscillate in several separated length scales. So, take a family of separated length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$, ie a family of smooth functions such that $l_i(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$, and

$$\lim_{\varepsilon \searrow 0} \frac{l_{i+1}(\varepsilon)}{l_i(\varepsilon)} \;=\; 0, \quad \text{for every} \quad 1 \leq i \leq N-1.$$

Consider the sequences $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$ defined in Ω by

$$a_\varepsilon(t) \;=\; a\left(t, \left\langle \frac{t}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{t}{l_N(\varepsilon)} \right\rangle\right)$$

and

$$b_\varepsilon(t) \;=\; b\left(t, \left\langle \frac{t}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{t}{l_N(\varepsilon)} \right\rangle\right)$$

where a and b are in $L^\infty(\Omega \times (0,1)^N)$, and $\langle y \rangle$ stands for the fractional part of y.

Proposition 6.2.1 *If u_ε is the solution of*

$$\begin{cases} -\frac{d}{dt}\, a\left(t, \left\langle \frac{t}{l_1(\varepsilon)} \right\rangle, \dots, \left\langle \frac{t}{l_N(\varepsilon)} \right\rangle\right) u_\varepsilon'(t) = \frac{d}{dt}\, b\left(t, \left\langle \frac{t}{l_1(\varepsilon)} \right\rangle, \dots, \left\langle \frac{t}{l_N(\varepsilon)} \right\rangle\right) & \text{in } \Omega \\ u_\varepsilon \in H_0^1(\Omega), \end{cases}$$

then the sequence $\{u_\varepsilon\}$ converges weakly to the solution of $(P_\star)$, where

$$a_0(t) = \left(\int_0^1 \dots \int_0^1 \frac{1}{a(t, y_1, \dots, y_N)}\, dy_1 \dots dy_N \right)^{-1}$$

and

$$g^\star(t) = $$
$$= \left(\int_0^1 \dots \int_0^1 \frac{1}{a(t, y_1, \dots, y_N)}\, dy_1 \dots dy_N \right)^{-1} \int_0^1 \dots \int_0^1 \frac{b(t, y_1, \dots, y_N)}{a(t, y_1, \dots, y_N)}\, dy_1 \dots dy_N,$$

for a.e. $t \in \Omega$.

Proof. It follows from Theorem 6.2.1 that

$$a_0(t) = \left(\int_{\mathbb{R}} \frac{1}{\alpha}\, d\sigma_t(\alpha) \right)^{-1} \quad \text{and} \quad k(t) = \int_{\mathbb{R}^2} \frac{\beta}{\alpha}\, d\eta_t(\alpha, \beta),$$

for a.e. $t \in \Omega$, where $\sigma = \{\sigma_t\}_{t \in \Omega}$ is the Young measure associated with $\{a_\varepsilon\}$, and $\eta = \{\eta_t\}_{t \in \Omega}$ is the Young measure associated with the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$.

For every continuous function $f : \mathbb{R} \to \mathbb{R}$ for which $\{f(a_\varepsilon(\cdot))\}$ is weak convergent in $L^1(\Omega)$, it holds

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \int_\Omega f(a_\varepsilon(t))\, dt &= \int_\Omega \int_0^1 \dots \int_0^1 f(a(t, y_1, \dots, y_N))\, dy_1 \dots dy_N dt \\ &= \int_\Omega \int_{\mathbb{R}} f(\alpha)\ d\sigma_t(\alpha)\, dt \end{aligned}$$

(see Proposition 4.7.1), so that

$$a_0(t) = \left(\int_0^1 \dots \int_0^1 \frac{1}{a(t, y_1, \dots, y_N)}\, dy_1 \dots dy_N \right)^{-1}$$

for a.e. $t \in \Omega$. Moreover, whenever $\{h(a_\varepsilon(\cdot), b_\varepsilon(\cdot))\}$ is weak convergent in $L^1(\Omega)$, it holds

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \int_\Omega h(a_\varepsilon(t), b_\varepsilon(t))\ dt = \\ &= \int_\Omega \int_0^1 \dots \int_0^1 h(a(t, y_1, \dots, y_N), b(t, y_1, \dots, y_N))\ dy_1 \dots dy_N\ dt \\ &= \int_\Omega \int_{\mathbb{R}^2} h(\alpha, \beta)\ d\eta_t(\alpha, \beta)\ dt, \end{aligned}$$

so that we conclude

$$k(t) \;=\; \int_0^1 \cdots \int_0^1 \frac{b(t,y_1,\ldots,y_N)}{a(t,y_1,\ldots,y_N)}\; dy_1 \ldots dy_N,$$

for a.e. $t \in \Omega$. $\square$

Notice that, for any family of separated length scales, the Young measure associated with the sequence

$$\left\{ \left(\left\langle \frac{t}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{t}{l_N(\varepsilon)} \right\rangle \right) \right\}$$

is the Lebesgue measure supported on $(0,1)^N$. Thus the previous result does not depend on the length scales.

On the other hand, we may consider sequences $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillating in distinct length scales, namely

$$a_\varepsilon(t) \;=\; a\left(t, \left\langle \frac{t}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{t}{l_N(\varepsilon)} \right\rangle \right)$$

and

$$b_\varepsilon(t) \;=\; b\left(t, \left\langle \frac{t}{l_{N+1}(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{t}{l_{N+M}(\varepsilon)} \right\rangle \right)$$

where $\{l_1(\varepsilon), \ldots, l_N(\varepsilon), l_{N+1}(\varepsilon), \ldots, l_{N+M}(\varepsilon)\}$ is a family of $N+M$ separated length scales. In this situation the source term $g^\star$ is indeed the weak* limit of $\{b_\varepsilon\}$.

Proposition 6.2.2 *If u_ε is the solution of*

$$\begin{cases} -\frac{d}{dt}\, a\left(t, \left\langle \frac{t}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{t}{l_N(\varepsilon)} \right\rangle \right) u_\varepsilon'(t) \;=\; \frac{d}{dt}\, b\left(t, \left\langle \frac{t}{l_{N+1}(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{t}{l_{N+M}(\varepsilon)} \right\rangle \right) \\ u_\varepsilon \;\in\; H_0^1(\Omega), \end{cases}$$

then the sequence $\{u_\varepsilon\}$ converges weakly to the solution of $(P_\star)$, where

$$a_0(t) = \left(\int_0^1 \cdots \int_0^1 \frac{1}{a(t,y_1,\ldots,y_N)}\; dy_1 \ldots dy_N \right)^{-1}$$

and

$$g^\star(t) \;=\; \int_0^1 \cdots \int_0^1 b(t,y_{N+1},\ldots,y_{N+M})\; dy_{N+1} \ldots dy_{N+M},$$

for a.e. $t \in \Omega$.

6.2.1. Examples

The full characterization of effective coefficients coming from the homogenization of the Dirichlet problem

$$\begin{cases} -\frac{d}{dt}\, a_\varepsilon(t)\, u_\varepsilon'(t) & = & \frac{d}{dt}\, b_\varepsilon(t) \qquad \text{in } \Omega \\ u_\varepsilon & \in & H_0^1(\Omega), \end{cases}$$

is obtained by means of the joint Young measure associated with such sequence of pairs, according with Theorem 6.2.1. In this section we present simple examples, in the non-periodic and periodic settings.

1. In particular, we may consider a sequence of second order derivatives, namely

$$a_\varepsilon(t) \;=\; v_\varepsilon''(t)$$

for any weak* convergent sequence of strictly convex functions $v_\varepsilon \in W^{2,\infty}(\Omega)$.

2. In the case of both sequences oscillate at the same length scale $l(\varepsilon)$, ie

$$a_\varepsilon(t) \;=\; a\left(\left\langle \frac{t}{l(\varepsilon)} \right\rangle\right) \quad \text{and} \quad b_\varepsilon(t) \;=\; b\left(t, \left\langle \frac{t}{l(\varepsilon)} \right\rangle\right),$$

we conclude that the leading coefficient

$$a_0 \;=\; \left(\int_0^1 \frac{1}{a(y)}\, dy\right)^{-1}$$

is constant, and the source term $g^\star : (0,1) \to \mathbb{R}$ is given by

$$g^\star(t) \;=\; a_0\, k(t) = \left(\int_0^1 \frac{1}{a(y)}\, dy\right)^{-1} \int_0^1 \frac{b(t,y)}{a(y)}\, dy.$$

Clearly the source term depends on both sequences $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$, because the oscillations take place at the same scale. When the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$ is a laminate given by

$$(a_\varepsilon(t), b_\varepsilon(t)) \;=\; (a_1, b_1)\, \chi_{(0,s(t))}\left(\left\langle \frac{t}{\varepsilon} \right\rangle\right) + (a_2, b_2)\left(1 - \chi_{(0,s(t))}\left(\left\langle \frac{t}{\varepsilon} \right\rangle\right)\right),$$

with $s \in (0,1)$, the effective coefficient is

$$a_0(t) \;=\; \frac{a_1\, a_2}{s(t)\, a_2 + (1 - s(t))a_1},$$

and the source term is

$$g^\star(t) \;=\; \frac{s(t)\, a_2\, b_1 + (1 - s(t))\, a_1\, b_2}{s(t)\, a_2 + (1 - s(t))\, a_1}.$$

3. When the sequences oscillate in different length scales, namely

$$a_\varepsilon(t) \;=\; a\left(\left\langle \frac{t}{\varepsilon}\right\rangle\right) \quad \text{and} \quad b_\varepsilon(t) \;=\; b\left(t, \left\langle \frac{t}{\sqrt{\varepsilon}}\right\rangle\right),$$

the leading coefficient is the same as before

$$a_0 \;=\; \left(\int_0^1 \frac{1}{a(y)}\, dy\right)^{-1},$$

while now the source term is indeed the weak* limit of $\{b_\varepsilon\}$, ie

$$g^\star(t) \;=\; \int_0^1 b(t,y)\, dy.$$

Notice that the source term does not depend on whether the sequence $\{b_\varepsilon\}$ oscillates at a length scale slower (or faster) than the sequence $\{a_\varepsilon\}$. In the case of laminates

$$a_\varepsilon\,(t) \;=\; a_1\, \chi_{(0,r)}\left(\left\langle \frac{t}{\varepsilon}\right\rangle\right) + a_2\left(1 - \chi_{(0,r)}\left(\left\langle \frac{t}{\varepsilon}\right\rangle\right)\right),$$

and

$$b_\varepsilon\,(t) \;=\; b_1\, \chi_{(0,s(t))}\left(\left\langle \frac{t}{\sqrt{\varepsilon}}\right\rangle\right) + b_2\left(1 - \chi_{(0,s(t))}\left(\left\langle \frac{t}{\sqrt{\varepsilon}}\right\rangle\right)\right),$$

we get

$$g^\star(t) \;=\; s(t)b_1 + (1-s(t))b_2.$$

4. When the sequence $\{b_\varepsilon\}$ oscillates at two separated length scales, while $\{a_\varepsilon\}$ oscilates only at the faster one, the source term reproduces the oscillatory behaviour of $\{a_\varepsilon\}$, besides the one of $\{b_\varepsilon\}$. Namely, if

$$a_\varepsilon(t) \;=\; a\left(\left\langle \frac{t}{\varepsilon}\right\rangle\right) \quad \text{and} \quad b_\varepsilon(t) \;=\; b\left(t, \left\langle \frac{t}{\sqrt{\varepsilon}}\right\rangle, \left\langle \frac{t}{\varepsilon}\right\rangle\right),$$

then

$$g^\star(t) \;=\; a_0 \int_0^1\!\!\int_0^1 \frac{b(t,y_1,y_2)}{a(y_2)}\, dy_1\, dy_2.$$

Here the leading coefficient is the same as in the previous example.

6.3. The periodic n-dimensional case: same scales

In the previous section we have concluded that, in one dimension, the homogenized source term $g^\star$ depends on the relationship between the separated length scales, where the oscillations, of the leading coefficient and the source terms, take place.

In this section the homogenization of multi-scale problems, in higher dimensions, is described in order to characterize the homogenized source term $g^\star$, when there is an interaction between the oscillatory behaviour of the sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$. Namely, we study the case when $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillate at the same length scale $l(\varepsilon)$. More general, the case of both sequences oscillating at the same family of separated length scales $\{l_1(\varepsilon), \ldots, l_N(\varepsilon)\}$ is also treated.

6.3.1. One scale

Consider the Dirichlet problem

$$(P_\varepsilon) \quad \begin{cases} -\operatorname{div} A\left(x, \left\langle \frac{x}{l(\varepsilon)} \right\rangle\right) \nabla u_\varepsilon(x) &= \operatorname{div} b\left(x, \left\langle \frac{x}{l(\varepsilon)} \right\rangle\right) \quad \text{in } \Omega \\ \qquad\qquad u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

with $A = [a_{ij}] \in [L^\infty(\Omega \times Q)]^{n \times n}$ symmetric such that, there exist $0 < \alpha \leq \beta$, $\alpha|\rho|^2 \leq \rho^T A \rho \leq \beta|\rho|^2$, for every $\rho \in \mathbb{R}^n$, and $b \in [L^\infty(\Omega \times Q)]^n$. Here any length scales $l(\varepsilon)$ may be considered so that the oscillatory behaviour of both coefficients is the same. The characterization of the effective coefficients may be deduced from the explicit characterization of the density of the Γ-limit of the sequence of associated functionals

$$I_\varepsilon(u) = \int_\Omega \left[\nabla u(x)^T \frac{A\left(x, \left\langle \frac{x}{l(\varepsilon)} \right\rangle\right)}{2} \nabla u(x) + b\left(x, \left\langle \frac{x}{l(\varepsilon)} \right\rangle\right) \cdot \nabla u(x) \right] dx.$$

Theorem 6.3.1 *The sequence $\{I_\varepsilon\}$ is Γ-convergent, with respect to the weak topology of $H_0^1(\Omega)$, to*

$$I(u) = \int_\Omega \psi(x, \nabla u(x))\, dx,$$

where $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi(x, \rho) = \rho^T \frac{A_0(x)}{2} \rho + g^\star(x) \cdot \rho + c(x).$$

The matrix function $A_0 : \Omega \to \mathbb{R}^{n \times n}$ is defined by

$$A_0(x) = \int_Q (I_n + [\nabla_y w_j(x,y)])^T A(x,y) (I_n + [\nabla_y w_j(x,y)])\, dy, \tag{6.4}$$

the field $g^\star : \Omega \to \mathbb{R}^n$ by

$$g^\star(x) = \int_Q (I_n + [\nabla_y w_j(x,y)])^T [\, A(x,y) \nabla_y z(x,y) + b(x,y) \,]\, dy,$$

and $c : \Omega \to \mathbb{R}$ by

$$c(x) = \int_Q \left[\nabla_y z(x,y)^T \frac{A(x,y)}{2} \nabla_y z(x,y) + b(x,y) \nabla_y z(x,y) \right] dy,$$

where I_n is the $n \times n$-identity matrix. The function $w_j(x,\cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_y A(x,y)(e_j + \nabla_y w_j(x,y)) = 0 & in \;\; Q, \\ w_j(x,\cdot) \in H^1_{per}(Q), \end{cases}$$

for every $1 \le j \le n$, for some basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$, and $z(x,\cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_y A(x,y)\nabla_y z(x,y) = \operatorname{div} b(x,y) & in \;\; Q, \\ z(x,\cdot) \in H^1_{per}(Q), \end{cases}$$

Notice that if we multiply the previous matrix function A_0 by any vector $\rho \in \mathbb{R}^n$, then we recover the well known expression for the vector $A_0(x)\rho$ given by (3.1) in Chapter 3.

Therefore the next corollary is an immediate consequence of the explicit characterization of the density of the Γ-limit of the sequence of functionals I_ε.

Corollary 6.3.2 *If u_ε is the solution of (P_ε), then the sequence $\{u_\varepsilon\}$ is weak convergent to the solution u_0 of the homogenized problem*

$$\begin{cases} -\operatorname{div} A_0(x)\nabla u_0(x) = \operatorname{div} g^\star(x) & in \;\; \Omega, \\ u_0 \in H^1_0(\Omega). \end{cases}$$

It follows that the homogenized source term $g^\star$ depends on the behaviour of the leading coefficient A_ε besides the source term b_ε, whenever both sequences oscillate at the same length scales.

Proof. (of Theorem 6.3.1) It follows from Theorem 4.7.2 that the sequence of functionals I_ε is Γ-convergent to

$$I(u) = \int_\Omega \psi(x, \nabla u(x))\, dx,$$

where $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi(x,\rho) =$$
$$= \inf_{v\in\Psi} \int_Q \left[(\rho + \nabla_y v(x,y))^T \frac{A(x,y)}{2} (\rho + \nabla_y v(x,y)) + b(x,y)\cdot(\rho + \nabla_y v(x,y)) \right] dy,$$

with

$$\Psi = L^2[\Omega; H^1_{per}(Q)].$$

The minimizer $v^\rho \in \Psi$ is the solution of

$$\begin{cases} -\operatorname{div}_y A(x,y)(\rho + \nabla_y v^\rho(x,y)) = \operatorname{div}_y b(x,y) & \text{in } \;\; Q \\ v^\rho(x,\cdot) \in H^1_{per}(Q), \end{cases}$$

which may be written as

$$v^\rho(x,y) \;=\; \sum_{j=1}^{n} w_j(x,y)\rho_j + z(x,y),$$

according with Lemma 6.3.5, where $w_j(x,\cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_y A\,(x,y)\,(e_j + \nabla_y w_j(x,y)) \;=\; 0 \quad \text{in } Q \\ w_j(x,\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

for every $1 \le j \le n$, and some basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$, and $z(x,\cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_y A\,(x,y)\,\nabla_y z(x,y) \;=\; \operatorname{div}_y b\,(x,y) \quad \text{in } Q \\ z(x,\cdot) \;\in\; H^1_{per}(Q). \end{cases}$$

If we replace the expression of $\nabla_y v^\rho(x,y)$ in $\psi(x,\rho)$, we obtain that

$$\psi(x,\rho) \;=\; \int_Q \Big[((I_n + [\nabla_y w_j(x,y)])\,\rho + \nabla_y z(x,y)\;)^T\; \frac{A(x,y)}{2}$$
$$((I_n + [\nabla_y w_j(x,y)])\,\rho + \nabla_y z(x,y)\,) + b(x,y)\cdot(\,(I_n + [\nabla_y w_j(x,y)])\,\rho + \nabla_y z(x,y))\,\Big]\,dy,$$

where I_n is the $n\times n$-identity matrix and $[\nabla_y w_j(x,y)]$ is the $n\times n$-matrix whose columns are the vectors $\nabla_y w_j(x,y)$, with $1 \le j \le n$. Then the claim is achieved, after some simplifications. □

6.3.2. Multi-scales

In this section we will generalize the previous results to a family of N separated length scales. So, consider the multi-scale problem

$$(P^N_\varepsilon)\;\begin{cases} -\operatorname{div} A\left(x, \left\langle \frac{x}{l_1(\varepsilon)}\right\rangle,\dots,\left\langle \frac{x}{l_N(\varepsilon)}\right\rangle\right)\nabla u_\varepsilon(x) = \operatorname{div} b\left(x, \left\langle \frac{x}{l_1(\varepsilon)}\right\rangle,\dots,\left\langle \frac{x}{l_N(\varepsilon)}\right\rangle\right) \\ u_\varepsilon \in H^1_0(\Omega), \end{cases}$$

with $A = [a_{ij}] \in [L^\infty(\Omega\times Q^N)]^{n\times n}$ symmetric, satisfying $\alpha|\rho|^2 \le \rho^T A\rho \le \beta|\rho|^2$, for some $0<\alpha\le\beta$, for every $\rho\in\mathbb{R}^n$, and $b \in [L^\infty(\Omega\times Q^N)]^n$. Let $\{l_1(\varepsilon),\dots,l_N(\varepsilon)\}$ be any family of separated length scales.

Theorem 6.3.3 *Let $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ be the previous sequences. Then the sequence of functionals*

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\; \nabla u(x)^T\,\frac{A_\varepsilon(x)}{2}\,\nabla u(x) \;+\; b_\varepsilon(x)\cdot\nabla u(x)\;\right]\,dx$$

is Γ-convergent to

$$I(u) \;=\; \int_\Omega \psi(x,\nabla u(x))\;dx,$$

where $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *is given by*

$$\psi(x,\rho) \;=\; \rho^T \, \frac{A_0(x)}{2} \, \rho \;+\; g^\star(x)\cdot\rho \;+\; c(x)$$

with

$$A_0(x) \;=\; \int_{Q^N} \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}(x,y_1,\ldots,y_{N-k}) \right] \right) \right)^T A(x,y_1,\ldots,y_N)$$
$$\left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}(x,y_1,\ldots,y_{N-k}) \right] \right) \right) dy_1 \ldots dy_N,$$

$$g^\star(x) = \int_{Q^N} \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}(x,y_1,\ldots,y_{N-k}) \right] \right) \right)^T A(x,y_1,\ldots,y_N)$$
$$\left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)}(x,y_1,\ldots,y_{N-(i-1)}) \right] \right) \nabla_{y_{N-k}} z_{N-k}(x,y_1,\ldots,y_{N-k}) \right.$$
$$\left. +\; \nabla_{y_N} z_N(x,y_1,\ldots,y_N) \,\right) \; dy_1 \ldots dy_N \;+$$
$$+ \int_{Q^N} b(x,y_1,\ldots,y_N) \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}(x,y_1,\ldots,y_{N-k}) \right] \right) \right) \; dy_1 \ldots dy_N,$$

and

$$c(x) = \int_{Q^N} \left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \nabla_{y_{N-k}} z_{N-k} \;+ \right.$$
$$\left. +\; \nabla_{y_N} z_N \,\right)^T \; \frac{A(x,y_1,\ldots,y_N)}{2}$$
$$\left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \nabla_{y_{N-k}} z_{N-k} + \nabla_{y_N} z_N \right) dy_1 \ldots dy_N \;+$$
$$+ \; \int_{Q^N} b(x,y_1,\ldots,y_N) \left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \right.$$
$$\left. \nabla_{y_{N-k}} z_{N-k} \;+\; \nabla_{y_N} z_N \,\right) \; dy_1 \ldots dy_N.$$

For $0 \le k \le N-1$*, the function* $w_j^{N-k}(x,y_1,\ldots,y_{N-k-1},\cdot)$ *is the solution of*

$$\begin{cases} -\,\mathrm{div}_{y_{N-k}} \; A^\star_{N-k}\,(x,y_1,\ldots,y_{N-k}) \left(e_j + \nabla_{y_{N-k}} w_j^{N-k} \right) \;=\; 0 \qquad in \;\; Q \\ \qquad\qquad\qquad\qquad w_j^{N-k}(x,y_1,\ldots,y_{N-k-1},\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

for every $1 \le j \le n$*, and* $z_{N-k}(x,y_1,\ldots,y_{N-k-1},\cdot)$ *is the solution of*

$$\begin{cases} -\,\mathrm{div}_{y_{N-k}} \; A^\star_{N-k}\,(x,y_1,\ldots,y_{N-k})\,\nabla_{y_{N-k}} z_{N-k} \;=\; \mathrm{div}\; g^\star_{N-k}\,(x,y_1,\ldots,y_{N-k}) \\ \qquad\qquad\qquad z_{N-k}(x,y_1,\ldots,y_{N-k-1},\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

with $A^\star_N(x, y_1, \ldots, y_N) = A(x, y_1, \ldots, y_N)$ *when* $k = 0$*, and*

$$A^\star_{N-k}(x, y_1, \ldots, y_{N-k}) = \int_{Q^k} \left(\prod_{i=0}^{k-1} \left(I_n + \left[\nabla_{y_{N-i}} w_j^{N-i} \right] \right) \right)^T A(x, y_1, \ldots, y_N) \left(\prod_{i=0}^{k-1} \left(I_n + \left[\nabla_{y_{N-i}} w_j^{N-i} \right] \right) \right) dy_{N-k+1} \ldots dy_N,$$

for $1 \leq k \leq N - 1$*, where* I_n *is the* $n \times n$*-identity matrix,* $\left[\nabla_{y_{N-i}} w_j^{N-i}(x, y_1, \ldots, y_{N-i}) \right]$ *is the* $n \times n$*-matrix whose columns are the vectors* $\nabla_{y_{N-i}} w_j^{N-i}(x, y_1, \ldots, y_{N-i})$*,* $1 \leq j \leq n$*, and* $g^\star_N(x, y_1, \ldots, y_N) = b(x, y_1, \ldots, y_N)$ *when* $k = 0$*, and otherwise*

$$g^\star_{N-k}(x, y_1, \ldots, y_{N-k}) = \int_{Q^k} \left(\prod_{i=0}^{k-1} \left(I_n + \left[\nabla_{y_{N-i}} w_j^{N-i} \right] \right) \right)^T A(x, y_1, \ldots, y_N) \left(\sum_{h=1}^{k-1} \prod_{i=1}^{h} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \nabla_{y_{N-h}} z_{N-h} + \nabla_{y_N} z_N \right) dy_{N-k+1} \ldots dy_N + \int_{Q^k} b(x, y_1, \ldots, y_N) \left(\prod_{i=0}^{k-1} \left(I_n + \left[\nabla_{y_{N-i}} w_j^{N-i} \right] \right) \right) dy_{N-k+1} \ldots dy_N.$$

In this way we get the characterization of the effective coefficients associated with the multi-scale problem (P^N_ε).

Corollary 6.3.4 *If* u_ε *is the solution of* (P^N_ε)*, then the sequence* $\{u_\varepsilon\}$ *converges weakly to the solution of*

$$\begin{cases} -\operatorname{div} A_0(x) \nabla u_0(x) = \operatorname{div} g^\star(x) & \text{in } \Omega \\ u_0 \in H^1_0(\Omega), \end{cases}$$

with A_0 *and* $g^\star$ *defined in the previous theorem.*

It follows that the homogenized source term $g^\star$ depends on the behaviour of the leading coefficient A_ε, whenever both sequences oscillate at the same family of separated length scales.

Proof. (of Theorem 6.3.3) Applying Theorem 4.7.2, the sequence of functionals I_ε is Γ-convergent to

$$I(u) = \int_\Omega \psi(x, \nabla u(x))\, dx,$$

where $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi(x,\rho) = \inf_{\substack{v_i \in \Psi_i \\ 1\le i\le N}} \int_{Q^N} \left[\left(\rho + \sum_{i=1}^{N} \nabla_{y_i} v_i\right)^T \frac{A(x,y_1,\dots,y_N)}{2} \left(\rho + \sum_{i=1}^{N} \nabla_{y_i} v_i\right) + b(x,y_1,\dots,y_N)\cdot \left(\rho + \sum_{i=1}^{N} \nabla_{y_i} v_i\right)\right] dy_1 \dots dy_N,$$

with

$$\Psi_i \;=\; L^2[\Omega \times Q^{i-1}; H^1_{per}(Q)], \quad \text{for every} \quad 1 \le i \le N.$$

Fix $v_i \in \Psi_i$, for every $1 \le i \le N-1$. Then the minimizer $v_N^\rho \in \Psi_N$ is the solution of

$$\begin{cases} -\mathrm{div}\, A\left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i(x,y_1,\cdot,y_i) + \nabla_{y_N} v_N^\rho(x,y_1,\cdot,y_N)\right) = \mathrm{div}\, b\,(x,y_1,\cdot,y_N) \\ v_N^\rho(x,y_1,\dots,y_{N-1},\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

which may be written as

$$\begin{aligned} & v_N^\rho(x,y_1,\dots,y_N) \;= \\ &= \sum_{j=1}^{d} \; w_j^N(x,y_1,\dots,y_N)\left(\rho_j + \sum_{i=1}^{N-1} \frac{\partial v_i}{\partial y_i^j}(x,y_1,\dots,y_i)\right) \;+\; z_N(x,y_1,\dots,y_N), \end{aligned}$$

where $w_j^N(x,y_1,\dots,y_{N-1},\cdot)$ is the solution of

$$\begin{cases} -\mathrm{div}_{y_N}\; A\,(x,y_1,\dots,y_N)\left(e_j + \nabla_{y_N} w_j^N(x,y_1,\dots,y_N)\right) \;=\; 0 \qquad \text{in } \; Q \\ w_j^N(x,y_1,\dots,y_{N-1},\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

for every $1 \le j \le d$, and some basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$, and $z_N(x,y_1,\dots,y_{N-1},\cdot)$ solves

$$\begin{cases} -\mathrm{div}_{y_N}\; A\,(x,y_1,\dots,y_N)\,\nabla_{y_N} z_N(x,y_1,\dots,y_N) \;=\; \mathrm{div}_{y_N}\; b\,(x,y_1,\dots,y_N) \;\text{ in } Q \\ z_N(x,y_1,\dots,y_{N-1},\cdot) \;\in\; H^1_{per}(Q). \end{cases}$$

If we replace the expression of $\nabla_{y_N} v_N^\rho$ in $\psi(x,\rho)$, we obtain that

$$\psi(x,\rho) =$$
$$= \inf_{\substack{v_i\in\Psi_i\\1\le i\le N-1}} \int_{Q^N} \Bigg[\Bigg(\left(I_n + \left[\nabla_{y_N} w_j^N(x,y_1,\dots,y_N)\right]\right) \left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i(x,y_1,\dots,y_i)\right)$$
$$+ \nabla_{y_N} z_N(x,y_1,\dots,y_N) \Bigg)^T \frac{A(x,y_1,\dots,y_N)}{2}$$
$$\Bigg(\left(I_n + \left[\nabla_{y_N} w_j^N(x,y_1,\dots,y_N)\right]\right) \left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i(x,y_1,\dots,y_i)\right)$$
$$+ \nabla_{y_N} z_N(x,y_1,\dots,y_N) \Bigg)$$
$$+ b(x,y_1,\dots,y_N)\cdot \Bigg(\left(I_n + \left[\nabla_{y_N} w_j^N(x,y_1,\dots,y_N)\right]\right) \left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i(x,y_1,\dots,y_i)\right)$$
$$+ \nabla_{y_N} z_N(x,y_1,\dots,y_N) \Bigg) \Bigg]\, dy_1\dots dy_N,$$

where I_n is the $n\times n-$identity matrix and $\left[\nabla_{y_N} w_j^N(x,y_1,\dots,y_N)\right]$ is the $n\times n-$matrix whose columns are the vectors $\nabla_{y_N} w_j^N(x,y_1,\dots,y_N)$, with $1\le j\le n$.

Now, fixed $v_i\in\Psi_i$, for every $1\le i\le N-2$, the minimizer $v_{N-1}^\rho\in\Psi_{N-1}$ is the solution of

$$\begin{cases} -\mathrm{div} A_{N-1}^\star \left(\rho + \sum_{i=1}^{N-2} \nabla_{y_i} v_i(x,y_1,\cdot,y_i) + \nabla_{y_{N-1}} v_{N-1}^\rho(x,y_1,\cdot,y_{N-1})\right) = \mathrm{div}\, g_{N-1}^\star \\ v_{N-1}^\rho(x,y_1,\dots,y_{N-2},\cdot) \in H^1_{per}(Q), \end{cases}$$

with $A_{N-1}^\star = A_{N-1}^\star(x,y_1,\dots,y_{N-1})$ given by

$$A_{N-1}^\star = \int_Q \left(I_n + \left[\nabla_{y_N} w_j^N(x,y_1,\dots,y_N)\right]\right)^T A \left(I_n + \left[\nabla_{y_N} w_j^N(x,y_1,\dots,y_N)\right]\right) dy_N,$$

and $g_{N-1}^\star = g_{N-1}^\star(x,y_1,\dots,y_{N-1})$ is defined by

$$g_{N-1}^\star = \int_Q \left(I_d + \left[\nabla_{y_N} w_j^N(x,y_1,\dots,y_N)\right]\right)^T \big(A(x,y_1,\dots,y_N)\nabla_{y_N} z_N(x,y_1,\dots,y_N)$$
$$+ b(x,y_1,\dots,y_N) \big)\, dy_N.$$

We may write $v_{N-1}^\rho\in\Psi_{N-1}$ as

$$v_{N-1}^\rho(x,y_1,\dots,y_{N-1}) =$$
$$= \sum_{j=1}^n w_j^{N-1}(x,y_1,\dots,y_{N-1}) \left(\rho_j + \sum_{i=1}^{N-2} \frac{\partial v_i}{\partial y_i^j}(x,y_1,\dots,y_i)\right) + z_{N-1}(x,y_1,\dots,y_{N-1}),$$

where $w_j^{N-1}(x,y_1,\dots,y_{N-2},\cdot)$ is the solution of

$$\begin{cases} -\mathrm{div}_{y_{N-1}} A_{N-1}^\star(x,y_1,\dots,y_{N-1}) \left(e_j + \nabla_{y_{N-1}} w_j^{N-1}(x,y_1,\dots,y_{N-1})\right) = 0 \\ w_j^{N-1}(x,y_1,\dots,y_{N-2},\cdot) \in H^1_{per}(Q), \end{cases}$$

for every $1 \leq j \leq n$, and $z_{N-1}(x, y_1, \ldots, y_{N-2}, \cdot)$ solves

$$\begin{cases} -\mathrm{div}_{y_{N-1}} A^{\star}_{N-1}(x, y_1, \ldots, y_{N-1}) \nabla_{y_{N-1}} z_{N-1} = \mathrm{div}_{y_{N-1}} g^{\star}_{N-1}(x, y_1, \ldots, y_{N-1}) \\ z_{N-1}(x, y_1, \ldots, y_{N-2}, \cdot) \in H^1_{per}(Q). \end{cases}$$

Thus we get

$$\begin{aligned} \psi(x,\rho) = \inf_{\substack{v_i \in \Psi_i \\ 1 \leq i \leq N-2}} \int_{Q^N} \Bigg[\Bigg(& \left(I_n + \left[\nabla_{y_N} w_j^N\right]\right) \left(\left(I_n + \left[\nabla_{y_{N-1}} w_j^{N-1}\right]\right) \right. \\ & \left. \left(\rho + \sum_{i=1}^{N-2} \nabla_{y_i} v_i\right) + \nabla_{y_{N-1}} z_{N-1} \right) + \nabla_{y_N} z_N \Bigg)^T \\ & \frac{A(x, y_1, \ldots, y_N)}{2} \left(\left(I_n + \left[\nabla_{y_N} w_j^N\right]\right) \left(\left(I_n + \left[\nabla_{y_{N-1}} w_j^{N-1}\right]\right) \right.\right. \\ & \left.\left. \left(\rho + \sum_{i=1}^{N-2} \nabla_{y_i} v_i\right) + \nabla_{y_{N-1}} z_{N-1} \right) + \nabla_{y_N} z_N \right) \\ & + b(x, y_1, \ldots, y_N) \cdot \left(\left(I_n + \left[\nabla_{y_N} w_j^N\right]\right) \left(\left(I_n + \left[\nabla_{y_{N-1}} w_j^{N-1}\right]\right) \right.\right. \\ & \left.\left. \left(\rho + \sum_{i=1}^{N-2} \nabla_{y_i} v_i\right) + \nabla_{y_{N-1}} z_{N-1} \right) + \nabla_{y_N} z_N \right) \Bigg] \, dy_1 \ldots dy_N. \end{aligned}$$

After some iterations, we conclude that

$$\begin{aligned} \psi(x,\rho) = \int_{Q^N} \Bigg[\Bigg(& \prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}\right]\right) \rho + \\ & + \sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)}\right]\right) \nabla_{y_{N-k}} z_{N-k} + \\ & + \nabla_{y_N} z_N \Bigg)^T \frac{A(x, y_1, \ldots, y_N)}{2} \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}\right]\right) \rho + \right. \\ & + \sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)}\right]\right) \nabla_{y_{N-k}} z_{N-k} + \\ & \left. + \nabla_{y_N} z_N \right) + b(x, y_1, \ldots, y_N) \cdot \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}\right]\right) \rho + \right. \\ & \left. + \sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)}\right]\right) \nabla_{y_{N-k}} z_{N-k} + \nabla_{y_N} z_N \right) \Bigg] \, dy_1 \ldots dy_N. \end{aligned}$$

More explicitly,

$$\psi(x,\rho) = \rho^T \frac{A_0(x)}{2} \rho + g^{\star}(x) \cdot \rho + c(x)$$

with

$$A_0(x) = \int_{Q^N} \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}(x, y_1, \ldots, y_{N-k}) \right] \right) \right)^T A(x, y_1, \ldots, y_N)$$
$$\left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k}(x, y_1, \ldots, y_{N-k}) \right] \right) \right) dy_1 \ldots dy_N,$$

$$g^\star(x) = \int_{Q^N} \left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \nabla_{y_{N-k}} z_{N-k} + \right.$$
$$\left. + \nabla_{y_N} z_N \right)^T \frac{A(x, y_1, \ldots, y_N)}{2} \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k} \right] \right) \right) dy_1 \ldots dy_N +$$
$$+ \int_{Q^N} \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k} \right] \right) \right)^T \frac{A(x, y_1, \ldots, y_N)}{2}$$
$$\left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \nabla_{y_{N-k}} z_{N-k} + \nabla_{y_N} z_N \right) dy_1 \ldots dy_N +$$
$$+ \int_{Q^N} b(x, y_1, \ldots, y_N) \left(\prod_{k=0}^{N-1} \left(I_n + \left[\nabla_{y_{N-k}} w_j^{N-k} \right] \right) \right) dy_1 \ldots dy_N.$$

and

$$c(x) = \int_{Q^N} \left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \nabla_{y_{N-k}} z_{N-k} + \nabla_{y_N} z_N \right)^T \frac{A}{2}$$
$$\left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \nabla_{y_{N-k}} z_{N-k} + \nabla_{y_N} z_N \right) dy_1 \ldots dy_N +$$
$$+ \int_{Q^N} b \left(\sum_{k=1}^{N-1} \prod_{i=1}^{k} \left(I_n + \left[\nabla_{y_{N-(i-1)}} w_j^{N-(i-1)} \right] \right) \nabla_{y_{N-k}} z_{N-k} + \nabla_{y_N} z_N \right) dy_1 \ldots dy_N.$$

Then the claim is achieved. □

The following lemma is an auxiliary result used in the previous proof.

Lemma 6.3.5 *For a.e.* $(x, y_1, \ldots, y_{N-1}) \in \Omega \times Q^{N-1}$ *and any* $\rho \in \mathbb{R}^n$*, if* $v_N^\rho(x, y_1, \ldots, y_{N-1}, \cdot)$ *is the solution of the cell problem*

$$\begin{cases} -\mathrm{div}_{y_N} A(x, y_1, \ldots, y_N) \left(\rho + \sum_{j=1}^{N} \nabla_{y_j} v_j(x, y_1, \ldots, y_j) \right) = \mathrm{div}_{y_N} \, b(x, y_1, \ldots, y_N) \\ v_N(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q), \end{cases}$$

then

$$v_N^\rho(x, y_1, \ldots, y_N) =$$
$$= \sum_{i=1}^{n} \left(\rho_i + \sum_{j=1}^{N-1} \frac{\partial v_j}{\partial y_j^i}(x, y_1, \ldots, y_j) \right) w_i(x, y_1, \ldots, y_N) + z(x, y_1, \ldots, y_N),$$

where $w_i \in L^\infty \left[\Omega \times Q^{N-1}; H^1_{per}(Q)\right]$ *is the solution of*

$$\begin{cases} -\operatorname{div}_{y_N} A(x, y_1, \ldots, y_N)(e_i + \nabla_{y_N} w_i(x, y_1, \ldots, y_N)) = 0 \quad \text{in } Q \\ w_i(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q), \end{cases}$$

for every $1 \le i \le n$, *and* $z \in L^\infty \left[\Omega \times Q^{N-1}; H^1_{per}(Q)\right]$ *is solution of*

$$\begin{cases} -\operatorname{div}_{y_N} A(x, y_1, \ldots, y_N) \nabla_{y_N} z(x, y_1, \ldots, y_N) = \operatorname{div}_{y_N} b(x, y_1, \ldots, y_N) \quad \text{in } Q \\ z(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q). \end{cases}$$

Proof. Let $\overline{v}_N^\rho(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q)$ be the solution of

$$\begin{cases} -\operatorname{div}_{y_N} A(x, y_1, \ldots, y_N)\left(\rho + \sum \nabla_{y_j} v_j(x, y_1, \ldots, y_j) + \nabla_{y_N} \overline{v}_N(x, y_1, \ldots, y_N)\right) = 0 \\ \overline{v}_N(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q). \end{cases}$$

Writing ρ and $\nabla_{y_j} v_j$, for every $1 \le j \le N-1$, using a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ as

$$\begin{aligned} \rho + \sum_{j=1}^{N-1} \nabla_{y_j} v_j(x, y_1, \ldots, y_j) &= \sum_{i=1}^{n} \left(\rho_i + \sum_{j=1}^{N-1} \frac{\partial v_j}{\partial y_j^i}(x, y_1, \ldots, y_j) \right) e_i \\ \nabla_{y_N} \overline{v}_N^\rho(x, y_1, \ldots, y_N) &= \sum_{i=1}^{n} \frac{\partial \overline{v}_N^\rho}{\partial y_N^i}(x, y_1, \ldots, y_N)\, e_i, \end{aligned}$$

and then replacing it in the equation, we get

$$-\operatorname{div} A(x, y_1, \ldots, y_N) \sum_{i=1}^{n} \left(\rho_i + \sum_{j=1}^{N-1} \frac{\partial v_j}{\partial y_j^i}(x, y_1, \ldots, y_j) + \frac{\partial \overline{v}_N^\rho}{\partial y_N^i}(x, y_1, \ldots, y_N) \right) e_i = 0.$$

Since the divergence operator is linear, the following is 0

$$-\operatorname{div}_{y_N} A(x, y_1, \ldots, y_N) \left[\left(\rho_i + \sum_{j=1}^{N-1} \frac{\partial v_j}{\partial y_j^i}(x, y_1, \ldots, y_j) \right) e_i + \frac{\partial \overline{v}_N^\rho}{\partial y_N^i}(x, y_1, \ldots, y_N) e_i \right]$$

and

$$-\sum_{i=1}^{n} \operatorname{div}_{y_N} A(x, y_1, \ldots, y_N) \left(e_i + \frac{1}{\overline{\rho}_i} \frac{\partial \overline{v}_N^\rho}{\partial y_N^i}(x, y_1, \ldots, y_N)\, e_i \right) = 0,$$

with

$$\overline{\rho}_i \;=\; \rho_i + \sum_{j=1}^{N-1} \frac{\partial v_j}{\partial y_j^i}(x, y_1, \ldots, y_j) \;\neq\; 0.$$

For every $1 \leq i \leq n$, let $w_i(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q)$ be the solution of

$$\begin{cases} -\operatorname{div}_{y_N} A(x, y_1, \ldots, y_N)\,(e_i + \nabla_{y_N} w_i(x, y_1, \ldots, y_N)) = 0 \quad \text{in } Q \\ w_i(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q). \end{cases}$$

Then

$$\frac{1}{\overline{\rho}_i}\,\frac{\partial \overline{v}_N^\rho}{\partial y_N^i}(x, y_1, \ldots, y_N)\, e_i \;=\; \nabla_{y_N} w_i(x, y_1, \ldots, y_N)$$

so that

$$\begin{aligned} \nabla_{y_N} \overline{v}_N^\rho(x, y_1, \ldots, y_N) &= \sum_{i=1}^{n} \overline{\rho}_i\; \nabla_{y_N} w_i(x, y_1, \ldots, y_N) = \\ &= \sum_{i=1}^{n} \left(\rho_i + \sum_{j=1}^{N-1} \frac{\partial v_j}{\partial y_j^i}(x, y_1, \ldots, y_j) \right) \nabla_{y_N} w_i(x, y_1, \ldots, y_N). \end{aligned}$$

Moreover, if $z(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q)$ is solution of

$$\begin{cases} -\operatorname{div}_{y_N} A(x, y_1, \ldots, y_N)\, \nabla_{y_N} z(x, y_1, \ldots, y_N) = \operatorname{div}_{y_N} b(x, y_1, \ldots, y_N) \quad \text{in } Q \\ z(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q), \end{cases}$$

we conclude that the solution $v_N^\rho(x, y_1, \ldots, y_{N-1}, \cdot)$ of the cell problem

$$\begin{cases} -\operatorname{div}_{y_N} A(x, y_1, \ldots, y_N) \left(\rho + \sum_{j=1}^{N} \nabla_{y_j} v_j(x, y_1, \ldots, y_j) \right) = \operatorname{div}_{y_N} b(x, y_1, \ldots, y_N) \\ v_N(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q), \end{cases}$$

may be written as

$$\begin{aligned} v_N^\rho(x, y_1, \ldots, y_N) &= \\ = \sum_{i=1}^{n} \left(\rho_i + \sum_{j=1}^{N-1} \frac{\partial v_j}{\partial y_j^i}(x, y_1, \ldots, y_j) \right) & w_i(x, y_1, \ldots, y_N) + z(x, y_1, \ldots, y_N), \end{aligned}$$

with

$$\nabla_{y_N} v_N^\rho(x, y_1, \ldots, y_N) = \sum_{i=1}^{n} \left(\rho_i + \sum_{j=1}^{N-1} \frac{\partial v_j}{\partial y_j^i}(x, y_1, \ldots, y_j) \right) \nabla_{y_N} w_i(x, y_1, \ldots, y_N) + \nabla_{y_N} z(x, y_1, \ldots, y_N).$$

□

6.3.3. Example

Consider the three-scale problem

$$(P_\varepsilon^3) \quad \begin{cases} -\operatorname{div} A\left(x, \left\langle \frac{x}{\varepsilon} \right\rangle\right) \nabla u_\varepsilon(x) &= \operatorname{div} b\left(x, \left\langle \frac{x}{\sqrt{\varepsilon}} \right\rangle, \left\langle \frac{x}{\varepsilon} \right\rangle, \left\langle \frac{x}{\varepsilon^2} \right\rangle\right) \quad \text{in } \Omega \\ u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

with A and b defined previously, where the leading coefficient oscillates at the length scale ε, while the source term oscillate at three separated length scales, $\sqrt{\varepsilon}$, ε, and ε^2. Let u_ε be the solution of problem (P_ε^3). Then it follows from Corollary 6.3.4 that the sequence $\{u_\varepsilon\}$ is weak convergent in $H_0^1(\Omega)$ to the solution u_0 of the homogenized problem

$$\begin{cases} -\operatorname{div} A_0(x) \nabla u_0(x) &= \operatorname{div} g^\star(x) \quad \text{in } \Omega \\ u_0 &\in H_0^1(\Omega), \end{cases}$$

where the effective coefficients are determined following the iterating process explained in Theorem 6.3.3. Namely, for a.e. $x \in \Omega$,

$$A_0(x) = \int_Q \left(I_n + \left[\nabla_{y_2} w_j^2(x, y_2)\right]\right)^T A(x, y_2) \left(I_n + \left[\nabla_{y_2} w_j^2(x, y_2)\right]\right) dy_2,$$

and

$$g^\star(x) = \int_Q \int_Q \int_Q \Big[\left(\left(I_n + \left[\nabla_{y_2} w_j^2\right]\right) \nabla_{y_1} z_1 + \nabla_{y_2} z_2 + \nabla_{y_3} z_3\right)^T$$
$$A(x, y_2) \left(I_n + \left[\nabla_{y_2} w_j^2\right]\right) + b(x, y_1, y_2, y_3) \left(I_n + \left[\nabla_{y_2} w_j^2\right]\right) \Big] \, dy_1 \, dy_2 \, dy_3.$$

Clearly the effective matrix function A_0 is defined through the integration over the unit cell corresponding to the oscillatory length scale ε.

The functions $z_i(x, y_1, \ldots, y_i)$, with $1 \le i \le 3$, are computed starting from the higher index i to the lower one, as follows, while the auxiliary functions $w_j^1(x, y_1)$ and $w_j^3(x, y_1, y_2, y_3)$ are not computed because the matrix function $A(x, y_2)$ does not depend on the variables y_1 and y_3. So, the function $z_3(x, y_1, y_2, \cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_{y_3} A(x, y_2) \nabla_{y_3} z_3(x, y_1, y_2, y_3) &= \operatorname{div}_{y_3} b(x, y_1, y_2, y_3) \quad \text{in } Q \\ z_3(x, y_1, y_2, \cdot) &\in H_{per}^1(Q). \end{cases}$$

Secondly, the vectors $\nabla_{y_2} w_j^2(x, y_2)$, for $1 \le j \le n$, which are the columns of the $n \times n-$matrix $\left[\nabla_{y_2} w_j^2(x, y_2)\right]$, are obtained as the solution of

$$\begin{cases} -\operatorname{div}_{y_2} A(x, y_2) \left(e_j + \nabla_{y_2} w_j^2(x, y_2)\right) &= 0 \quad \text{in } Q \\ w_j^2(x, \cdot) &\in H_{per}^1(Q), \end{cases}$$

while the function $z_2(x, y_1, \cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_{y_2} A(x, y_2) \nabla_{y_2} z_2(x, y_1, y_2) &= \operatorname{div}_{y_2} g_2^\star(x, y_1, y_2) \quad \text{in } Q \\ z_2(x, y_1, \cdot) &\in H_{per}^1(Q), \end{cases}$$

with

$$g_2^\star(x, y_1, y_2) \;=\; \int_Q b(x, y_1, y_2, y_3)\ dy_3.$$

Finally, $z_1(x, \cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_{y_1}\ A_1^\star(x)\nabla_{y_1} z_1(x, y_1) &= \operatorname{div}_{y_1}\ g_1^\star(x, y_1) \quad \text{in } Q \\ z_1(x, \cdot) &\in H^1_{per}(Q), \end{cases}$$

with

$$A_1^\star(x) = \int_Q \left(I_n + \left[\nabla_{y_2} w_j^2(x, y_2)\right]\right)^T A(x, y_2) \left(I_n + \left[\nabla_{y_2} w_j^2(x, y_2)\right]\right)\ dy_2$$

and

$$g_1^\star(x, y_1) = \int_Q \int_Q \left(I_n + \left[\nabla_{y_2} w_j^2(x, y_2)\right]\right)^T A(x, y_2) \left(\nabla_{y_2} z_2 + \nabla_{y_3} z_3\right)\ dy_2\ dy_3\ + \\ + \int_Q \int_Q b(x, y_1, y_2, y_3) \left(I_n + \left[\nabla_{y_2} w_j^2(x, y_2)\right]\right)\ dy_2\ dy_3,$$

for a.e. $(x, y_1) \in \Omega \times Q$.

6.4. The periodic n-dimensional case: distinct scales

Here we are interested in studying the asymptotic behaviour of quadratic functionals, with oscillatory linear perturbations, in the case when the quadratic and linear coefficients oscillate at different length scales. More precisely, we would like to answer the question : does the homogenized linear coefficient $g^\star$ really depend on the sequence $\{A_\varepsilon\}$?

6.4.1. Two scales

Let us consider the two-scale problem

$$(P_{2\varepsilon}) \quad \begin{cases} -\operatorname{div}\ A\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle\right) \nabla u_\varepsilon(x) &= \operatorname{div}\ b\left(x, \left\langle \frac{x}{l_2(\varepsilon)} \right\rangle\right) \quad \text{in } \Omega \\ u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

where $A = [a_{ij}] \in [L^\infty(\Omega \times Q)]^{n \times n}$ is symmetric such that, there exist $0 < \alpha \le \beta$, $\alpha|\rho|^2 \le \rho^T A \rho \le \beta|\rho|^2$, for every $\rho \in \mathbb{R}^n$, and $b \in [L^\infty(\Omega \times Q)]^n$. Suppose that $l_1(\varepsilon)$ and $l_2(\varepsilon)$ are two separated length scales. Since the characterization of the effective coefficients may be deduced from the explicit characterization of the density of the Γ-limit of the sequence of associated functionals, we study the Γ-convergence of such functionals.

Theorem 6.4.1 *The sequence of functionals*

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\nabla u(x)^T \, \frac{A\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle\right)}{2} \, \nabla u(x) \;+\; b\left(x, \left\langle \frac{x}{l_2(\varepsilon)} \right\rangle\right) \cdot \nabla u(x) \right] \, dx$$

is Γ-convergent, with respect to the weak topology of $H_0^1(\Omega)$, to the functional I whose density is given by

$$\psi(x, \rho) \;=\; \rho^T \, \frac{A_0(x)}{2} \, \rho \;+\; b_0(x) \cdot \rho \;+\; d(x)$$

where the matrix function A_0 is given by (6.4), the linear coefficient b_0 is the weak* limit of $\{b_\varepsilon\}$ given by

$$b_0(x) \;=\; \int_Q b(x, y_2) \; dy_2,$$

and

$$d(x) \;=$$

$$= \int_Q \int_Q \left[\nabla_{y_2} v(x, y_1, y_2)^T \frac{A(x, y_1)}{2} \nabla_{y_2} v(x, y_1, y_2) + b(x, y_2) \nabla_{y_2} v(x, y_1, y_2) \right] dy_1 dy_2.$$

The function $v(x, y_1, \cdot)$ is the solution of the cell problem

$$\begin{cases} -\operatorname{div}_{y_2} A(x, y_1) \nabla_{y_2} v(x, y_1, y_2) \;=\; \operatorname{div}_{y_2} b(x, y_2) & \text{in } Q \\ v(x, y_1, \cdot) \;\in\; H^1_{per}(Q). \end{cases}$$

Thus, whenever the coefficients oscillate at distinct length scales, we conclude that the homogenized linear coefficient b_0 is the weak* limit of the sequence $\{b_\varepsilon\}$, so that it does not depend on the sequence $\{A_\varepsilon\}$ at all.

Corollary 6.4.2 *If u_ε is the solution of $(P_{2\varepsilon})$, then the sequence $\{u_\varepsilon\}$ is weak convergent to the solution of*

$$\begin{cases} -\operatorname{div} A_0(x) \nabla u_0(x) \;=\; \operatorname{div} b_0(x) & \text{in } \Omega \\ u_0 \;\in\; H^1_0(\Omega). \end{cases}$$

Now, we present the proof of Theorem 6.4.1, which is obtained following closely the proof of Theorem 6.3.1.

Proof. (of Theorem 6.4.1) We already known that the sequence of functionals I_ε is Γ-convergent to

$$I(u) \;=\; \int_\Omega \psi(x, \nabla u(x)) \; dx,$$

where $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi(x,\rho) = \inf_{\substack{v_i \in \Psi_i \\ i=1,2}} \int_{Q^2} \Big[(\rho + \nabla_{y_1} v_1 + \nabla_{y_2} v_2)^T \, \frac{A(x,y_1)}{2} \, (\rho + \nabla_{y_1} v_1 + \nabla_{y_2} v_2) + b(x,y_2) \cdot (\rho + \nabla_{y_1} v_1 + \nabla_{y_2} v_2) \Big] \; dy_1 dy_2,$$

with

$$\Psi_i \;=\; L^2[\Omega \times Q^{i-1}; H^1_{per}(Q)], \quad \text{for} \quad 1 \le i \le 2.$$

Computing explicitly the expression of $\psi(x,\rho)$ as a quadratic function, we have finished the proof. Namely, fix $v_1 \in \Psi_1$, then the minimizer $v_2^\rho \in \Psi_2$ is the solution of

$$\begin{cases} -\operatorname{div}_{y_2} \; A\,(x,y_1)\,(\rho + \nabla_{y_1} v_1(x,y_1) + \nabla_{y_2} v_2^\rho(x,y_1,y_2)) \;=\; \operatorname{div}_{y_2} \; b\,(x,y_2) \quad \text{in} \;\; Q \\ v_2^\rho(x,y_1,\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

which is equivalent to

$$\begin{cases} -\operatorname{div}_{y_2} \; A\,(x,y_1)\,\nabla_{y_2} v_2(x,y_1,y_2) \;=\; \operatorname{div}_{y_2} \; b\,(x,y_2) \quad \text{in} \;\; Q \\ v_2(x,y_1,\cdot) \;\in\; H^1_{per}(Q). \end{cases}$$

Now let us minimize over Ψ_1, so that the minimizer v_1^ρ is solution of

$$\begin{cases} -\operatorname{div}_{y_1} \; A\,(x,y_1)\,(\rho + \nabla_{y_1} v_1^\rho(x,y_1)) \;=\; 0 \quad \text{in} \;\; Q \\ v_1^\rho(x,\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

which may be written as

$$v_1^\rho(x,y_1) \;=\; \sum_{j=1}^{d} w_j(x,y_1)\rho_j$$

where $w_j(x,\cdot)$ is the solution of

$$\begin{cases} -\operatorname{div}_{y_1} \; A\,(x,y_1)\,(e_j + \nabla_{y_1} w_j(x,y_1)) \;=\; 0 \qquad \text{in} \;\; Q \\ \\ w_j(x,\cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

for every $1 \le j \le n$, and some basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. If we replace the expression of $\nabla_{y_1} v_1^\rho$ in $\psi(x,\rho)$, we obtain that

$$\psi(x,\rho) \;=\; \int_Q \int_Q \Big[((I_n + [\nabla_{y_1} w_j])\,\rho + \nabla_{y_2} v_2 \;)^T \; \frac{A(x,y_1)}{2} \\ ((I_n + [\nabla_{y_1} w_j])\,\rho + \nabla_{y_1} v_2) + b(x,y_2) \cdot ((I_n + [\nabla_{y_1} w_j])\,\rho + \nabla_{y_2} v_2) \Big] \; dy_1 \; dy_2,$$

where $[\nabla_{y_1} w_j]$ is the $n \times n$-matrix whose columns are the vectors $\nabla_{y_1} w_j(x,y_1)$, with $1 \le j \le n$. □

6.4.2. Multi-scales

When the sequences $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillate in distinct separated length scales, namely

$$A_\varepsilon(x) \; = \; A\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_N(\varepsilon)} \right\rangle\right)$$

and

$$b_\varepsilon(x) \; = \; b\left(x, \left\langle \frac{x}{l_{N+1}(\varepsilon)} \right\rangle, \ldots, \left\langle \frac{x}{l_{N+M}(\varepsilon)} \right\rangle\right)$$

where $\{l_1(\varepsilon), \ldots, l_{N+M}(\varepsilon)\}$ is a family of $N+M$ separated length scales, clearly the source term in the homogenized equation is the weak* limit of the sequence $\{b_\varepsilon\}$, as follows from the theorem below, provided there is no interaction between the oscillatory behaviours of $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$. Notice that the oscillations of $\{b_\varepsilon\}$ take place at faster length scales than $\{A_\varepsilon\}$. However this is irrelevant in the following result. The separability of the family of length scales $\{l_1(\varepsilon), \ldots, l_{N+M}(\varepsilon)\}$, and the fact that $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ oscillate at different length scales, are sufficient to obtain the result.

Theorem 6.4.3 *Let $\{A_\varepsilon\}$ and $\{b_\varepsilon\}$ be the previous sequences. Then the sequence of functionals $\{I_\varepsilon\}$ is Γ-convergent to the functional I whose density is given by*

$$\psi(x,\rho) \; = \; \rho^T \, \frac{A_0(x)}{2} \, \rho \; + \; b_0(x) \cdot \rho \; + \; d(x)$$

with

$$b_0(x) \; = \; \int_{Q^M} b(x, y_{N+1}, \ldots, y_{N+M}) \; dy_{N+1} \ldots dy_{N+M},$$

and

$$d(x) = \int_{Q^{N+M}} \left(\sum_{i=1}^{M} \nabla_{y_{N+i}} v_{N+i} \right)^T \frac{A(x, y_1, \ldots, y_N)}{2} \left(\sum_{i=1}^{M} \nabla_{y_{N+i}} v_{N+i} \right)$$
$$+ \; b(x, y_{N+1}, \ldots, y_{N+M}) \left(\sum_{i=1}^{M} \nabla_{y_{N+i}} v_{N+i} \right) \; dy_1 \ldots dy_{N+M}.$$

For any $1 \le i \le M$, the function v_{N+i} is the solution of the problem

$$\left\{ \begin{array}{rcl} -\operatorname{div}_{y_{N+i}} A\left(x, y_1, \ldots, y_N\right) \nabla_{y_{N+i}} v_{N+i} & = & \operatorname{div} b_{N+i}\left(x, y_1, \ldots, y_{N+i}\right) \\ v_{N+i}(x, y_1, \ldots, y_{N+i-1}, \cdot) & \in & H^1_{per}(Q), \end{array} \right.$$

with

$$b_{N+i}\left(x, y_1, \ldots, y_{N+i}\right) \; = \; \int_{Q^{M-i}} b\left(x, y_1, \ldots, y_{N+M}\right) \; dy_{N+i+1} \ldots dy_{N+M}.$$

Corollary 6.4.4 *If u_ε is the solution of*

$$\begin{cases} -\operatorname{div} A\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle, \dots, \left\langle \frac{x}{l_N(\varepsilon)} \right\rangle\right) \nabla u_\varepsilon(x) = \operatorname{div}\ b\left(x, \left\langle \frac{x}{l_{N+1}(\varepsilon)} \right\rangle, \dots, \left\langle \frac{x}{l_{N+M}(\varepsilon)} \right\rangle\right) \\ u_\varepsilon \ \in \ H_0^1(\Omega), \end{cases}$$

then the sequence $\{u_\varepsilon\}$ is weak convergent to the solution of

$$\begin{cases} -\operatorname{div}\ A_0(x)\, \nabla u_0(x) \ = \ \operatorname{div}\ b_0(x) \quad in \ \ \Omega \\ u_0 \ \in \ H_0^1(\Omega). \end{cases}$$

Remark 6.4.1 *The claim of the previous result is the same if we consider the sequences*

$$A_\varepsilon(x) \ = \ A\left(x, \left\langle \frac{x}{l_{N+1}(\varepsilon)} \right\rangle, \dots, \left\langle \frac{x}{l_{N+M}(\varepsilon)} \right\rangle\right)$$

and

$$b_\varepsilon(x) \ = \ b\left(x, \left\langle \frac{x}{l_1(\varepsilon)} \right\rangle, \dots, \left\langle \frac{x}{l_N(\varepsilon)} \right\rangle\right),$$

whenever $\{l_1(\varepsilon), \dots, l_{N+M}(\varepsilon)\}$ is a family of separated length scales.

Proof. (of Theorem 6.4.3) It follows from Theorem 4.7.2 that the sequence of functionals I_ε is Γ-convergent to

$$I(u) \ = \ \int_\Omega \psi(x, \nabla u(x))\, dx,$$

where $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi(x,\rho) \ = \ \inf_{\substack{v_i \in \Psi_i \\ 1 \le i \le N+M}} \int_{Q^{N+M}} \left[\left(\rho + \sum_{i=1}^{N+M} \nabla_{y_i} v_i \right)^T \frac{A(x, y_1, \dots, y_N)}{2} \right.$$
$$\left. \left(\rho + \sum_{i=1}^{N+M} \nabla_{y_i} v_i \right) + b(x, y_{N+1}, \dots, y_{N+M}) \cdot \left(\rho + \sum_{i=1}^{N+M} \nabla_{y_i} v_i \right) \right] dy_1 \dots dy_{N+M},$$

with

$$\Psi_i \ = \ L^2[\Omega \times Q^{i-1}; H^1_{per}(Q)], \ \ \text{for every} \ \ 1 \le i \le N+M.$$

We will proceed to minimize the previous integral functional from the higher index i to the lower one. So, let us fix $v_i \in \Psi_i$, for every $1 \le i \le N+M-1$, and minimize in $v_{N+M} \in \Psi_{N+M}$. For each $\rho \in \mathbb{R}^n$, the minimizer v^ρ_{N+M} is the solution of the problem

$$\begin{cases} -\operatorname{div} A\left(\rho + \sum_{i=1}^{N+M-1} \nabla_{y_i} v_i(x, y_1, \dots, y_i) + \nabla_{y_{N+M}} v^\rho_{N+M}(x, y_1, \dots, y_{N+M})\right) \\ = \operatorname{div} b\,(x, y_{N+1}, \dots, y_{N+M}) \\ v^\rho_{N+M}(x, y_1, \dots, y_{N+M-1}, \cdot) \ \in \ H^1_{per}(Q). \end{cases}$$

Since the leading term does not depend on the variable y_{N+M}, the minimizer does not depend on ρ so that $v^{\rho}_{N+M} = v_{N+M}$ is the solution of

$$\begin{cases} -\text{div} A\,(x, y_1, \ldots, y_N)\,\nabla_{y_{N+M}} v_{N+M}(x, y_1, \ldots, y_{N+M}) = \text{div}\; b\,(x, y_{N+1}, \ldots, y_{N+M}) \\ v_{N+M}(x, y_1, \ldots, y_{N+M-1}, \cdot) \;\in\; H^1_{per}(Q). \end{cases}$$

Therefore, for every $1 \leq i \leq M$, the minimizer $v_{N+i} \in \Psi_{N+i}$ is the solution of the problem

$$\begin{cases} -\;\text{div}\, A\,(x, y_1, \ldots, y_N)\,\nabla_{y_{N+i}} v_{N+i}(x, y_1, \ldots, y_{N+i}) \;=\; \text{div}\,\overline{b}_i(x, y_{N+1}, \ldots, y_{N+i}) \\ v_{N+i}(x, y_1, \ldots, y_{N+i-1}, \cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

with

$$\overline{b}_i(x, y_{N+1}, \ldots, y_{N+i}) \;=\; \int_{Q^{M-i}} b\,(x, y_{N+1}, \ldots, y_{N+M})\;\, dy_{N+i+1} \ldots dy_{N+M},$$

provided the matrix function $A\,(x, y_1, \ldots, y_N)$ does not depend on the variables y_{N+i}.

Now, fixed $v_i \in \Psi_i$, for every $1 \leq i \leq N-1$, the minimizer $v^{\rho}_N \in \Psi_N$ is the solution of

$$\begin{cases} -\;\text{div}_{y_N}\, A\,(x, y_1, \ldots, y_N)\left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i(x, y_1, \ldots, y_i) + \nabla_{y_N} v^{\rho}_N(x, y_1, \ldots, y_N)\right) \\ \qquad\qquad = \text{div}_{y_N}\, b_0(x) \\ v^{\rho}_N(x, y_1, \ldots, y_{N-1}, \cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

with

$$b_0(x) \;=\; \int_{Q^M} b\,(x, y_{N+1}, \ldots, y_{N+M})\;\, dy_{N+1} \ldots dy_{N+M}.$$

Notice that there is not a dependence on the previous minimizers v_{N+k} because it holds

$$\sum_{k=1}^{M} \int_{Q^M} \nabla_{y_{N+k}} v_{N+k}(x, y_1, \ldots, y_{N+k})\; dy_{N+1} \ldots dy_{N+M} \;\;=\;\; 0,$$

for every $v_{N+k}(x, y_1, \ldots, y_{N+k-1}, \cdot) \;\in\; H^1_{per}(Q)$. Thus the minimizer v^{ρ}_N is the solution of

$$\begin{cases} -\text{div}_{y_N} A\,(x, y_1, \ldots, y_N)\,\left(\rho + \sum \nabla_{y_i} v_i(x, y_1, \ldots, y_i) + \nabla_{y_N} v^{\rho}_N(x, y_1, \ldots, y_N)\right) = 0 \\ v^{\rho}_N(x, y_1, \ldots, y_{N-1}, \cdot) \;\in\; H^1_{per}(Q), \end{cases}$$

which may be written as a function of ρ by putting

$$v^{\rho}_N(x, y_1, \ldots, y_N) = \sum_{j=1}^{d}\; w^N_j(x, y_1, \ldots, y_N)\left(\rho_j + \sum_{i=1}^{N-1} \frac{\partial v_i}{\partial y^j_i}(x, y_1, \ldots, y_i)\right),$$

so that

$$\nabla_{y_N} v_N^\rho(x, y_1, \ldots, y_N) = \sum_{j=1}^{d} \nabla_{y_N} w_j^N(x, y_1, \ldots, y_N) \left(\rho_j + \sum_{i=1}^{N-1} \frac{\partial v_i}{\partial y_i^j}(x, y_1, \ldots, y_i) \right).$$

The function $w_j^N(x, y_1, \ldots, y_{N-1}, \cdot)$ is the solution of the unit cell problem

$$\begin{cases} -\operatorname{div}_{y_N} A(x, y_1, \ldots, y_N)\left(e_j + \nabla_{y_N} w_j^N(x, y_1, \ldots, y_N)\right) = 0 \quad \text{in } Q \\ w_j^N(x, y_1, \ldots, y_{N-1}, \cdot) \in H^1_{per}(Q), \end{cases}$$

for every $1 \le j \le n$ and some basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$.

Replacing the expression of $\nabla_{y_N} v_N^\rho$ in $\psi(x,\rho)$, we obtain that

$$\begin{aligned} \psi(x,\rho) = \inf_{\substack{v_i \in \Psi_i \\ 1 \le i \le N-1}} \int_{Q^{N+M}} \Bigg[\Bigg(& \left(I_n + [\nabla_{y_N} w_j^N]\right) \left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i\right) \\ & + \sum_{k=1}^{M} \nabla_{y_{N+k}} v_{N+k} \Bigg)^T \frac{A(x, y_1, \ldots, y_N)}{2} \\ & \left(\left(I_n + [\nabla_{y_N} w_j^N]\right) \left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i\right) + \sum_{k=1}^{M} \nabla_{y_{N+k}} v_{N+k} \right) \\ + b(x, y_1, \ldots, y_N) \cdot & \left(\left(I_n + [\nabla_{y_N} w_j^N]\right) \left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i\right) + \sum_{k=1}^{M} \nabla_{y_{N+k}} v_{N+k} \right) \Bigg] dy \end{aligned}$$

where $\left[\nabla_{y_N} w_j^N(x, y_1, \ldots, y_N)\right]$ is the $n \times n$-matrix whose columns are the vectors $\nabla_{y_N} w_j^N(x, y_1, \ldots, y_N)$, with $1 \le j \le n$.

Iterating one more time, fix $v_i \in \Psi_i$, with $1 \le i \le N-2$, and let us minimize in $v_{N-1} \in \Psi_{N-1}$. The minimizer v_{N-1}^ρ is the solution of

$$\begin{cases} -\operatorname{div}_{y_{N-1}} A^\star_{N-1} \left(\rho + \sum_{i=1}^{N-2} \nabla_{y_i} v_i(x, y_1, \ldots, y_i) + \nabla_{y_{N-1}} v_{N-1}^\rho(x, y_1, \ldots, y_{N-1})\right) \\ \qquad = \operatorname{div}_{y_{N-1}} b_0(x) \\ v_{N-1}^\rho(x, y_1, \ldots, y_{N-2}, \cdot) \in H^1_{per}(Q), \end{cases}$$

where the matrix function $A^\star_{N-1}(x, y_1, \ldots, y_{N-1})$ defined by

$$A^\star_{N-1} = \int_Q \left(I_n + [\nabla_{y_N} w_j^N]\right)^T A(x, y_1, \ldots, y_N) \left(I_n + [\nabla_{y_N} w_j^N]\right) dy_N,$$

and

$$b_0(x) = \int_{Q^{1+M}} \left(I_n + [\nabla_{y_N} w_j^N(x, y_1, \ldots, y_N)]\right)^T A(x, y_1, \ldots, y_N)$$

$$\sum_{k=1}^{M} \nabla_{y_{N+k}} v_{N+k} + b(x, y_{N+1}, \ldots, y_{N+M}) \left(I_n + [\nabla_{y_N} w_j^N(x, y_1, \ldots, y_N)]\right) dy_N \cdot dy_{N+M}.$$

Then we conclude that the minimizer v^{ρ}_{N-1} is the solution of

$$\begin{cases} -\mathrm{div} A^{\star}_{N-1}\left(\rho + \sum_{i=1}^{N-2} \nabla_{y_i} v_i(x, y_1, \dots, y_i) + \nabla_{y_{N-1}} v^{\rho}_{N-1}(x, y_1, \dots, y_{N-1})\right) = 0 \\ v^{\rho}_{N-1}(x, y_1, \dots, y_{N-2}, \cdot) \in H^1_{per}(Q), \end{cases}$$

and may be written as

$$v^{\rho}_{N-1}(x, y_1, \dots, y_{N-1}) = \sum_{j=1}^{n} w^{N-1}_j(x, y_1, \dots, y_{N-1}) \left(\rho_j + \sum_{i=1}^{N-2} \frac{\partial v_i}{\partial y^j_i}(x, y_1, \dots, y_i)\right),$$

so that

$$\nabla_{y_{N-1}} v^{\rho}_{N-1} = \sum_{j=1}^{n} \nabla_{y_{N-1}} w^{N-1}_j(x, y_1, \dots, y_{N-1}) \left(\rho_j + \sum_{i=1}^{N-2} \frac{\partial v_i}{\partial y^j_i}(x, y_1, \dots, y_i)\right).$$

The function $w^{N-1}_j(x, y_1, \dots, y_{N-2}, \cdot)$ is the solution of the unit cell problem

$$\begin{cases} -\mathrm{div}_{y_{N-1}} A^{\star}_{N-1}(x, y_1, \dots, y_{N-1}) \left(e_j + \nabla_{y_{N-1}} w^{N-1}_j(x, y_1, \dots, y_{N-1})\right) = 0 \\ w^{N-1}_j(x, y_1, \dots, y_{N-2}, \cdot) \in H^1_{per}(Q), \end{cases}$$

for every $1 \le j \le n$.

It follows

$$\begin{aligned} &\psi(x, \rho) = \\ &= \inf_{\substack{v_i \in \Psi_i \\ 1 \le i \le N-2}} \int_{Q^{N+M}} \Bigg[\Bigg(\left(I_n + [\nabla_{y_N} w^N_j]\right) \left(I_n + \left[\nabla_{y_{N+1}} w^{N+1}_j\right]\right) \left(\rho + \sum_{i=1}^{N-2} \nabla_{y_i} v_i\right) \\ &\qquad + \sum_{k=1}^{M} \nabla_{y_{N+k}} v_{N+k} \Bigg)^T \frac{A(x, y_1, \dots, y_N)}{2} \\ &\left(\left(I_n + [\nabla_{y_N} w^N_j]\right) \left(I_n + \left[\nabla_{y_{N+1}} w^{N+1}_j\right]\right) \left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i\right) + \sum_{k=1}^{M} \nabla_{y_{N+k}} v_{N+k} \right) \\ &+ b(x, y_1, \dots, y_N) \cdot \Bigg(\left(I_n + [\nabla_{y_N} w^N_j]\right) \left(I_n + \left[\nabla_{y_{N+1}} w^{N+1}_j\right]\right) \left(\rho + \sum_{i=1}^{N-1} \nabla_{y_i} v_i\right) + \\ &\qquad + \sum_{k=1}^{M} \nabla_{y_{N+k}} v_{N+k} \Bigg) \Bigg] \, dy_1 \dots dy_{N+M}. \end{aligned}$$

Calculating other minimizers v^{ρ}_i, with $1 \le i \le N-2$, we achieve the explicit expression of $\psi(x, \rho)$. □

6.5. The non-periodic multi-dimensional case

Consider the problem

$$(P_\varepsilon) \quad \begin{cases} -\text{div } A_\varepsilon(x)\nabla u_\varepsilon(x) &= \text{div}\, b_\varepsilon(x) \qquad \text{in } \Omega \\ u_\varepsilon &\in H_0^1(\Omega), \end{cases}$$

with $\{A_\varepsilon\} \subset [L^\infty(\Omega)]^{n\times n}$ symmetric and $\{b_\varepsilon\} \subset [L^\infty(\Omega)]^n$ satisfying the following conditions:

(H1) $c_1|\rho|^2 \le \rho^T A_\varepsilon(x)\rho \le c_2|\rho|^2$, for some $c_2 \ge c_1 > 0$,

(H2) $\{b_\varepsilon\}$ is uniformly bounded in $[L^\infty(\Omega)]^n$,

(H3) $\{(A_\varepsilon, b_\varepsilon)\}$ satisfies the Composition Gradient Property (CGP).

The CGP is a sufficient condition to obtain the explicit characterization of the density of the Γ-limit of functionals

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\nabla u(x)^T \, \frac{A_\varepsilon(x)}{2} \, \nabla u(x) \;+\; b_\varepsilon(x)\cdot\nabla u(x) \right] dx \tag{6.5}$$

defined in $H_0^1(\Omega)$. It was introduced in the previous chapter to characterize the Γ-limit of general sequences of functionals, in the non-periodic setting. Let us recall its definition for sequences of pairs $\{(A_\varepsilon, b_\varepsilon)\}$.

Definition 6.5.1 *A sequence of pairs $(A_\varepsilon, b_\varepsilon) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n\times n} \times \mathbb{R}^n$, with associated Young measure $\eta = \{\eta_x\}_{x\in\Omega}$, satisfies the CGP (with respect to the exponent q) if and only if there exists a Carathéodory map $\phi : \Omega \times \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for a.e. $x \in \Omega$,*

i) $\phi(x,\cdot,\cdot)$ is one-to-one over the support of η_x;

ii) $\{\phi(x, A_\varepsilon(x + r_\varepsilon\cdot), b_\varepsilon(x + r_\varepsilon\cdot))\}$ is "essentially a sequence of gradients", in the sense

$$\|\,\text{curl } \phi(x, A_\varepsilon(x + r_\varepsilon\cdot), b_\varepsilon(x + r_\varepsilon\cdot))\,\|_{W^{-1,q}(B)} \underset{\varepsilon\searrow 0}{\longrightarrow} 0,$$

for some sequence $r_\varepsilon \searrow 0$.

Under the CGP condition, the explicit characterization of the density of the Γ-limit of functionals (6.5) was achieved by means of the Young measure associated with $\{(A_\varepsilon, b_\varepsilon)\}$.

Theorem 6.5.1 *Let $\{A_\varepsilon\} \subset [L^\infty(\Omega)]^{n\times n}$ and $\{b_\varepsilon\} \subset [L^\infty(\Omega)]^n$ satisfying (H1), (H2) and (H3). Then the sequence of functionals in (6.5) is Γ-convergent, in the weak topology of $H_0^1(\Omega)$, to the functional I defined by*

$$I(u) \;=\; \int_\Omega \psi(x, \nabla u(x))\; dx$$

with

$$\psi(x,\rho) \;=\; \inf_{\varphi\in\mathcal{A}_x} \left\{ \int_{\mathbb{R}^{n\times n+n}} \left[\varphi(\Lambda,\beta)^T \frac{\Lambda}{2}\varphi(\Lambda,\beta) \;+\; \beta\cdot\varphi(\Lambda,\beta) \right] d\eta_x(\Lambda,\beta) \;: \right.$$
$$\left. \rho = \int_{\mathbb{R}^{d\times d+d}} \varphi(\Lambda,\beta)\, d\eta_x(\Lambda,\beta) \right\}$$

and

$$\mathcal{A}_x = \left\{ \varphi : \mathbb{R}^{n\times n}\times\mathbb{R}^n \to \mathbb{R}^n \;:\; \|\,\mathrm{curl}\; \varphi(A_\varepsilon(x+r_\varepsilon\cdot), b_\varepsilon(x+r_\varepsilon\cdot))\,\|_{W^{-1,q}(B)} \to 0 \right\}$$

for any $q>2$, whenever the sequence $\{(A_\varepsilon(x+r_\varepsilon\cdot), b_\varepsilon(x+r_\varepsilon\cdot))\}$ defined in the unit ball, for some sequence $r_\varepsilon \searrow 0$, generates the homogenous Young measure η_x.

Proof. It is a particular case of Theorem 5.1.1 for quadratic functionals. □

Here our aim is to characterize the leading coefficient and the source term, coming from the homogenization of problem (P_ε), through the Γ-limit of its associated sequence of functionals I_ε. However, in the general non-periodic setting of Thereom 6.5.1, it is not easy to write explicitly the homogenized equation separating the leading coefficient from the source term. Indeed, if u_ε is the solution of (P_ε), then the sequence $\{u_\varepsilon\}$ is weak convergent in $H_0^1(\Omega)$ to the solution u_0 of

$$\begin{cases} -\,\mathrm{div}\left(\frac{\partial\psi}{\partial\rho}(x,\nabla u_0(x))\right) \;=\; 0 & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u_0 \;\in\; H_0^1(\Omega), & \end{cases}$$

where ψ is defined in the previous theorem.

Basically, to turn the homogenized equation into a more explicit divergence form depends on the characterization itself of the admissible fields $\varphi \in \mathcal{A}_x$, ie the fields φ for which the composition sequence

$$\{\,\varphi(A_\varepsilon(x+r_\varepsilon\cdot), b_\varepsilon(x+r_\varepsilon\cdot))\,\},$$

defined in the unit ball, is essentially a sequence of gradients, whenever the sequence $\{(A_\varepsilon(x+r_\varepsilon\cdot), b_\varepsilon(x+r_\varepsilon\cdot))\}$ generates the homogenous Young measure η_x.

6.5.1. Examples

1. Consider the function $v \in W_0^{1,\infty}(D)$ so that $c_1|\rho|^2 \leq \rho^T \nabla v \otimes \nabla v\, \rho \leq c_2|\rho|^2$. Let us consider a Vitali covering $\{x_k^{(\varepsilon)} + r_k^{(\varepsilon)} D\}_k$ of Ω, with $\{x_k^{(\varepsilon)}\} \subset \Omega$ and $r_k^{(\varepsilon)} > 0$, and define the sequence of functions $v_\varepsilon : \Omega \to \mathbb{R}$ by putting

$$v_\varepsilon(x) \;=\; r_k^{(\varepsilon)} v\left(\frac{x - x_k^{(\varepsilon)}}{r_k^{(\varepsilon)}}\right) \quad if \quad x \in x_k^{(\varepsilon)} + r_k^{(\varepsilon)} D.$$

The sequence of pairs $\{(\nabla v_\varepsilon \otimes \nabla v_\varepsilon, \nabla v_\varepsilon)\}$ generates the homogenous Young measure η given by

$$\langle \varphi, \eta \rangle \;=\; \frac{1}{|D|} \int_D \varphi(\nabla v(y) \otimes \nabla v(y), \nabla v(y))\, dy,$$

for every $\varphi \in C_0(\mathbb{R}^{n\times n} \times \mathbb{R}^n)$, and it verifies the CGP provided, if we take the injective field ϕ defined on the support of η by $\phi(b \otimes b, b) = b$, the sequence of functions $\phi(\nabla v_\varepsilon \otimes \nabla v_\varepsilon, \nabla v_\varepsilon) = \nabla v_\varepsilon$ is a gradient sequence. Thus it follows from Theorem 6.5.1 that the sequence of functionals I_ε defined by

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\; \nabla u(x)^T \,\frac{\nabla v_\varepsilon(x) \otimes \nabla v_\varepsilon(x)}{2}\, \nabla u(x) \;+\; \nabla v_\varepsilon(x) \cdot \nabla u(x) \;\right] dx$$

is Γ-convergent to the functional I whose homogenous density $\psi : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi(\rho) \;=$$

$$= \inf_z \frac{1}{|D|} \int_D \left[(\rho + \nabla z(y))^T \,\frac{\nabla v(y) \otimes \nabla v(y)}{2}\,(\rho + \nabla z(y)) + \nabla v(y) \cdot (\rho + \nabla z(y))\right] dy.$$

Indeed, taking into account the previous definition of η and after the change of variable $\varphi = \nabla z + \rho$, provided the constraint $\operatorname{curl} \varphi(\nabla v \otimes \nabla v, \nabla v) = 0$ is met, we achieve the expression above. Notice that, for each $\rho \in \mathbb{R}^n$, the minimizer $z^\rho \in H_0^1(D)$ may be written so that

$$\nabla z^\rho(y) \;=\; \sum_{j=1}^n \nabla f_j(y)\, \rho_j + \nabla g(y),$$

where f_j is the solution of

$$\begin{cases} -\operatorname{div}\, \nabla v(y) \otimes \nabla v(y)\, (e_j + \nabla f_j(y)) \;=\; 0 & \text{in } D, \\ \qquad\qquad\qquad\qquad f_j \;\in\; H_0^1(D), & \end{cases}$$

for every $1 \leq j \leq n$, and $g \in H_0^1(D)$ is the solution of

$$\begin{cases} -\operatorname{div}\, \nabla v(y) \otimes \nabla v(y) \nabla g(y) \;=\; \operatorname{div}\, \nabla v(y) & \text{in } D \\ \qquad\qquad\qquad\qquad g \;\in\; H_0^1(D). & \end{cases}$$

Replacing the expression of ∇z^ρ, and after some calculus, the explicit expression of ψ as a quadratic function is achieved, ie

$$I(u) \;=\; \int_\Omega \left[\; \nabla u(x)^T \,\frac{A_0}{2}\, \nabla u(x) \;+\; g^\star \cdot \nabla u(x) \;+ c \;\right] \,dx,$$

with

$$\begin{aligned}
A_0 \;&=\; \frac{1}{|D|}\int_D (I_d + [\nabla f_j(y)])^T\, \nabla v(y) \otimes \nabla v(y)\, (I_d + [\nabla f_j(y)]) \;dy,\\
g^\star \;&=\; \frac{1}{|D|}\int_D \left(\nabla g(y)^T \nabla v(y) \otimes \nabla v(y) + \nabla v(y)\right) (I_d + [\nabla f_j(y)]) \;dy,\\
c \;&=\; \frac{1}{|D|}\int_D \left(\nabla g(y)^T \frac{\nabla v(y) \otimes \nabla v(y)}{2} + \nabla v(y)\right) \cdot \nabla g(y)\; dy.
\end{aligned}$$

If u_ε is the solution of

$$\left\{ \begin{array}{rcll} -\text{ div } \nabla v_\varepsilon(x) \otimes \nabla v_\varepsilon(x) \nabla u_\varepsilon(x) &=& \text{div } \nabla v_\varepsilon(x) & \text{ in } \Omega, \\ u_\varepsilon &\in& H^1_0(\Omega), & \end{array} \right.$$

then the sequence $\{u_\varepsilon\}$ converges weakly to the solution u_0 of

$$\left\{ \begin{array}{rcll} -\text{ div } A_0 \nabla u_0(x) &=& 0 & \text{ in } \Omega \\ u_0 &\in& H^1_0(\Omega), & \end{array} \right.$$

provided $g^\star$ is constant.

2. For a more explicit example (which may also be considered within the framework of periodic homogenization), define the sequence of pairs $(a_\varepsilon, b_\varepsilon) : \Omega \to (1, +\infty) \times \mathbb{R}^n$ by putting

$$(a_\varepsilon(x), b_\varepsilon(x)) \;=\; (a_1, b_1)\chi_{(0,t(x))}\left(\frac{x}{\varepsilon}\cdot \overrightarrow{n}\right) \;+\; (a_2, b_2)\left(1 - \chi_{(0,t(x))}\left(\frac{x}{\varepsilon}\cdot \overrightarrow{n}\right)\right),$$

where $\chi_{(0,t(x))}(s)$ is the characteristic function of the interval $(0, t(x))$ over $(0, 1)$, extended by periodicity to $\mathbb{R}$. There exists an injective field $\phi : (1, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying the continuity condition on the interface

$$\phi(a_1, b_1) - \phi(a_2, b_2) \;\parallel\; \overrightarrow{n},$$

for the normal vector $\overrightarrow{n}$, ie the sequence of pairs $\{(a_\varepsilon, b_\varepsilon)\}$ satisfies the CGP.

In this case, we are interested on the homogenization of a Dirichlet problem for a laminate composite material of type

$$\left\{ \begin{array}{rcll} -\text{ div } a_\varepsilon(x) \nabla u_\varepsilon(x) &=& \text{div } b_\varepsilon(x) & \text{ in } \Omega \\ u_\varepsilon &\in& H^1_0(\Omega), & \end{array} \right.$$

which is associated with the family of functionals

$$I_\varepsilon(u) \;=\; \int_\Omega \left[\frac{a_\varepsilon(x)}{2}\, |\nabla u(x)|^2 + b_\varepsilon(x)\cdot \nabla u(x) \right] dx.$$

It follows from Theorem 6.5.1 that the sequence $\{I_\varepsilon\}$ is Γ-convergent to the functional I whose density ψ is defined by

$$\begin{aligned}
\psi(x,\rho) \;&=\; \min_{A,B\in\mathbb{R}^n} \Big\{ t(x)\left(\frac{a_1}{2}|A|^2 + b_1\cdot A\right) + (1-t(x))\left(\frac{a_2}{2}|B|^2 + b_2\cdot B\right) \;: \\
&\qquad\qquad \rho = t(x)A + (1-t(x))B,\;\; (B-A)\parallel \vec{n} \Big\} \;= \\
&=\; \min_{c\in\mathbb{R}} \Big\{ t(x)\left[\frac{a_1}{2}|\rho-(1-t(x))c\vec{n}|^2 + b_1\cdot(\rho-(1-t(x))c\vec{n}) \right] + \\
&\qquad + (1-t(x))\left[\frac{a_2}{2}|\rho+t(x)c\vec{n}|^2 + b_2\cdot(\rho+t(x)c\vec{n}) \right] \Big\},
\end{aligned}$$

provided the sequence $\{(a_\varepsilon, b_\varepsilon)\}$ generates the Young measure $\eta = \{\eta_x\}_{x\in\Omega}$ given by

$$\eta_x \;=\; t(x)\,\delta_{(a_1,b_1)} + (1-t)\,\delta_{(a_2,b_2)}.$$

After some computations, we conclude that the integrand ψ is a quadratic function defined by

$$\psi(x,\rho) \;=\; \frac{a_0(x)}{2}\,|\rho|^2 \;+\; g^\star(x)\cdot\rho \;+\; c(x),$$

with

$$\begin{aligned}
a_0(x) \;&=\; \frac{a_1\,a_2}{(1-t(x))\,a_1 + t(x)\,a_2}, \\
g^\star(x) \;&=\; \frac{t(x)\,a_2}{(1-t(x))\,a_1 + t(x)\,a_2}\,b_1 \;+\; \frac{(1-t(x))\,a_1}{(1-t(x))\,a_1 + t(x)\,a_2}\,b_2,
\end{aligned}$$

and

$$c(x) \;=\; \frac{(t(x)-1)\,t(x)}{(1-t(x))\,a_1 + t(x)\,a_2}\,|b_1-b_2|^2.$$

The associated homogenized equation may be written explicitly as

$$\begin{cases} -\operatorname{div}\, a_0(x)\,\nabla u_0(x) \;=\; \operatorname{div} g^\star(x) \quad \text{in } \Omega \\ \qquad u_0 \;\in\; H_0^1(\Omega). \end{cases}$$

Notice that, thanks to Theorem 6.5.1 and the laminate structure of the domain Ω, we obtain so explicit expressions for the effective coefficients.

Chapter 7

Γ-convergence of laminates with non-standard growth conditions

7.1. Introduction

As we have remarked, Γ-convergence is a very useful tool to study composite materials and determine their effective properties. Namely, if we have that the elastic internal energy density of a certain material i is $W_i(\rho)$, for a matrix variable ρ, and we mix several of these materials periodically in the unit cell Q, in a prescribed way given by characteristic functions χ_i of a partition of Q, then the behaviour of the mixture will have an energy density

$$W(x,\rho) \;=\; \sum_i \chi_i(x)\, W_i(\rho).$$

If we now perform a refinement on a scale $1/j$, the effective behaviour of the resulting composite material will be determined by the Γ- limit of the sequence of functionals associated with the densities

$$W_j(x,\rho) \;=\; \sum_i \chi_i(jx)\, W_i(\rho).$$

In this chapter, we restrict attention to the situation where the densities are given by

$$W_j(x,\rho) \;=\; \chi_{(0,t)}(jx\cdot\vec{n})\, W_1(\rho) \;+\; (1-\chi_{(0,t)}(jx\cdot\vec{n}))\, W_2(\rho),$$

where $\chi_{(0,t)}(y\cdot\vec{n})$ is the characteristic function of the interval $(0,t)$ over $(0,1)$, extended by periodicity to $\mathbb{R}$, and $\vec{n}$ is a unit vector in $\mathbb{R}^n$. A laminate composite material, with layers normal to $\vec{n}$, is considered. Notice that we may write

$$W(a_j(x),\rho) \;=\; \chi_{(0,t)}(jx\cdot\vec{n})\, W_1(\rho) \;+\; (1-\chi_{(0,t)}(jx\cdot\vec{n}))\, W_2(\rho),$$

whenever the sequence of functions $a_j : \Omega \to (1, +\infty)$ is defined by

$$a_j(x) \;=\; p\, \chi_{(0,t)}(jx \cdot \vec{n}) \;+\; q\left(1 - \chi_{(0,t)}(jx \cdot \vec{n})\right),$$

with $1 < p \le q < \infty$, and $W_1(\rho) = W(p, \rho)$, $W_2(\rho) = W(q, \rho)$ for every $\rho \in \mathbb{R}^n$. So, we investigate the explicit representation of the Γ- limit of the sequence of functionals I_j defined by

$$I_j(u) \;=\; \int_\Omega W(a_j(x), \nabla u(x))\, dx, \tag{7.1}$$

where the sequence $\{a_j\}$ is the previous one, and $W : (1, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, convex in the second variable, satisfying the non-standard growth condition

$$c|\rho|^\lambda \;\le\; W(\lambda, \rho) \;\le\; C(1 + |\rho|^\lambda) \qquad \text{for every } (\lambda, \rho) \in (1, \infty) \times \mathbb{R}^n, \tag{7.2}$$

with $C \ge c > 0$, which implies

$$c|\rho|^{a_j(x)} \;\le\; W(a_j(x), \rho) \;\le\; C(1 + |\rho|^{a_j(x)}) \quad \text{for a.e. } x \in \Omega, \text{ every } \rho \in \mathbb{R}^n, \; j \in \mathbb{N} \tag{7.3}$$

This means that, for each $j \in \mathbb{N}$, the functional I_j given by (7.1) is well defined in the generalized Sobolev space

$$W^{1,a_j(x)}(\Omega) \;=\; \left\{ u \in L^{a_j(x)}(\Omega) \;:\; \int_\Omega |\eta \nabla u_j(x)|^{a_j(x)}\, dx < +\infty, \text{ for some } \eta > 0 \right\},$$

with

$$L^{a_j(x)}(\Omega) \;=\; \left\{ u \in L^1(\Omega) \;:\; \int_\Omega |\eta\, u(x)|^{a_j(x)}\, dx < +\infty, \text{ for some } \eta > 0 \right\},$$

and $W^{1,q}(\Omega) \subset W^{1,a_j(x)}(\Omega) \subset W^{1,p}(\Omega)$. For more details on these spaces, see Appendix.

Our main result, on the Γ-convergence of functionals I_j satisfying the $a_j(x)$-growth condition, is the explicit characterization of the limit density, without any restriction on the upper exponent q. Namely, in Theorem 7.2.1 below, we conclude that the sequence $\{I_j\}$ is Γ-convergent (in the weak topology of $W^{1,p}(\Omega)$) to the functional I given by

$$I(u) \;=\; \int_\Omega \psi_t(\nabla u(x))\, dx,$$

where the density $\psi_t : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\psi_t(\rho) \;=\; \min_{A,B \in \mathbb{R}^n} \left\{ t\, W(p, A) + (1-t)\, W(q, B) \;:\; \rho = tA + (1-t)B, \;\; \vec{n} \parallel B - A \right\}.$$

The limit energy density is defined through a finite minimization problem in $\mathbb{R}^n$, or, obviously, through an one-dimensional minimization problem, ie

$$\psi_t(\rho) \;=\; \min_{c\in\mathbb{R}} \;\Big\{\; t\,W(p,\rho-(1-t)\,c\,\vec{n}) \;+\; (1-t)\,W(q,\rho+t\,c\,\vec{n}) \;\Big\}$$

for every $\rho \in \mathbb{R}^n$.

An important issue is the definition of the domain of the Γ-limit, that is the class of functions $u:\Omega\to\mathbb{R}$ for which $\int_\Omega \psi_t(\nabla u(x))\,dx < \infty$, denoted by $\Psi_t(\Omega)$. This class is an intermediate set of functions between $W^{1,q}(\Omega)$ and $W^{1,p}(\Omega)$, which depends on the fixed value $t\in(0,1)$. Indeed, depending on the value of t in the interval $(0,1)$, $\Psi_t(\Omega)$ is closer to $W^{1,p}(\Omega)$ than to $W^{1,q}(\Omega)$.

Notice that, if one considers the Γ-convergence in different topologies and with different structures on Ω, interesting and surprising phenomena may occur. For instance, the Lavrentiev phenomenon may appear when we take the chess-board structure on the plane and power-law materials, with different powers, as in [35] and [73]. Here we are considering laminated structures in $\mathbb{R}^n$ with two materials with different growth.

In the particular situation of power-law composite materials, where the density is a combination of powers of the norm of ρ, it is known that if the corresponding exponents of all of the materials are the same, e.g. $p=2=q$, then the resulting homogenized density will also be a power-law material with the same exponent. In this chapter, we focus on the situation where these exponents are different. Specifically, in the last section, we consider the example of power-law materials with energy densities

$$W(\lambda,\rho) \;=\; f(\lambda)\;|\rho|^{\lambda},$$

where $f(\lambda)\ge c>0$, and composites with energy densities

$$W(a_j(x),\rho) \;=\; f(p)|\rho|^p\,\chi_{(0,t)}(jx\cdot\vec{n}) \;+\; f(q)|\rho|^q\,(1-\chi_{(0,t)}(jx\cdot\vec{n})).$$

which are obtained by mixing layers, with the same direction, of each material. Our conclusion is that the limit energy density does not correspond to a power-law material. Indeed, according to Corollary 7.4.1, the sequence of functionals

$$I_j(u) \;=\; \int_\Omega f(a_j(x))\,|\nabla u(x)|^{a_j(x)}\,dx$$

is Γ-convergent (with respect to the weak topology of $W^{1,p}(\Omega)$) to the functional I whose density is given by

$$\psi_t(\rho) = \min_{A,B\in\mathbb{R}^n}\Big\{\,t\,f(p)\,|A|^p + (1-t)f(q)\,|B|^q \,:\, \rho = t\,A+(1-t)\,B,\; \vec{n}\parallel B-A\Big\}.$$

The domain of the Γ-limit is the set

$$\Psi_t(\Omega) =$$
$$= \left\{ u \in W^{1,p}(\Omega) \,:\, \nabla u = tA + (1-t)B, \int_\Omega [\, t\,|A(x)|^p + (1-t)|B(x)|^q \,]\, dx < \infty \right\}.$$

An interesting application of this result is the homogenization of Dirichlet problems with $a_j(x)$-laplacian of type

$$\begin{cases} -\operatorname{div}\ \left[\, a_j(x)\, f(a_j(x))\, |\nabla u_j(x)|^{a_j(x)-2} \nabla u_j(x) \,\right] &= 0 \quad \text{in } \Omega \\ u_j &= 0 \quad \text{on } \partial\Omega \end{cases}$$

discussed in the last section.

7.2. Main result

We are interested in the explicit characterization of the Γ-limit of non-linear functionals of the type

$$I_j(u) = \int_\Omega W(a_j(x), \nabla u(x))\, dx,$$

where the integrand $W(\lambda, \rho)$ is continuous in $(1, \infty) \times \mathbb{R}^n$, convex in $\mathbb{R}^n$, and satisfies the non-standard growth condition (7.2), with exponents depending on the first variable. The sequence of functions $a_j : \Omega \to (1, +\infty)$ defined by

$$a_j(x) = p\, \chi_{(0,t)}(jx \cdot \vec{n}) + q \left(1 - \chi_{(0,t)}(jx \cdot \vec{n})\right) \tag{7.4}$$

stands for a first order laminate, where $\chi_{(0,t)}$ is the characteristic function of $(0, t)$ over the interval $(0, 1)$, extended by periodicity to $\mathbb{R}$, $\vec{n} \in \mathbb{R}^n$ is the normal unit vector, and $1 < p \le q < \infty$. Thus, we focus on a family of functionals whose densities satisfy

$$c\,|\rho|^{a_j(x)} \le W(a_j(x), \rho) \le C\,(1 + |\rho|^{a_j(x)}), \qquad \text{for a.e. } x \in \Omega, \text{ every } \rho \in \mathbb{R}^n.$$

The following Theorem is the main result of this chapter.

Theorem 7.2.1 *Let Ω be an open bounded set in $\mathbb{R}^n$. Let $W(\lambda, \rho) : (1, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ be continuous in both variables, and convex in the second one, such that there exist $C \ge c > 0$,*

$$c|\rho|^\lambda \le W(\lambda, \rho) \le C(|\rho|^\lambda + 1), \qquad \textit{for every } (\lambda, \rho) \in (1, +\infty) \times \mathbb{R}^n.$$

Then the sequence of functionals I_j defined by

$$I_j(u) = \int_\Omega W(a_j(x), \nabla u(x))\, dx,$$

with $\{a_j\}$ given by (7.4), is Γ-convergent (in the weak topology of $W^{1,p}(\Omega)$) to the functional

$$I(u) \;=\; \int_\Omega \psi_t(\nabla u(x))\; dx,$$

where $\psi_t : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi_t(\rho) = \min_{A,B\in\mathbb{R}^n} \Big\{ t\,W(p,A) + (1-t)\,W(q,B) \;:\; \rho = tA + (1-t)B,\; \vec{n} \;\|\; B - A \Big\}.$$

The main achievement relies on the definition of the homogenized density ψ_t through a finite minimization problem in $\mathbb{R}^n$ under two linear restrictions, which may be even written as a minimization problem in $\mathbb{R}$, ie

$$\psi_t(\rho) \;=\; \min_{c\in\mathbb{R}} \;\Big\{\; t\,W(p, \rho - (1-t)\,c\,\vec{n}) \;+\; (1-t)\,W(q, \rho + t\,c\,\vec{n}) \;\Big\}$$

for every $\rho \in \mathbb{R}^n$.

Remark 7.2.1 *Theorem 7.2.1 may be obtained if we consider a second order laminate given by*

$$a_j(x) \;=\; \\ = p_1\chi_{(0,t)}(jx\cdot\vec{n}) + \Big(1 - \chi_{(0,t)}(jx\cdot\vec{n})\Big)\Big(p_2\chi_{(0,s)}(jx\cdot\vec{m}) + p_3\Big(1 - \chi_{(0,s)}(jx\cdot\vec{m})\Big)\Big)$$

for some unit vector $\vec{m} \in \mathbb{R}^n$, and $1 < p_1 \le p_2 \le p_3 < \infty$. In this case, the homogenized density ψ_t is defined by

$$\psi_t(\rho) \;=\; \min_{A,B,C\in\mathbb{R}^n} \{\; t\,W(p_1,A) + (1-t)s\,W(p_2,B) + (1-t)(1-s)\,W(p_3,C) \;: \\ \rho = tA + (1-t)sB + (1-t)(1-s)C, \;\; \vec{n} \,\|\, [A - (sB + (1-s)C)]\,, \;\; \vec{m} \,\|\, C - B \;\Big\}$$

for every $\rho \in \mathbb{R}^n$.

7.3. Proof

This section is entirely dedicated to the proof of the main result.

Proof. (of Theorem 7.2.1)

Step 1: (the lower limit inequality) Let $u \in W^{1,p}(\Omega)$ and $\{u_j\}$ be any weak convergent sequence to u in $W^{1,p}(\Omega)$. Let $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure associated with the sequence of pairs $\{(a_j, \nabla u_j)\}$, with support in $(1, +\infty) \times \mathbb{R}^n$. Then it holds

$$\liminf_{j\to\infty} \int_\Omega W(a_j(x), \nabla u_j(x))\; dx \;\ge\; \int_\Omega \int_{\mathbb{R}^{n+1}} W(\lambda,\rho)\; d\nu_x(\lambda,\rho)\; dx.$$

The Young measure ν may be decomposed as the product

$$\nu_x \;=\; \mu_{\lambda,x} \otimes \sigma \;=\; t\,\mu_{p,x} \otimes \delta_p + (1-t)\,\mu_{q,x} \otimes \delta_q, \qquad \text{for a.e. } x \in \Omega,$$

where the homogenous measure σ given by

$$\sigma \;=\; t\,\delta_p + (1-t)\,\delta_q,$$

is the Young measure associated with the sequence $\{a_j\}$, and μ_p, μ_q are some probability measures supported on $\mathbb{R}^n$. Thus

$$\liminf_{j\to\infty} \int_\Omega W(a_j(x), \nabla u_j(x))\; dx \;\geq\; \int_\Omega \int_{\mathbb{R}} \int_{\mathbb{R}^n} W(\lambda, \rho)\; d\mu_{\lambda,x}(\rho)\; d\sigma(\lambda)\; dx.$$

Since $W(\lambda, \cdot)$ is convex in $\mathbb{R}^n$, for every $\lambda \in (1, +\infty)$, we may apply the Jensen inequality so that

$$\int_\Omega \int_{\mathbb{R}} \int_{\mathbb{R}^n} W(\lambda, \rho)\; d\mu_{\lambda,x}(\rho)\; d\sigma(\lambda)\; dx \;\geq\; \int_\Omega \int_{\mathbb{R}} W\left(\lambda, \int_{\mathbb{R}^n} \rho\; d\mu_{\lambda,x}(\rho)\right) d\sigma(\lambda)\; dx.$$

Let us define the function $\varphi : \Omega \times (1, +\infty) \to \mathbb{R}^n$ by putting

$$\varphi(x, \lambda) \;=\; \int_{\mathbb{R}^n} \rho\; d\mu_{\lambda,x}(\rho).$$

Then

$$\liminf_{j\to\infty} \int_\Omega W(a_j(x), \nabla u_j(x))\; dx \;\geq\; \int_\Omega \int_{\mathbb{R}} W\left(\lambda, \varphi(x, \lambda)\right) d\sigma(\lambda)\; dx,$$

and

$$\nabla u(x) \;=\; \int_{\mathbb{R}} \varphi(x, \lambda)\; d\sigma(\lambda) \;=\; t\,\varphi(x, p) + (1-t)\,\varphi(x, q) \qquad \text{a.e. in } \Omega.$$

Notice that the sequence $\{\nabla u_j\}$ generates the gradient Young measure $\theta = \{\theta_x\}_{x\in\Omega}$ given by

$$\theta_x \;=\; t\,\mu_{p,x} + (1-t)\,\mu_{q,x},$$

and $\varphi(x, p)$ (resp. $\varphi(x, q)$) is the first moment of $\mu_{p,x}$ (resp. $\mu_{q,x}$). Recall that, for fixed $x \in \Omega$, if we put $A = \varphi(x, p)$, $B = \varphi(x, q)$, thus θ_x is a homogenous gradient Young measure whenever $B - A$ is parallel to the unit vector $\vec{n}$. In this way, it holds

$$\begin{aligned}\int_{\mathbb{R}} W\left(\lambda, \varphi(x, \lambda)\right) d\sigma(\lambda) \;&=\; t\,W(p, \varphi(x, p)) + (1-t)\,W(q, \varphi(x, q)) \;\geq \\ \geq\; &\min_{A,B\in\mathbb{R}^n} \left\{ t\,W(p, A) + (1-t)\,W(q, B) \,:\, \nabla u(x) = tA + (1-t)B,\; \vec{n} \parallel B - A \right\}.\end{aligned}$$

Therefore we achieve the inequality

$$\liminf_{j\to\infty} \int_\Omega W(a_j(x), \nabla u_j(x))\ dx\ \geq\ \int_\Omega \psi_t(\nabla u(x))\ dx,$$

for any weak convergent sequence $\{u_j\}$ to u in $W^{1,p}(\Omega)$.

Step 2: *(the recovering sequence)* It remains to prove that, for each $u \in W^{1,p}(\Omega)$ such that $\int_\Omega \psi_t(\nabla u(x))\, dx < \infty$, there exists a sequence $\{u_j\}$, weak converging to u in $W^{1,p}(\Omega)$, for which

$$\lim_{j\to\infty} \int_\Omega W(a_j(x), \nabla u_j(x))\ dx\ =\ \int_\Omega \psi_t(\nabla u(x))\ dx.$$

Let us consider $u \in W^{1,p}(\Omega)$ such that $\int_\Omega \psi_t(\nabla u(x))\, dx < \infty$. From the definition of ψ_t, for each $\rho \in \mathbb{R}^n$, there exists an optimal pair $(A(\rho), B(\rho)) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\begin{aligned} \psi_t(\rho) &= t\, W(p, A(\rho))\ +\ (1-t)\, W(q, B(\rho)) \\ \rho &= t\, A(\rho) + (1-t) B(\rho) \\ c(\rho)\, \vec{n} &= B(\rho) - A(\rho), \end{aligned}$$

with $c(\rho) = |B(\rho) - A(\rho)|$, ie

$$\psi_t(\rho)\ =\ t\, W(p, \rho - (1-t)c(\rho)\vec{n})\ +\ (1-t)\, W(q, \rho + tc(\rho)\vec{n}).$$

Therefore, for a.e. $x \in \Omega$, if we take $\rho = \nabla u(x)$, there exists $(A(\nabla u(x)), B(\nabla u(x))) \in \mathbb{R}^n \times \mathbb{R}^n$ so that

$$\begin{aligned} \psi_t(\nabla u(x)) &= t\, W(p, A(\nabla u(x)))\ +\ (1-t)\, W(q, B(\nabla u(x))) \\ \nabla u(x) &= t\, A(\nabla u(x)) + (1-t) B(\nabla u(x)) \\ c(x)\, \vec{n} &= B(\nabla u(x)) - A(\nabla u(x)), \end{aligned}$$

with $c(x) = |B(\nabla u(x)) - A(\nabla u(x))|$,

$$\int_\Omega \psi_t(\nabla u(x))\, dx\ =$$

$$= \int_\Omega \Big[\, t\, W(p, \nabla u(x) - (1-t)c(x)\vec{n}) + (1-t)\, W(q, \nabla u(x) + tc(x)\vec{n}) \Big]\, dx < \infty \tag{7.5}$$

and, due to the coercivity of $W(\lambda, \rho)$,

$$C \int_\Omega \Big[\, t\, |\nabla u(x) - (1-t)c(x)\vec{n}|^p + (1-t)\, |\nabla u(x) + tc(x)\vec{n}|^q \Big]\, dx \leq \int_\Omega \psi_t(\nabla u(x)) dx$$

for some constant $C > 0$.

Consider the Young measure $\eta = \{\eta_x\}_{x\in\Omega}$ supported on $\mathbb{R}^n$, given by

$$\eta_x \;=\; t\,\delta_{\nabla u(x)-(1-t)c(x)\vec{n}} \;+\; (1-t)\,\delta_{\nabla u(x)+tc(x)\vec{n}}$$

for a.e. $x \in \Omega$, whose barycenter is ∇u, and the integral

$$\begin{aligned} &\int_\Omega \int_{\mathbb{R}^n} |\rho|^p \; d\eta_x(\rho)\; dx \;= \\ &= \int_\Omega \Big[t\,|\nabla u(x) - (1-t)c(x)\vec{n}|^p + (1-t)\,|\nabla u(x) + tc(x)\vec{n}|^p \Big]\, dx \end{aligned}$$

is finite. Thus there exists a weak convergent sequence $\{u_j\}$ to u in $W^{1,p}(\Omega)$ such that η is the gradient Young measure associated with $\{\nabla u_j\}$. From (7.5) it follows

$$\lim_{j\to\infty} \int_\Omega W(a_j(x), \nabla u_j(x))\; dx \;=\; \int_\Omega \psi_t(\nabla u(x))\; dx.$$

□

7.4. Example: $a_j(x)$-laplacian

The homogenization of layers composite power-law materials is the main application of Theorem 7.2.1 which we are concerned on. Namely, consider the integrand $W : (1,+\infty)\times\mathbb{R}^n \to \mathbb{R}$ defined by

$$W(\lambda,\rho) \;=\; f(\lambda)\; |\rho|^\lambda,$$

for some continuous function $f:\mathbb{R}\to(0,+\infty)$, so that there exist $C \ge c > 0$

$$c\,|\rho|^\lambda \;\le\; f(\lambda)\,|\rho|^\lambda \;\le\; C\,(1+|\rho|^\lambda), \qquad \text{for every } \rho\in\mathbb{R}^n \text{ and } \lambda\in(1,\infty).$$

Corollary 7.4.1 *The sequence of functionals*

$$I_j(u) \;=\; \int_\Omega f(a_j(x))\; |\nabla u(x)|^{a_j(x)}\; dx$$

is Γ-convergent (in the weak topology of $W^{1,p}(\Omega)$) to the functional I whose density $\psi_t : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi_t(\rho) = \min_{A,B\in\mathbb{R}^n} \Big\{ t\, f(p)\, |A|^p + (1-t)\, f(q)\, |B|^q \;:\; \rho = tA + (1-t)B,\; \vec{n} \parallel B - A \Big\}.$$

The limit energy density ψ_t may be even written as an one-dimensional minimization problem

$$\psi_t(\rho) \;=\; \min_{c\in\mathbb{R}} \;\Big\{\; t\, f(p)\, |\rho - c\,(1-t)\,\vec{n}|^p + (1-t)\, f(q)\, |\rho + c\,t\,\vec{n}|^q \;\Big\}.$$

Moreover, this explicit characterization enables to know how looks like the homogenized equation of optimality associated with this family of functionals. Indeed, we may look at the equations of optimality associated with the sequence of functionals I_j, ie the Dirichlet problem with the $a_j(x)$-laplacian

$$\begin{cases} -\text{ div }\left[a_j(x)\, f(a_j(x))\, |\nabla u_j(x)|^{a_j(x)-2}\, \nabla u_j(x)\right] & = \ 0 \ \text{ in } \Omega \\ u_j & = \ 0 \ \text{ on } \partial\Omega, \end{cases} \tag{7.6}$$

which may be written as

$$\begin{cases} -\text{ div }\left[p\, f(p)\, |\nabla u_j^1(x)|^{p-2}\, \nabla u_j^1(x)\right] & = \ 0 \quad \text{in } \Omega_j \\ -\text{ div }\left[q\, f(q)\, |\nabla u_j^2(x)|^{q-2}\, \nabla u_j^2(x)\right] & = \ 0 \quad \text{in } \Omega_j^c \\ \nabla u_j^1 - \nabla u_j^2 & \parallel \ \vec{n} \quad \text{on } \partial\Omega_j \cap \partial\Omega_j^c \\ u_j^1,\ u_j^2 & = \ 0 \quad \text{on } \partial\Omega, \end{cases}$$

with

$$\Omega_j \ = \ \left\{ x \in \Omega \ : \ \chi_{(0,t)}(jx\cdot\vec{n}) \ = \ 1 \right\} \quad \text{and} \quad \Omega_j^c \ = \ \left\{ x \in \Omega \ : \ \chi_{(0,t)}(jx\cdot\vec{n}) \ = \ 0 \right\}.$$

From the previous corollary, it follows that there exists a sequence of solutions $\{u_j\}$ weak converging to the minimum point u of the Γ-limit, which is solution of the equation

$$\begin{cases} -\text{ div } \nabla\psi_t(\nabla u(x)) & = \ 0 \quad \text{in } \Omega \\ u & = \ 0 \quad \text{on } \partial\Omega. \end{cases}$$

Let us compute the gradient of ψ_t. Fixed $\rho \in \mathbb{R}^n$, if $c = c(\rho) \in \mathbb{R}$ is the minimum point, then

$$\begin{aligned} \nabla\psi_t(\rho) \ = \ & t\, f(p)\, p\, |\rho + c\,(1-t)\,\vec{n}|^{p-2}(\rho + c\,(1-t)\,\vec{n})\,(I_d + (1-t)\, diag(n\otimes\nabla c)) \ + \\ & + \ (1-t)\, f(q)\, q\, |\rho - c\, t\,\vec{n}|^{q-2}(\rho - c\, t\,\vec{n})\,(I_d - t\, diag(n\otimes\nabla c))\,, \end{aligned}$$

where I_n is the $n\times n-$identity matrix, and $diag(n\otimes\nabla c)$ is a $n\times n-$diagonal matrix with values $n_i\frac{\partial c}{\partial\rho_i}(\rho)$, $1\le i\le n$. Clearly the homogenized equation is neither a p- nor a q-laplacian, but instead it is similar to the convex combination of both.

Besides, an interesting issue on the definition of the limit density ψ_t is the definition itself of the domain of the Γ-limit, ie the subset of $W^{1,p}(\Omega)$ where the functional I is finite, denoted by $\Psi_t(\Omega)$, for each $t\in(0,1)$. Indeed, in the particular case where the coefficient $f(\lambda)\equiv 1$, the study of the Γ-convergence of functionals

$$I_j(u) \ = \ \int_\Omega |\nabla u(x)|^{a_j(x)}\, dx$$

leads to the subset

$$\Psi_t(\Omega) \ = \\ = \left\{ u \in W^{1,p}(\Omega) : \nabla u = tA + (1-t)B,\ \int_\Omega \left[\, t|A(x)|^p + (1-t)|B(x)|^q\,\right] dx < \infty \right\}$$

which is an intermediate class of functions between $W^{1,q}(\Omega)$ and $W^{1,p}(\Omega)$. Moreover, depending on the value of $t \in (0,1)$, the set $\Psi_t(\Omega)$ is closer to $W^{1,q}(\Omega)$ than $W^{1,p}(\Omega)$.

Appendix A

A.1. Slicing decomposition of measures

Definition A.1.1 *Let $E \subset \mathbb{R}^m$, $F \subset \mathbb{R}^n$ be open sets, σ be a positive Radon measure on E and μ_λ be a finite Radon measure on F. Assume that*

$$\int_{\overline{E}} |\mu_\lambda(F)| \; d\sigma(\lambda) \; < \; \infty,$$

for all open $\overline{E} \subset\subset E$. The generalized product of measures is the Radon measure $\sigma \otimes \mu_\lambda$ on $E \times F$ for which

$$(\sigma \otimes \mu_\lambda)(B) \; = \; \int_E \left(\int_F \chi_B(\lambda, \rho) \; d\mu_\lambda(\rho) \right) d\sigma(\lambda)$$

for all $B \subset \mathcal{B}(K \times F)$, where $K \subset E$ is any compact set.

Notice that the integration formula

$$\int_{E \times F} f(\lambda, \rho) \; d(\sigma \otimes \mu_\lambda)(\lambda, \rho) \; = \; \int_E \left(\int_F f(\lambda, \rho) \; d\mu_\lambda(\rho) \right) d\sigma(\lambda)$$

holds for every bounded Borel function $f : E \times F \to \mathbb{R}$ with support in $\overline{E} \times F$, with $\overline{E} \subset\subset E$.

Lemma A.1.1 (See [8]) *Let $E \subset \mathbb{R}^m$ and $F \subset \mathbb{R}^n$ be open sets, ν be a positive Radon measure on $E \times F$ and σ be its projection onto E, which is also a Radon measure. Then, for σ-a.e. $\lambda \in E$, there exists a probability measure μ_λ on F such that, for every bounded continuous function f,*

1. *the map*

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{R} \\ \lambda & \longmapsto & \displaystyle\int_F f(\lambda, \rho) \; d\mu_\lambda(\rho) \end{array}$$

is σ-measurable;

2. it holds

$$\int_{E\times F} f(\lambda,\rho)\ d\nu(\lambda,\rho)\ =\ \int_E \left(\int_F f(\lambda,\rho)\ d\mu_\lambda(\rho)\right)\ d\sigma(\lambda).$$

Definition A.1.2 *Let $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ be measure spaces, and let $f: X\to Y$ be such that $f^{-1}(F)\in\mathcal{E}$ whenever $F\in\mathcal{F}$. For any positive measure μ on $(X,\mathcal{E})$, we say that $f_\sharp\mu$ is the push-forward measure of μ through the function f if it is defined in $(Y,\mathcal{F})$ by*

$$f_\sharp\mu(F)\ =\ \mu\left(f^{-1}(F)\right),\quad \forall\ F\in\mathcal{F}.$$

From the definition of push-forward measure $f_\sharp\mu$, if w is a function in Y summable with respect to $f_\sharp\mu$, then

$$\int_Y w(\rho)\ d(f_\sharp\mu)(\rho)\ =\ \int_X w(f(\lambda))\ d\mu(\lambda).$$

A.2. Carathéodory and convex functions

Theorem A.2.1 (See [28, Lusin Theorem]) *A function $f:\Omega\to\overline{\mathbb{R}}$ is measurable if and only if, for every compact set $K\subset\Omega$ and all $\epsilon>0$, there exists a compact set $K_\epsilon\subset K$ such that $|K\setminus K_\epsilon|\le\epsilon$ for which the restriction of f to K_ϵ is continuous.*

Definition A.2.1 *A function $f:\Omega\times\mathbb{R}^d\to\overline{\mathbb{R}}$ is a Carathéodory function if*

i) $f(\cdot,\rho)$ is measurable in Ω, for every $\rho\in\mathbb{R}^d$,

ii) $f(x,\cdot)$ is continuous in $\mathbb{R}^d$, for a.e. $x\in\Omega$.

Theorem A.2.2 (See [28, Scorza-Dragoni theorem]) *A function $f:\Omega\times\mathbb{R}^d\to\overline{\mathbb{R}}$ is a Carathéodory function if and only if, for all compact sets $K\subset\Omega$ and all $\epsilon>0$, there exists a compact set $K_\epsilon\subset K$ such that $|K\setminus K_\epsilon|\le\epsilon$ for which the restriction of f to $K_\epsilon\times\mathbb{R}^d$ is continuous.*

Theorem A.2.3 (See [64, Nemytskiĭ operator]) *If $\varphi:\Omega\times\mathbb{R}^m\to\mathbb{R}^d$ is a Carathéodory map, and $1\le p<+\infty$, $1\le q<+\infty$, then the following statements are equivalent:*

i) if $\{u_j\}$ is bounded in $L^p(\Omega;\mathbb{R}^m)$, then $\{\varphi(\cdot,u_j(\cdot))\}$ is bounded in $L^q(\Omega;\mathbb{R}^d)$,

ii) there exist $a\in L^q(\Omega)$ and $b\in\mathbb{R}$ such that

$$|\varphi(x,\lambda)|\ \le\ a(x)+b\,|\lambda|^{p/q},$$

iii) if $\{u_j\}$ *is strong convergent in* $L^p(\Omega;\mathbb{R}^m)$, *then* $\{\varphi(\cdot,u_j(\cdot))\}$ *is strong convergent in* $L^q(\Omega;\mathbb{R}^d)$,

Theorem A.2.4 (See [28]) *Let* V *be a real vector space,* $g : V \to \overline{\mathbb{R}}$ *be a sub-linear*[1] *function,* U *a vector subspace of* V, *and* $f : U \to \overline{\mathbb{R}}$ *a linear function such that* $f < g$. *Then there exists a linear function* $\tilde{f} : V \to \overline{\mathbb{R}}$ *such that* $\tilde{f} = f$ *in* U, *and* $\tilde{f} < g$.

Corollary A.2.5 *Let* V *be a normed space,* U *a topological vector subspace, and* $f : U \to \overline{\mathbb{R}}$ *a continuous linear functional. Then* f *can be extended into a continuous linear functional over* V *with the same norm.*

Lemma A.2.6 (See [63]) *Let* Ω *have Lipschitz boundary, and* $u \in W_0^{1,\infty}(\Omega)$. *Then there exist piecewise affine functions* $u_j \in W_0^{1,\infty}(\Omega)$ *such that* $\{u_j\}$ *converges strongly to* u *in* $W^{1,p}(\Omega)$, *for any* $1 \le q < \infty$, *and* $\|u_j\|_{L^\infty} \le C\|\nabla u\|_{L^\infty}$, *for some constant* C *independent of* j.

Theorem A.2.7 (See [49]) *Let* μ *be a positive Radon measure with support in* $\mathbb{R}^d$, *such that* $\mu(\Omega) = 1$, *for some* Ω, *and let* f *be a vector field in* $L^1(\Omega;\mu)$, *such that* K *is a convex subset of* $\mathbb{R}^d$ *and* $f(x) \in K$ *for* μ*-a.e.* $x \in \Omega$. *Let* $\varphi : K \to \mathbb{R}$ *be a convex function. Then*

$$\varphi\left(\int_\Omega f \; d\mu\right) \;\le\; \int_\Omega \varphi(f) \; d\mu.$$

Definition A.2.2 *The convexification of* $\varphi : \Omega \to \mathbb{R}$ *is the function* $C\varphi$ *defined by*

$$C\varphi(y) \;=\; \sup\{\, g(y) \;:\; g \textit{ is a convex function and } g(x) \;\le\; \varphi(x)\ \forall x \in \Omega\},$$

for every $y \in \Omega$, *ie it is the greatest convex function not greater than* φ.

Lemma A.2.8 (See [49]) *Let* $\varphi : \Omega \to \mathbb{R}^d$ *be a continuous function. Then, for every* $y \in \Omega$,

$$C\varphi(y) \;=\; \inf\left\{\, \int_{\mathbb{R}^d} \varphi(\lambda)\; d\eta(\lambda) \;:\; \eta \in M_y \,\right\},$$

where

$$M_y \;=\; \Big\{ \nu \textit{ probability measure over } \mathbb{R}^d : \textit{there exists } z \in L^g(\Omega;\mathbb{R}^d) \qquad \textit{such that } \; \nu = \overline{\delta}_{z(x)} \textit{ and } y = \frac{1}{|\Omega|}\int_\Omega z(x)\; dx \Big\},$$

[1] $f : V \to \overline{\mathbb{R}}$ is said to be sub-linear if $g(\alpha u) = \alpha g(u)$, $\forall \alpha > 0$, and $g(u+v) \le g(u) + g(v)$, $\forall u, v \in V$.

and

$$L^g(\Omega;\mathbb{R}^d) \;=\; \left\{ z:\Omega\to\mathbb{R}^d \ \textit{measurable} \ :\ \int_\Omega g(|z(x)|)\ dx<\infty \right\}.$$

Notice that the above infimum is attained when φ is a coercive function. M_y is the set of all probability measures ν with support in $\mathbb{R}^d$ for which there exists a function z in $L^g\left(\Omega;\mathbb{R}^d\right)$ such that

- ν is the homogeneous Young measure associated with the sequence $(z)_j$, ie

$$\langle \nu,\varphi\rangle \;=\; \frac{1}{|\Omega|}\int_\Omega \varphi(z(x))\ dx \;=\; \int_{\mathbb{R}^d}\varphi(\lambda)\ d\nu(\lambda);$$

- $y=\frac{1}{|\Omega|}\int_\Omega z(x)\ dx$, ie the value $y\in\mathbb{R}^d$ is the average of z over Ω.

Thus M_y is the set of all homogeneous Young measures associated with constant sequences $(z)_j\subset L^g\left(\Omega;\mathbb{R}^d\right)$ whose average over Ω is y.

Proposition A.2.1 (See [49]) *M_y is a convex set of probability measures.*

A.3. Curl and Green's formula

For $1<p<\infty$, the dual space of $W_0^{1,p}(\Omega)$, denoted by $W^{-1,p'}(\Omega)$, may be characterized as the completion of $L^{p'}(\Omega)$ with respect to the norm

$$\|v\|_{W^{-1,p'}} \;=\; \sup_{\|\varphi\|_{W_0^{1,p}}\le 1}\left|\int_\Omega v(x)\,\varphi(x)\ dx\right|,$$

which means $L^{p'}(\Omega)$ is a dense subset of $W^{-1,p'}(\Omega)$. So the imbedding $L^{p'}(\Omega)\subset W^{-1,p'}(\Omega)$ is compact[2], and for any $v\in L^{p'}(\Omega),\ \varphi\in W_0^{1,p}(\Omega)$ it holds

i) $\|v\|_{W^{-1,p'}} \;\le\; \|v\|_{L^{p'}}$,

ii) $\left|\int_\Omega v(x)\,\varphi(x)\ dx\right| \;\le\; \|v\|_{W^{-1,p'}}\,\|\varphi\|_{W_0^{1,p}}$.

Moreover, $V\in W^{-1,p}(\Omega;\mathbb{R}^d)$ if and only if there exist $v_1,\dots,v_n\in L^p(\Omega;\mathbb{R}^d)$ such that

$$\langle V,\varphi\rangle \;=\; \sum_{i=1}^n\int_\Omega v_i(x)\,\frac{\partial\varphi}{\partial x_i}(x)\ dx \qquad \text{for every } \varphi\in W_0^{1,p}(\Omega;\mathbb{R}^d).$$

See [2] for more details.

[2] Any bounded sequence in $L^{p'}(\Omega)$ has a strongly convergent subsequence in $W^{-1,p'}(\Omega)$.

Definition A.3.1 *Let $v \in L^p(\Omega;\mathbb{R}^n)$. The skew-symmetric matrix* curl v *is a continuous linear operator in $W^{-1,p}(\Omega)$ defined by the dual pair*

$$\begin{aligned}\langle \text{curl } v, \varphi\rangle &= \int_\Omega \left(\frac{\partial v_i}{\partial x_j}(x) - \frac{\partial v_j}{\partial x_i}(x)\right)_{i,j} \varphi(x)\, dx = \\ &= \int_\Omega [\, v(x)\otimes \nabla\varphi(x) - \nabla\varphi(x)\otimes v(x)\,]\, dx = \\ &= \left(\int_\Omega \left[v_i(x)\frac{\partial \varphi}{\partial x_j}(x) - \frac{\partial \varphi}{\partial x_i}(x) v_j(x)\right] dx\right)_{i,j} \in \mathbb{R}^{n\times n}\end{aligned}$$

for every $\varphi \in W_0^{1,p'}(\Omega)$. Its norm is defined by

$$\|\,\text{curl } v\,\|_{W^{-1,p}} = \sup_{\|\varphi\|_{W_0^{1,p'}}\leq 1} |\,\langle \text{curl } v, \varphi\rangle\,|.$$

Notice that

- for some $v \in L^p_{loc}(\mathbb{R}^n;\mathbb{R}^n)$, $\text{curl}\, v = 0$ in $\mathbb{R}^n$ if
$$\int_{\mathbb{R}^n} \left[v_i(x)\frac{\partial \varphi}{\partial x_j}(x) - \frac{\partial \varphi}{\partial x_i}(x) v_j(x)\right] dx = 0, \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n),$$
for every $i, j = 1, \dots, n$.
- if $v = \nabla u$, for some $u \in W^{1,p}(\Omega)$, then $\text{curl}\, v = \text{curl}\, \nabla u = 0$ and we say that v is irrotational.
- if $v \in L^p(\Omega;\mathbb{R}^n)$, where Ω is a bounded open set with Lipschitz boundary and simply connected, and $\text{curl}\, v = 0$, then there exists $u \in W^{1,p}(\Omega)$ such that $v = \nabla u$.

(For more details see, for instance [29, 34].)

The generalized Green formula for functions of bounded variation[3] stands as follows. (See [8, 10].)

Theorem A.3.1 (generalized Green's formula) *Let $\Omega \subset \mathbb{R}^n$ have Lipschitz boundary Γ. There exists a linear continuous map $\gamma : BV(\Omega) \to L^1_{\mathcal{H}^{n-1}}(\Gamma)$ such that*

i) for all $u \in C(\overline{\Omega}) \cap BV(\Omega)$, $\gamma(u) = u\,|_\Gamma$,

ii) for all $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$,

$$\int_\Omega \varphi(x)\cdot dDu(x) = -\int_\Omega \text{div}\,(\varphi(x))\, u(x)\, dx + \int_\Gamma \gamma(u(x))\varphi(x)\cdot \vec{n}(x)\, d\mathcal{H}^{n-1}(x)$$

where $\vec{n}(x)$ is the outer normal vector at $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

[3] $BV = \{u \in L^1(\Omega) : |Du|(\Omega) < \infty\}$ is the set of all L^1-functions whose distributional derivative $Du = \nabla u \mathcal{L}^n + D^s u$ is a measure with finite variation in Ω.

A.4. Generalized Lebesgue and Sobolev spaces

In this section we introduce the generalized Lebesgue and Sobolev spaces, and discuss some of their properties, like reflexivity, separability, and density of smooth functions. For more details, see [65] and the references therein. The generalized Sobolev spaces are the natural spaces where the functionals I_j, considered throughout Chapter 7, are defined. We give a general idea of these spaces, even though the following results will not be used.

Definition A.4.1 *Let $p : \Omega \to (1, \infty)$ be a measurable function such that $1 < p_1 \leq p(x) \leq p_2 < \infty$, for a.e. $x \in \Omega$. The generalized Lebesgue space $L^{p(x)}(\Omega; \mathbb{R}^d)$ is defined by*

$$L^{p(x)}(\Omega; \mathbb{R}^d) \;=\; \left\{ u \in L^1(\Omega; \mathbb{R}^d) \;:\; \int_\Omega |\eta\, u(x)|^{p(x)}\; dx < \infty,\; for\ some\ \eta > 0 \right\}.$$

The mapping

$$\|u\|_{L^{p(x)}} \;=\; \inf \left\{ \eta > 0 \;:\; \int_\Omega \left| \frac{u(x)}{\eta} \right|^{p(x)} dx \;\leq\; 1 \right\}$$

is a norm in the space $L^{p(x)}(\Omega; \mathbb{R}^d)$, such that

$$\|u\|_{L^{p(x)}} \;\leq\; 1 - \frac{1}{p_2} + \frac{1}{p_1} \int_\Omega |u(x)|^{p(x)}\; dx.$$

Moreover, the space $L^{p(x)}(\Omega; \mathbb{R}^d)$ endowed with this norm is a Banach space. Let $p'(x)$ be the conjugate function of $p(x)$, defined by $p'(x) = \frac{p(x)}{p(x)-1}$ in Ω. The following generalized Hölder inequality is true in these spaces.

Lemma A.4.1 (See [65]) *Let $p : \Omega \to (1, \infty)$ be a measurable function such that $1 < p_1 \leq p(x) \leq p_2 < \infty$, for a.e. $x \in \Omega$. Then*

$$\int_\Omega |u(x) v(x)|\; dx \;\leq\; \left(1 + \frac{1}{p_1} - \frac{1}{p_2}\right) \|u\|_{L^{p(x)}} \|v\|_{L^{p'(x)}},$$

for every $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

Lemma A.4.2 (See [65])

1. *The dual space of $L^{p(x)}(\Omega)$ is the space $L^{p'(x)}(\Omega)$ if and only if $p \in L^\infty(\Omega; (1, \infty))$.*
2. *If $p \in L^\infty(\Omega; (1, \infty))$, then the set $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$.*

Now we may introduce the generalized Sobolev spaces.

Definition A.4.2 *Let $p : \Omega \to (1, \infty)$ be a measurable function such that $1 < p_1 \leq p(x) \leq p_2 < \infty$, for a.e. $x \in \Omega$, and $m \in \mathbb{N}$. The generalized Sobolev space $W^{m,p(x)}(\Omega)$ is defined by*

$$W^{m,p(x)}(\Omega) \; = \; \left\{ \; u \in L^{p(x)}(\Omega) \; : \; |D^\alpha u(x)| \in L^{p(x)}(\Omega), \; for \; all \; 0 \leq |\alpha| \leq m \; \right\}.$$

The space $W^{m,p(x)}(\Omega)$, endowed with the norm

$$\|u\|_{W^{m,p(x)}} \; = \; \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(x)}},$$

is a Banach space. The embeddings

$$L^{p(x)}(\Omega) \; \subset \; L^{q(x)}(\Omega) \qquad \text{and} \qquad W^{m,p(x)}(\Omega) \; \subset \; W^{m,q(x)}(\Omega)$$

are true if and only if $p(x) \leq q(x)$ a.e. in Ω.

A.5. Auxiliary lemmas

Lemma A.5.1 (See [9]) *If $f(\epsilon, \delta) : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \overline{\mathbb{R}}$, then there exists a sequence $\delta(\epsilon) \searrow 0$ as $\epsilon \searrow 0$ such that*

$$\limsup_{\epsilon \searrow 0} \; f(\epsilon, \delta(\epsilon)) \; \leq \; \limsup_{\epsilon \searrow 0} \; \limsup_{\delta \searrow 0} \; f(\epsilon, \delta).$$

Lemma A.5.2 (See [49]) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $|\partial\Omega| = 0$ and let $N \subset \Omega$ be a null measure subset. Let $(f_j)_j$ be a sequence of functions in $L^p(\Omega)$ and $(r_k)_k$ be a sequence of functions $r_k : \Omega \setminus N \to \mathbb{R}^+$. Then there exists a set of points $\{a_{ki}\} \subset \Omega \setminus N$ and positive numbers $\{\varepsilon_{ki}\}$, $\varepsilon_{ki} \leq r_k(a_{ki})$, such that*

1. *$\left\{a_{ki} + \varepsilon_{ki}\overline{\Omega}\right\}_i$ is a family of pairwise disjoint sets, for every $k \in \mathbb{N}$;*
2. *$\overline{\Omega} = \bigcup_i \left(a_{ki} + \varepsilon_{ki}\overline{\Omega}\right) \cup N_k$, with $|N_k| = 0$, for every $k \in \mathbb{N}$;*
3. *for every $\xi \in L^{p'}(\Omega)$ and $j \in \mathbb{N}$,*

$$\int_\Omega \xi(x) f_j(x) \; dx \; = \; \lim_{k \to \infty} \; \sum_{i=1}^{\infty} \; f_j(a_{ki}) \; \int_{a_{ki} + \varepsilon_{ki}\Omega} \xi(x) \; dx.$$

The previous lemma is true for any positive Radon measure σ, with open bounded support, instead of the Lebesgue measure.

Lemma A.5.3 (See [23]) *The sequence* $\{f_j\}$ *is weak convergent to* f *in* $L^1\left(\Omega;\mathbb{R}^d\right)$ *if and only if*

1. $\|f_j\|_{L^1} \;\leq\; K,$ *for all* $j \in \mathbb{N}$*;*
2. *for every cube* $D \subset \Omega$*,*
$$\lim_{j\to\infty}\int_D |f_j(x) - f(x)|\;dx \;=\; 0;$$
3. (equi-integrability) *for every* $\varepsilon > 0$*, there exists* $\lambda = \lambda(\varepsilon) > 0$*, such that if* E *is a measurable subset of* Ω *with* $|E| < \lambda(\varepsilon)$*, then*
$$\int_E |f_j(x)|\;dx \;<\; \varepsilon, \qquad \text{for every } j \in \mathbb{N}.$$

Lemma A.5.4 (See [23]) *The bounded sequence* $\{f_j\}$ *in* $L^1(\Omega)$ *is equi-integrable if and only if*

$$\lim_{k\to\infty}\left(\sup_{j\in\mathbb{N}}\int_{E_j^k} |f_j(x)|\;dx\right) \;=\; 0,$$

with $E_j^k = \{\, x \in \Omega \;:\; |f_j(x)| \geq k \,\}$*.*

Lemma A.5.5 (De La Vallée-Poussin criterion) (See [23]) *The sequence* $\{f_j\}$ *is equiintegrable in* $L^1\left(\Omega;\mathbb{R}^d\right)$ *if and only if*

$$\sup_{j\in\mathbb{N}}\int_\Omega \psi\left(|f_j(x)|\right)\;dx \;<\; \infty, \tag{A.1}$$

for some continuous function $\psi : [0,\infty) \to \mathbb{R}$ *such that*

$$\lim_{\lambda\to\infty}\frac{\psi(\lambda)}{\lambda} \;=\; \infty. \tag{A.2}$$

Lemma A.5.6 (See [40]) *Let* $f : \Omega \times \mathbb{R}^{d\times n} \to \mathbb{R}$ *be a quasiconvex function such that, for some* $c_1 > 0$*,*

$$0 \;\leq\; f(x,A) \;\leq\; c_1\left(1 + |A|^q\right) \qquad \text{for a.e. } x \in \Omega \text{ and every } A \in \mathbb{R}^{d\times n}.$$

Then there exists a constant $c_2 > 0$ *for which*

$$|f(x,A_1) - f(x,A_2)| \;\leq\; c_2\left(1 + |A_1| + |A_2|\right)^{q-1}|A_1 - A_2| \qquad \forall A_1, A_2 \in \mathbb{R}^{d\times n}.$$

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