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UNEMPLOYMENT, OPTIMAL WAITING AND QUEUES

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Abstract

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1 INTRODUCTION

In this article we portray the state of a technically diverse economy as the outcome of a probabilistic equilibrium. We use an optimization approach embedded in a stochastic Markovian framework. Firms and job searchers are ex ante and ex post heterogeneous. Each firm finds its optimal strategy given the distributions of technical expertise among entrepreneurial options and of relative abilities among job searchers. A strategy encompasses a technology, and the labor qualification to aim for, together with a salary offer and a duration for the chosen business. The connection between advertising firms and job searchers is embedded within a stochastic dynamical system, also encompassing the creation and destruction of firms. The steady state of the system is a distribution of unemployment and advertised vacancies which, given the strategies of all firms, accounts for the complete equilibrium solution. We analyze how a variation in the distribution of relative labor abilities affects the levels of unemployment, advertised vacancies, salaries and the duration of jobs.

We assume a closed network of firms, like that of Blanchard and Diamond (1994), where a given number of options is divided up into three possible positions for each stage in the life of all firms. But while those authors obtain the equilibrium solution of the network by equalizing the mean flows in and out of every node, we find endogenously the distribution ruling the equilibrium behavior of the system and are able to determine moments of any order. We also get the steady state distributions of the matching, creation and destruction functions of firms. We analyze the random dependence between unemployment and advertised vacancies and obtain the usual Beveridge relationship between their mean values as a particular case. We prove that, for large economies, there may exist a positive covariance between advertised vacancies and unemployment. Therefore a policy designed to decrease expected unemployment, through the creation of new advertised jobs, need not be judged a failure if the data shows an increase in unemployment figures. With our approach it becomes possible to tell apart changes in the state of the economy due to variations in policy parameters and occurrences due to the inherent randomness of events.

Aghion and Howitt (1994) also use a network for the options of firms. Like them we allow firms to leave the market when, given the technology adopted, the increasing price of human capital eliminates all profits. They assume innovations, which are immediately adopted, occur at a given Poisson rate and determine the creation of new advertising firms. In contrast, we model when it is optimal for new firms to appear in the market and the type of technology they select from a given menu of profitable options. At any point in time a menu encompasses a continuum of technologies. The technological choice of each firm depends on the its ex ante ability to absorb new technology and in this regard firms differ. In J. Zeira (1998) technological innovations are related to initial productivity. To assume that, except for the rights given by patents, all new techniques immediately supersede the old is counterfactual. In most economies, and at all times, many firms of different degrees of technical sophistication and labor expertise, start business and coexist with various degrees of profitability. We model the adoption of technique and the rate at which new advertising firms appear as independent and endogenous decisions. Both take into account how the distributions of initial expertise in unproductive options, and of relative labor qualifications among job searchers, affect the character of the labor market. We thus fall in line with recent articles in the literature relating labor variables to technology, wages and employment. This is generally done by introducing the idea of skill biased technical change either directly, through the availability of highly qualified labor in the economy, or through the level of R&D in firms (See for instance D. Acemoglu (1998), E. Bergman, J. Bound and S. Machin (1998) and S. Machin and J. Van Reeden, (1998)). In this literature new methods of production, requiring high skilled labor are created by individuals who posses those skills. This widens the gap between their labor prospects and those of the lowly skilled. The ratio between high and low skilled labor explains for Acemoglu (1999) the type of new jobs firms create. In this article we neither predetermine nor model how innovations occur but let the skill composition of the labor force play a role in the choice of technique by each new firm. It also informs when a firm appears as an advertising option, given the succession of profitable techniques open over time.

The character of the labor market partly depends on a possible mismatch between the distributions of qualification requirements of firms and the labor abilities of job searchers. In Acemoglu (1997), it depends on the spectrum of labor productivity and on whether or not the technologies adopted can efficiently employ high skilled labor. The empirical evidence of countries having or lacking markets for high skill complementary technologies seems to validate this approach. The US is often cited as a country with large markets of this type while the economies of Eastern Europe seem to be in dire need of them. If the capacity of potential firms to adopt technologically sophisticated methods is very limited, an improvement in the general levels of ability does not encourage firms to create adequate employment for well educated workers and eventually forces them to emigrate. Educational policies are successful in increasing employment only when they result in an alignment with the existing technical levels of firms.

As in Kahn (1987) we take into account the use and effectiveness of incentive salary schemes offered by firms. We use a continuum of abilities and prove that when the unemployed are willing to work in qualifications below their own (as in Ours and Ridder (1995)), and firms offer high salaries to attract the better qualified, there may be an even wider gap than usually predicted between the salaries of low and high skilled individuals. We show that an improvement in qualification may result in higher salaries and job durations but it need not decrease employment. Whether or not this is the case depends on the technologies adopted, on the induced changes in the fluidity of job contracting and on how the average hired qualification reacts to changes in educational levels and salaries. The overall response is very sensitive to the character of the existing distribution of relative labor qualifications.

The article is organized as follows. In the next section we describe labor supply. In section three we account for the optimal decisions of firms and obtain the corresponding market rates. In section four we incorporate these rates into a Markovian queuing network and determine the steady state probability distribution associated to the stochastic equilibrium of the economy. In section five we analyze the consequences of a change in the shape of the distribution of qualified labor on the equilibrium solution. All proofs are given in the appendix.

2 CHARACTERIZATION OF UNEMPLOYED LABOR

Consider an economy with a constant number L of individual workers at all times. We denote by $E(t)$ and $U(t)$ the numbers of employed and unemployed workers at t. That is,

$$L = E(t) + U(t), \quad \forall t. \tag{1}$$

All workers have a unit of labor of a particular qualification. When employed, it matches with one firm. We assume that only unemployed individuals search for jobs. Let $\Delta_u(t)$ and $\theta_u(t)$ denote respectively the absolute and relative levels of labor qualification of a particular unemployed individual at t. We assume, $\Delta_u^a(t) \leq \Delta_u(t) \leq \Delta_u^b(t)$, where the bounds are the minimum and maximum levels of $\Delta_u(t)$ at t. These bounds generally change over time. However since $\theta_u(t) = (\Delta_u(t) - \Delta_u^a(t))/(\Delta_u^b(t) - \Delta_u^a(t))$, the support of $\theta_u(t)$ is $[0, 1]$, $\forall t$. We think of $\Delta_u(t)$ and $\theta_u(t)$ as stochastic processes and assume the latter to have a limiting distribution $Lim_{t\to\infty} \Pr(\theta_u(t) \leq \theta_u) = F(\theta_u; \overline{\alpha})$, where $\overline{\alpha}$ is a vector of parameters. In this article we analyze the behavior of the labor market in steady state. In consequence the relative qualification of unemployed labor is defined by the random variable $\Theta \in [0, 1]$ with distribution $F(\theta_u, \overline{\alpha})$.

The level of absolute qualification at $t = 0$ corresponding to $\Theta = \theta_u$ is,

$$\Delta_u(0) = \Delta_u(\theta_u, 0). \tag{2}$$

We associate a reservation wage to each absolute labor qualification. It is assumed to be an increasing and differentiable function $W_R(\Delta_u(0)) = W_R(\theta_u, 0)$, $\forall \theta_u \in [0, 1]$. Without loss of generality we normalize this so that $W_R(\theta_u, 0) = \theta_u$ at $t = 0$. All reservation wages are assumed to change by $\sigma(T)$ if absolute qualifications change over time. We also assume that the reservation wage of any absolute qualification increases over time by $\zeta(t) = \exp(gt)$, where g is the productivity growth rate for that type of labor. That is, at $t = T + \tau$,

$$W_R(\theta_u, T+\tau) = \theta_u \exp(g\tau)\sigma(T), \forall\theta_u \in [0,1] \tag{3}$$

where (3) gives the reservation wage corresponding to the absolute qualification $\Delta_u(T)$ at $t = T + \tau$.

Firms differ in their salary offers and in the minimum labor qualification they demand from all applicants. Salaries can incorporate an incentive element used to attract more qualified job applicants. Well qualified individuals may apply to lowly qualified jobs if the salary is good enough (this fact has been reported by Ours and Ridder (1995)). But they are assumed to apply for jobs only in firms offering to pay at least their reservation wage and requiring a level of labor qualification less than or equal to their own. The salary w offered by the firm to anyone with qualification $\theta_u \geq \theta_v$ at T, determines the range of labor qualifications $[\theta_v, \theta_u^m]$ for all possible job applicants. Given (3) the upper limit of this interval is $\theta_u^m = \min(w/\sigma(T), 1)$. See Fig. 1.

Figure 1

The qualification of potential acceptable applicants to this firm is a random variable with distribution, $\mathcal{F}(\theta_u, \theta_v; \overline{\alpha}) = \Pr(\Theta \leq \theta_u \mid \theta_v \leq \Theta \leq \theta_u^m)$. We express the response to a particular job offer in terms of the time x a firm has to wait for the first acceptable applicant. We assume x to be a function of $\overline{\alpha}$ and of the minimum relative qualification θ_v required by a firm at T. We also make x dependent on the tightness of the labor market as represented by the expected levels of unemployment $\widehat{U}$ and advertised vacancies $\widehat{v}_a$. That is,

$$x \equiv x(\widehat{U}, \widehat{v}_a, \theta_v, \overline{\alpha}), \ \forall\theta_v, 0 \leq \theta_v \leq 1 \tag{4}$$

where $\partial x\ /\partial\widehat{U} < 0$ and $\partial x\ /\partial\widehat{v}_a > 0$. We assume that the first job applicant to arrive at the firm is made an offer, which he immediately accepts. All hired workers remain in the same firm till it goes out of business.

Proposition 1 *Assume that at $t = T$ a firm advertises a job opening for anyone with relative labor qualification $\theta_u \geq \theta_v$ at T, where θ_v is the minimum requirement. Let w be the salary offer. Assume all posted salaries grow at rate g. Then, the interval of acceptable qualifications of potential job applicants is $[\theta_v, \theta_u^m]$. The expected relative labor qualification $\widehat{\theta}_u$ at T of the first acceptable job applicant the firm gets and hires at $t = T + x$ is*

$$\widehat{\theta}_u \equiv \widehat{\theta}_u(\theta_v, T, w, \overline{\alpha}) = \int_{\theta_v}^{\theta_u^m} \theta_u d\mathcal{F}(\theta_u, \theta_v; \overline{\alpha}), \tag{5}$$

where $\partial\widehat{\theta}_u/\partial\theta_v > 0$, $\partial\widehat{\theta}_u/\partial w > 0$ and $\partial\widehat{\theta}_u/\partial T < 0$ when $\sigma(T)$ increases.

Proof. It is obvious that (5) provides the expected qualification of anyone arriving at $t = T$. His absolute qualification is $\Delta_u(\widehat{\theta}_u, T)$. Since salaries and reservation wages corresponding to any absolute labor qualification grow at rate g, the interval of job candidates at $t = T + x$ represents qualifications which at $t = T$ belonged to $[\theta_v, \theta_u^m]$. Then (5) also gives the relative expected qualification at $t = T$ of the candidate hired at $t = T + x$. Taking derivatives we get $\partial\widehat{\theta}_u/\partial\theta_v > 0$, $\partial\widehat{\theta}_u/\partial w > 0$ and $\partial\widehat{\theta}_u/\partial T < 0$ when $\sigma(T)$ increases. ■

3 OPTIMAL ENTREPRENEURIAL DECISIONS

As in Blanchard and Diamond (1994) we consider three alternative positions in the life of all entrepreneurial options. They can be in an unproductive state, which we call latent or dormant, advertising for a job applicant and fully operative. A latent firm represents potential entrepreneurial talent which, for the time being, is associated to a technology that was once profitable but is no longer so. That technology marks the initial position of the firm and its ability to incorporate any existing profitable method of production. A latent firm uses no labor and incurs in no costs. When a latent firm invests in a profit making technology, it becomes an advertising vacancy. When it hires a unit of labor, it becomes a filled firm and runs a business till its profits run out. It then goes into a latent stage and the process starts anew. The order of events for the firm is represented below,

$\longrightarrow\downarrow$ _ _ _ _ _ _ _ _ $\downarrow$ _ _ _ _ _ _ _ _ _ . $\downarrow$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ $\downarrow\longrightarrow$.

$t = 0.$ $\quad t = T$ $\quad t = T + x$ $\quad t = T + x + S$

Decisions taken Technology chosen Employee hired Firm destroyed.

We represent a latent firm by the minimum relative level of labor qualification $\theta_v^0 \in [0, 1]$ it would be required at $t = 0$ to operate its unprofitable technology. A superscript indicates that the firm is in a latent state. The value of the superscript indicates the period in reference to which the relative labor qualification is measured. The level of θ_v^0 represents the initial technical expertise of the firm and therefore marks its ability to adapt to methods of production that may not be new in the economy but are certainly unfamiliar for the firm at this stage. We assume that there is a known distribution $G(\theta_v^0)$ of unprofitable technologies.

Assume that at $t = 0$ a latent firm θ_v^0 decides to become an operating one. It then considers the profit making technologies it can adopt, given the menus of profitable technical options available at each subsequent date. We assume that a given time lag, denoted by T and measured from $t = 0$, determines a unique menu $M(T)$ of technological options which are profitable at T. Each latent firm is assumed to choose optimally the value of T and with it the available menu. A profitable technology is an element of $M(T)$. It is defined by the minimum level of relative labor qualification θ_v, $0 \leq \theta_v \leq 1$, required to operate it at T. This corresponds to an absolute labor qualification level $\Delta_v(\theta_v, T)$. We allow a latent firm to choose independently the menu $M(T)$ and the particular technology $\theta_v \in M(T)$. If both were dependent there would be a unique technology associated to a given waiting time. This is generally not the case here. It is a common in the literature to assume that technological shocks occur at an exogenous rate determining the technology to be used by firms. There is then a unique relationship between time and technology. We allow this as a particular case but have in mind a more general set up. In this article a given technology may remain profitable for a long time and in consequence belong to different menus. The time a technology is adopted by a firm determines the minimum relative labor qualification it requires to operate it. If there is any technical progress, or any improvement in the qualification of labor, more modern menus will generally associate a lower minimum relative labor qualification of a later date to a technology that first appeared in an earlier menu. A particular timing may be more convenient to a particular firm adopting the technology, provided it is still profitable. Any two latent firms deciding on which technology to adopt at T make their selection from an identical menu, but may choose different technologies θ_v if their initial latent positions differ. This set up allows for very different technologies to be used in the economy at any given time. The choices of all firms are made taking into account the distribution of relative labor qualifications $F(\theta_u, \overline{\alpha})$ and the speed of contracting given in (4).

Assume a latent firm θ_v^0 chooses $\theta_v \in M(T)$. It therefore becomes an advertising option at T. It requires an employee having at T a relative labor qualification level $\theta_u \geq \theta_v$. Given (3), the salary offer w must be such that $w \geq W_R(\theta_v, T)$. We assume that better qualified workers are preferred because, for any technology, their contribution to output is higher. Given (4) a latent firm knows at $t = 0$ that at $t = T + x$, it will have its first applicant and he will be hired. If the firm offers a salary w the expected qualification $\widehat{\theta}_u$ of that applicant is given in (5). The production of output Y starts immediately after hiring. The level of output produced per unit of time remains constant till the firm goes out of business.

We assume Y to be an increasing function of the technology adopted and of the expected labor qualification hired. Given (2), and (5), we can write it as,

$$Y \equiv Y(\theta_v, \widehat{\theta}_u(\theta_v, w, T, \overline{\alpha}), T) = \mathcal{Y}(\Delta_v(\theta_v, T), \Delta_u(\widehat{\theta}_u(\theta_v, w, T, \overline{\alpha}), T)) \qquad (6)$$

where $Y_T > 0$, $Y_{\theta_v} > 0, Y_{\widehat{\theta}_u} > 0$, $Y_{\widehat{\theta}_u^2} < 0$. Since we have assumed that all salaries associated to any absolute labor qualification level grow at rate g, this rate must be,

$$g = (\partial \mathcal{Y}/\partial \Delta_v)(\partial \Delta_v/\partial T)/Y$$

Note that in this article the rate of growth for the economy is not g but the rate $\mathcal{G}$ given by,

$$\begin{aligned} \mathcal{G} &= g + (\partial \mathcal{Y}/\partial \Delta_u)\left(\left(\partial \Delta_u/\partial \widehat{\theta}_u\right)\left(\partial \widehat{\theta}_u/\partial T\right) + \partial \Delta_u/\partial T\right)/Y = \\ &= (Y_T + Y_{\widehat{\theta}_u}(\partial \widehat{\theta}_u/\partial T))/Y \end{aligned} \tag{7}$$

where the second term in the first part of this equality represents the change in output due to variations in expected hired labor qualification over time.

We call $C(\theta_v, T; \theta_v^0) \equiv \mathcal{C}(\Delta_v(\theta_v, T); \theta_v^0)$ the cost for a latent firm θ_v^0 of incorporating at T a technology $\theta_v \in M(T)$. We assume $\partial C/\partial \theta_v^0 < 0$, $\partial C/\partial \theta_v > 0$, $\partial C/\partial T > 0$. That is, more advanced technologies within a given menu are more costly and all the more so the more backward is the latent position of the firm at T.

If we assume the interest rate $r > 0$ to be given, we can write the discounted value at $t = 0$ of accumulated profits to be obtained by this firm as,

$$\begin{aligned} \Pi(T, \theta_v, w, S; \theta_v^0, \overline{\alpha}, r, g) &= e^{-r(T+x)}[YQ(S) - we^{gx}P(S)] - \\ &\quad -e^{-rT}C(\theta_v, T; \theta_v^0) \end{aligned} \tag{8}$$

where

$$Q(S) = \int_0^S e^{-r\tau} d\tau \ \text{ and } P(S) = \int_0^S e^{(g-r)\tau} d\tau.$$

We assume the firm chooses the values of $T \geq 0$, $\theta_v \in [0, 1]$, $w \geq W_R(\theta_v, T)$, and $S \geq 0$, in a way that maximizes the value of (8). From now onwards, and for ease of notation, we omit the parameters r and g in the profit function.

To ensure the existence of a solution we impose the following conditions,

- A1) All latent firms are potentially profitable. That is, $\forall\theta_v^0$, $\exists T \geq 0$, $S \geq 0$ and $\delta > 0$ such that, for some $\theta_v \in [0,1]$, we have that $\Pi(T,\theta_v,w,S;\theta_v^0,\overline{\alpha}) > 0$, $\forall w \in [\theta_v\sigma(T), \theta_v\sigma(T)+\delta)$
- A2) The profits of any selected technology eventually run to zero. That is , $\forall T \geq 0$, $\theta_v \in M(T)$ and $w \geq \theta_v\sigma(T)$ satisfying (A1), $lim_{S\to\infty}\ \Pi(T,\theta_v,w,S;\theta_v^0,\overline{\alpha}) < 0$. This is always true if $g > 0$.
- A3) The adaptation cost of any given technology θ_v^0 is such that to wait too much wipes out the possibility of obtaining positive profits with any technology in any menu. That is, $\forall\theta_v \in M(T)$, and S and w satisfying (A1), $lim_{T\to\infty}\Pi(T,\theta_v,w,S;\theta_v^0,\overline{\alpha}) < 0$.
- A4) The function $\Pi(T,\theta_v,w,S;\theta_v^0,\overline{\alpha})$ is a twice differentiable and strictly concave function.

Theorem 2 *Under conditions (A1), (A2) and (A3), there exists a solution* $(\widehat{T},\widehat{\theta}_v,\widehat{w},\widehat{S})$ *for*

$$\max_{T,\theta_v,w,S}\Pi(T,\theta_v,w,S;\theta_v^0,\overline{\alpha}). \tag{9}$$

where $\widehat{T} \geq 0$, $\widehat{\theta}_v \in [0,1]$, $\widehat{w} \in [0,1]$ *and* $\widehat{S} \geq 0$.

Proof. See appendix. ■

The complementarity in the production of output between better technology and more qualified labor will generally lead to a multiplicity of solutions as in Acemoglu (1997). Assumption (A4) ensures uniqueness. But even if the equilibrium is unique there does not have to be a unique salary associated to any given qualification. Because of the incentive component firms use to attract the better qualified, different salaries may be paid to individuals with the same qualification.

Theorem 3 *Given (A1), (A2), (A3) and (A4), the solution to (9) is unique. Moreover, an interior differentiable solution*

$$\left(\widehat{T}(\theta_v^0;\widehat{U},\widehat{v}_a,\overline{\alpha}),\widehat{\theta}_v(\theta_v^0;\widehat{U},\widehat{v}_a,\overline{\alpha}),\widehat{w}(\theta_v^0;\widehat{U},\widehat{v}_a,\overline{\alpha}),\widehat{S}(\theta_v^0;\widehat{U},\widehat{v}_a,\overline{\alpha})\right), \tag{10}$$

must satisfy

$$Q(S)[Y_T + Y_{\widehat{\theta}_u}(\partial\widehat{\theta}_u/\partial T) - rY] + rwe^{gx}P(S) = e^{rx}(C_T - rC), \tag{11}$$

$$Q(S)[Y_{\theta_v} + Y_{\widehat{\theta}_u}(\partial\widehat{\theta}_u/\partial\theta_v) - rY\partial x/\partial\theta_v] - wP(S)e^{gx}(g-r)\partial x/\partial\theta_v = e^{rx}C_{\theta_v}, \tag{12}$$

$$Y_{\widehat{\theta}_u}(\partial\widehat{\theta}_u/\partial w)Q(S) = e^{gx}P(S) \tag{13}$$

$$Ye^{-g(x+S)} = w. \tag{14}$$

Proof. *See appendix.* ■

The above first order conditions establish an equality between the marginal revenue and the marginal cost associated to each choice variable. Note that (14) establishes that it is optimal for the firm to close business and go into a latent state when the instantaneous rate of profit hits the zero mark. Since output is constant and salaries grow at rate g, business length is finite. In this we resemble Aghion and Howitt (1994). According to (14), the salary paid is a proportion of output. A higher expected labor qualification increases the value of output and hence induces higher salaries, as in Acemoglu (1999). But for us the proportion of output the salary represents is itself dependant on $\widehat{\theta}_u$. This can be regarded as an incentive which would reinforce the wage inequality reported by Acemoglu (1999).

All firms are assumed to solve an equivalent optimization problem, given their initial position θ_v^0 at $t = 0$. Since the distribution $G(\theta_v^0)$ of latent vacancies at $t = 0$ is known, the optimal market values for all endogenous variables are,

$$\widehat{T}^M \equiv \widehat{T}^M(\widehat{U},\widehat{v}_a,\overline{\alpha}) = \int_0^1 \widehat{T}(\theta_v^0;\widehat{U},\widehat{v}_a,\overline{\alpha})dG(\theta_v^0), \tag{15}$$

$$\widehat{\theta}_v^M \equiv \widehat{\theta}_v^M(\widehat{U}, \widehat{v}_a, \overline{\alpha}) = \int_0^1 \widehat{\theta}_v(\theta_v^0; \widehat{U}, \widehat{v}_a, \overline{\alpha}) dG(\theta_v^0), \tag{16}$$

$$\widehat{w}^M \equiv \widehat{w}^M(\widehat{U}, \widehat{v}_a, \overline{\alpha}) = \int_0^1 \widehat{w}(\theta_v^0; \widehat{U}, \widehat{v}_a, \overline{\alpha}) dG(\theta_v^0), \tag{17}$$

and

$$\widehat{S}^M \equiv \widehat{S}^M(\widehat{U}, \widehat{v}_a, \overline{\alpha}) = \int_0^1 \widehat{S}(\theta_v^0; \widehat{U}, \widehat{v}_a, \overline{\alpha}) dG(\theta_v^0). \tag{18}$$

Given (4) and (5) we have

$$\widehat{x}^M \equiv \widehat{x}^M(\widehat{U}, \widehat{v}_a, \overline{\alpha}) = \int_0^1 x(\widehat{\theta}_v(\theta_v^0; \widehat{U}, \widehat{v}_a, \overline{\alpha}), \widehat{U}, \widehat{v}_a, \overline{\alpha}) dG(\theta_v^0) \tag{19}$$

and

$$\widehat{\theta}_u^M = \widehat{\theta}_u^M(\widehat{U}, \widehat{v}_a, \overline{\alpha}) = \int_0^1 \widehat{\theta}_u(\widehat{T}, \widehat{\theta}_v, \widehat{w}, \overline{\alpha}) dG(\theta_v^0). \tag{20}$$

All individual and market values depend on the average level of unemployment and advertised vacancies through their dependence on (4). In the next section we determine the endogenous distribution of the stationary random vector $(\widetilde{U}, \widetilde{v}_a)$ having $(\widehat{U}, \widehat{v}_a)$ as mean.

4 EQUILIBRIUM

To obtain the distribution of $(\widetilde{U}, \widetilde{v}_a)$, we envisage the economy as a closed network of interconnected nodes. Each one of them corresponds to a possible position for every firm. There is a node for latent firms, another one for advertising firms and also a node for filled firms. We assume that at all times there is a given total number V of possible options. We denote by $v_d(t)$, $v_a(t)$ and $v_f(t)$ the number of latent, advertised and filled firms in the economy at time t. That is,

$$V = v_d(t) + v_a(t) + v_f(t), \ \forall t. \tag{21}$$

We assume that the time each firm spends in each node is an exponentially distributed random variable. Therefore the number of items in each node is also random. The interrelationship between any two nodes is represented in figure 2.

Figure 2

The terms $c(v_a, v_f, v_d)$, $m(v_a, v_f, v_d)$, and $d(v_a, v_f, v_d)$, in figure 2, refer to the creation, matching and destruction market transition functions of firms. Since one worker matches with one firm, $v_f(t) = E(t)$ and, given (1), $U(t) = L - v_f(t)$, $\forall t$. The associated unemployment network is given in figure 3.

Figure 3

These networks describe the behavior of a dynamical system. They can be considered an stochastic extension of the deterministic system used by Blanchard and Diamond (1997).

We characterize the steady state behavior of the network by the limiting distribution $\left\{p_{v_a,v_f}\right\}$ of $X(t) = (v_a(t), v_f(t))$. Given (21) and (1), the values of $v_d(t)$ and $U(t)$ are determined once $X(t)$ is known. The state space E of $X(t)$ is finite and irreducible, where $\mathsf{E} = \{(v_a, v_f) \in \mathrm{IN} \times \mathrm{IN} : v_a + v_f \leq V\}$. The ergodic theorem for finite Markov processes (see Asmussen , theorem 4.2) ensures the existence and uniqueness of a stationary probability distribution $\left\{p_{v_a,v_f}\right\}$, where

$$p_{v_a,v_f} = \lim_{t\to\infty} \Pr\left[X(t) = (v_a, v_f)\right] = \Pr[(\widetilde{v}_a, \widetilde{v}_f) = (v_a, v_f)]. \tag{22}$$

When all firms optimize in the manner described in the previous section, the transition functions ruling the distribution are proportional to the number of items in each node. The factor of proportionality is given by (15), (18) and (19). That is

$$c(v_a, v_f, v_d) = v_d\lambda,\ m(v_a, v_f, v_d) = \mu v_a,\ d(v_a, v_f, v_d) = \gamma v_f, \tag{23}$$

where $v_d = V - v_f - v_a$, $v_f = L - U$, and

$$\lambda = 1/\widehat{T}^M(\widehat{U}, \widehat{v}_a; \overline{\alpha}),\ \mu = 1/\widehat{x}^M(\widehat{U}, \widehat{v}_a; \overline{\alpha}),\ \gamma = 1/\widehat{S}^M(\widehat{U}, \widehat{v}_a; \overline{\alpha}). \tag{24}$$

Note that although λ, μ and γ in (24) are deterministic, the corresponding creation, matching and destruction functions given in (23) are random since the number of firms in each node is considered random. We now characterize the equilibrium probability distribution of $(\widetilde{v}_f, \widetilde{v}_a)$,and therefore of $(\widetilde{U}, \widetilde{v}_a)$. We have to treat separately the cases, $L \geq V$ and $L < V$. We first consider that $L \geq V$, thereby assuming the existence of positive unemployment.

Proposition 4 *Assume $L \geq V$. Let $(v_f(t), v_a(t))$ be a Markovian process with transition functions defined in (23). Then there exists a unique stochastic equilibrium $(\widetilde{v}_f, \widetilde{v}_a)$ characterized by a generating function $P(z, y)$. This function is the unique solution to the partial differential equation:*

$$0 = \lambda V(1-z)P(z,y) + \left((\mu - \lambda)\, z + \lambda z^2 - \mu y\right) P'_z(z,y) +$$
$$+ \left((y-1)\, \gamma + \lambda y\, (z-1)\right) P'_y(z,y). \tag{25}$$

The first order moments are

$$E\left[\widetilde{v}_a\right] \equiv \widehat{v}_a = \lambda\gamma V/(\lambda\gamma + \mu\gamma + \lambda\mu), \tag{26}$$

$$E\left[\widetilde{v}_f\right] \equiv \widehat{v}_f = \mu\widehat{v}_a/\gamma = \lambda\mu V/(\lambda\gamma + \mu\gamma + \lambda\mu), \tag{27}$$

$$E\left[\widetilde{v}_d\right] \equiv \widehat{v}_d = \mu\gamma V/(\lambda\gamma + \mu\gamma + \lambda\mu). \tag{28}$$

The variances, covariances and correlation coefficient of $(\widetilde{v}_f, \widetilde{v}_a)$ are,

$$Var[\widetilde{v}_a] \equiv \sigma_a^2 = \frac{\mu\,(\lambda + \gamma)}{\lambda\gamma + \mu\gamma + \lambda\mu}\widehat{v}_a, \quad Var[\widetilde{v}_f] \equiv \sigma_f^2 = \frac{\mu\,(\lambda + \mu)}{\lambda\gamma + \mu\gamma + \lambda\mu}\widehat{v}_a \tag{29}$$

$$Cov[\widetilde{v}_f, \widetilde{v}_a] = -\frac{\lambda\mu\widehat{v}_a}{\lambda\gamma + \mu\gamma + \lambda\mu}, \tag{30}$$

$$Cov\left[\widetilde{v}_d, \widetilde{v}_a\right] = -\frac{\mu\gamma\widehat{v}_a}{\lambda\gamma + \mu\gamma + \lambda\mu},\ Cov\left[\widetilde{v}_f, \widetilde{v}_d\right] = -\frac{\mu^2\widehat{v}_a}{\lambda\gamma + \mu\gamma + \lambda\mu} \tag{31}$$

$$\rho^2_{v_a v_f} = \frac{\lambda^2}{(\lambda+\mu)(\lambda+\gamma)} < 1. \tag{32}$$

Proof. See Appendix. ■

Proposition 5 *Assume $L < V$. Let $(v_f(t), v_a(t))$ be a Markovian process with transition functions defined in (23). Then, there exists a unique stochastic equilibrium $(\widetilde{v}_f, \widetilde{v}_a)$ characterized by the generating function $Q(z,y)$ determined by (50) in the Appendix. The mean numbers of advertised, filled and latent vacancies are respectively,*

$$\widehat{v}_a = \frac{\lambda\gamma V + \mu(\gamma+\lambda)\widehat{v}_{a,L}}{\lambda\gamma+\mu\gamma+\lambda\mu}, \quad \widehat{v}_f = \frac{\lambda\mu(V-\widehat{v}_{a,L})}{\lambda\gamma+\mu\gamma+\lambda\mu}, \quad \widehat{v}_d = \frac{\mu\gamma(V-\widehat{v}_{a,L})}{\lambda\gamma+\mu\gamma+\lambda\mu}. \tag{33}$$

The covariance between advertised and filled vacancies is

$$Cov[\widetilde{v}_f, \widetilde{v}_a] = \frac{\lambda\mu(V-1)}{\lambda\gamma+\mu\gamma+\lambda\mu}\widehat{v}_a + \frac{\xi}{(\lambda\gamma+\mu\gamma+\lambda\mu)(\gamma+\lambda+\mu)} - \widehat{v}_a\widehat{v}_f \tag{34}$$

where

$$\xi = (\lambda+\mu)(L\gamma+\lambda\mu(L-V+1))\widehat{v}_{a,L} + (\mu^2-\lambda-\mu)\gamma\widehat{v}^2_{a,L}$$

$$\widehat{v}_{a,L} = \sum\nolimits_{v_a=1}^{v_a=V-L} v_a p_{v_a L} \quad \textit{and} \quad \widehat{v}^2_{a,L} = \sum\nolimits_{v_a=1}^{v_a=V-L} v_a(v_a-1)p_{v_a L}. \tag{35}$$

Proof. See Appendix ■

Notice that all covariances are non zero. That is, all random variables are dependent. When $L \geq V$, the square correlation coefficient $\rho^2_{v_a v_f}$ is an increasing function of λ and a decreasing function of γ and μ. For low values of λ, the random variables $\widetilde{v}_a$ and $\widetilde{v}_f$ are almost uncorrelated. Moreover, the random positive effect on unemployment of an increase in the value of $\widetilde{v}_a$ is insignificant. Also, $Cov[\widetilde{v}_f, \widetilde{v}_a] \uparrow 0$, when $\lambda \downarrow 0$. The relationship between these variables is almost linear when λ is large enough. In that case the random positive effect on unemployment of an increase in the value of $\widetilde{v}_a$ is not insignificant.

When $L \geq V$, $Cov[\widetilde{v}_f, \widetilde{v}_a] < 0$. That is, $Cov[\widetilde{U}, \widetilde{v}_a] > 0$. This is due to the closed structure imposed on the network, together with the restriction $L \geq V$. It shows that, in probability, an increase in the number of items in any node must simultaneously entail a decrease in the number of items in any other node. This contradicts the figures in Andolfatto (1996) in regard to the covariance between unemployment and advertised vacancies in the USA. But in a large economy it is likely that $L < V$. In that case, the sign of the covariance is indeterminate. It can be proved that when μ is high enough, jobs last a long time and λ is low, the $Cov(\widetilde{v}_a, \widetilde{U}) < 0$ when V is sufficiently larger than L. This happens when the appearance of new jobs is not accompanied by a rapid destruction of filled jobs and the newly adopted technologies fit in with existing labor qualifications. On the other hand, the fact that $Cov(\widetilde{v}_a, \widetilde{U}) > 0$ may be due to a mismatch between the technical requirements of the new emerging firms and the qualification of the unemployed population. A positive covariance indicates that the random variables $(\widetilde{U}, \widetilde{v}_a)$ may increase or decrease together. However there is always an inverse relationship between their mean values $(\widehat{U}, \widehat{v}_a)$. See (27).

Note that, given (29), all variances are proportional to V when $L \geq V$. If we define the unemployment rate by $\widetilde{u} = \widetilde{U}/L$, then $Var[\widetilde{u}] = O(V/L^2) \approx O(1/L)$ if $L \approx V$. For L large enough, $Var[\widetilde{u}] \approx 0$. In this case there is very low stochastic variability in the unemployment rate. Hence $\widetilde{u} \approx \widehat{u}$. It could then be argued that a deterministic model centered on the expected rates of unemployment could give a good approximation to the actual behavior of the rate. Notice however that if L is large, so is $Var[\widetilde{U}]$ indicating that $\widetilde{U}$ has very high stochastic variability. To know this can be of the essence in any long run policy decision dealing with unemployment relief funding. Furthermore if V and L are not similar, and $L < V$, then $Var[\widetilde{u}] = O(V/L^2)$. In this case the variance of the unemployment rate is high when V is large. In every case it would be a mistake to identify the actual values of unemployment and advertised vacancies with their average values. We may observe economies with identical equilibrium expected values for all variables, but with different equilibrium distributions and therefore with different actual equilibrium behavior.

We are now in a position to prove a theorem characterizing the full equilibrium of the system.

Theorem 6 *Assume that conditions (A1)-(A4) hold. Let* $(v_f(t), v_a(t))$ *be a Markovian process* *with intensity transitions defined in (23), where λ, μ and γ are given in (24). Then there exists a unique stochastic equilibrium $(\widetilde{U}^*, \widetilde{v}_a^*)$ characterized by the probability distribution $\{p^*_{v_a,v_f}\}$, $p^*_{v_a,v_f} \equiv p_{v_a,v_f}(\overline{\alpha})$. Hence we get the optimal market times $\left(\widehat{T}^{*M}(\overline{\alpha}), \widehat{x}^{*M}(\overline{\alpha}), \widehat{S}^{*M}(\overline{\alpha})\right)$, the probability distributions of the equilibrium transition functions (23) and the mean market values of technologies in advertised vacancies, salaries and hired relative labor qualifications $\left(\widehat{\theta}_v^{*M}(\overline{\alpha}), \widehat{w}^{*M}(\overline{\alpha}), \widehat{\theta}_u^{*M}(\overline{\alpha})\right)$. All these functions are differentiable in $\overline{\alpha}$.*

The optimal individual firm values are denoted by a hat. The corresponding market values are given the M-superscript. When starred they denote the equilibrium solution.

Proof. We consider the case $L \geq V$. From (26) and (27), together with (15), (18) and (19), we get $\widehat{U}^*(\overline{\alpha})$, $\widehat{v}_a^*(\overline{\alpha})$, $\widehat{T}^{*M}(\overline{\alpha})$, $\widehat{x}^{*M}(\overline{\alpha})$ and $\widehat{S}^{*M}(\overline{\alpha})$. Substituting $(\widehat{U}^*(\overline{\alpha}), \widehat{v}_a^*(\overline{\alpha}))$ in (16) and (17), we get $\widehat{\theta}_v^{*M}(\overline{\alpha})$ and $\widehat{w}^{*M}(\overline{\alpha})$ and, given (5) we get $\widehat{\theta}_u^{*M}(\overline{\alpha})$. Because of the implicit function theorem, they are all differentiable functions of $\overline{\alpha}$. To obtain the probability distribution $p^*_{v_a,v_f}$ of the equilibrium random vector $(\widetilde{U}^*, \widetilde{v}_a^*) = \left(V - \widetilde{v}_f^*, \widetilde{v}_a^*\right)$, we substitute $\widehat{v}_f^*(\overline{\alpha})$ and $\widehat{v}_a^*(\overline{\alpha})$ in (24) and then in (25). The equilibrium probability distribution, together with $\widehat{T}^{*M}(\overline{\alpha})$, $\widehat{x}^{*M}(\overline{\alpha})$ and $\widehat{S}^{*M}(\overline{\alpha})$, determine the distribution of the three transition functions given in (23). The proof of the case when $L < V$ is analogous. ■

Note that an equivalent theorem can be proved for each particular latent firm using (10) instead of the market values. In the next section we analyze how the equilibrium solution responds to changes in the distribution of relative qualification levels among the unemployed.

5 DISTRIBUTIONAL COMPARISONS

We now analyze the effect on the equilibrium solution of policies which alter the distribution $F(\theta_u; \overline{\alpha})$. In this section we assume $\overline{\alpha} = \alpha \in \mathbb{R}$. We also assume $F(\theta_u; \alpha)$ to be such that a higher value of α represents a greater proportion of highly qualified job searchers. For simplicity, we consider all latent firms to be identical. To make our results comparable with the existing literature, we also assume the values of $\widehat{T}^{*M}$ and $\widehat{\theta}_v^{*M}$ to be given constants. This is precisely what many authors do when they envisage an economy undergoing innovation shocks at an exogenous Poisson rate $\lambda = 1/\widehat{T}^{*M}$. These shocks determine the technology $\theta_v = \theta_v(\lambda, \theta_v^0)$ to be used by firms. In this section the technology adopted is not a choice variable and is therefore independent of α. Technical progress can still take place but its implementation becomes exogenous and leads to no alteration over time in the minimum relative qualification required by operating firms. All other endogenous variables are allowed to change.

Proposition 7 *Let $\overline{\alpha} = \alpha \in \mathbb{R}$ and $L = V$. For given $\widehat{T}^{*M}$ and $\widehat{\theta}_v^{*M}$, let $\widehat{w}^{*M}$ and $\widehat{S}^{*M}$ be the equilibrium solutions obtained in theorem (6). Let the matching function $m(\widehat{U}, \widehat{v}_a, \overline{\alpha}) = \widehat{v}_a(\widehat{x}^M(\widehat{v}_a, \widehat{U}, \overline{\alpha}))^{-1}$ be both, homogeneous of degree one in $\widehat{v}_a$ and $\widehat{U}$, and non decreasing in both variables. Then,*

$$\frac{\partial \widehat{S}^{*M}}{\partial \alpha} = a_1 \frac{\partial \widehat{\theta}_u^M}{\partial \alpha} - b_1 \frac{\partial^2 \widehat{\theta}_u^M}{\partial w \partial \alpha} - c_1 \frac{\partial \widehat{x}^M}{\partial \alpha}, \tag{36}$$

$$\frac{\partial \widehat{w}^{*M}}{\partial \alpha} = -a_2 \frac{\partial \widehat{\theta}_u^M}{\partial \alpha} + b_2 \frac{\partial^2 \widehat{\theta}_u^M}{\partial w \partial \alpha} - c_2 \frac{\partial \widehat{x}^M}{\partial \alpha}, \tag{37}$$

$$\frac{\partial \widehat{v}_a^*}{\partial \alpha} = -a_3 \frac{\partial \widehat{\theta}_u^M}{\partial \alpha} + b_3 \frac{\partial^2 \widehat{\theta}_u^M}{\partial w \partial \alpha} + c_3 \frac{\partial \widehat{x}^M}{\partial \alpha}, \tag{38}$$

$$\frac{\partial \widehat{v}_f^*}{\partial \alpha} = a_4 \frac{\partial \widehat{\theta}_u^M}{\partial \alpha} - b_4 \frac{\partial^2 \widehat{\theta}_u^M}{\partial w \partial \alpha} - c_4 \frac{\partial \widehat{x}^M}{\partial \alpha}, \tag{39}$$

where $a_i, b_i > 0$, $i = 1, 2, 3, 4$, are given in the appendix. The signs of the coefficients $c_i, \forall i$, depend on

$$A^* = -gY \frac{\partial^2 Y}{\partial w^2} + g \frac{\partial Y}{\partial w} \left(\frac{\partial Y}{\partial w} - \frac{Y}{w} \right)$$

and

$$B^* = -gY \frac{\partial^2 Y}{\partial w^2} - \frac{\exp(-rS)}{Q(S)} \left(\frac{\partial Y}{\partial w} - \frac{Y}{w} \right)^2 > 0.$$

We have two cases:

a) If $A^* > 0$*, then* $c_i > 0$, $i = 1, 2, 3, 4$.

b) If $A^* < 0$*, then* $c_1 < 0$, $c_2 > 0$*. We also get* $c_3 > 0 \iff |A^*|\,\widehat{v}_a^* < (V - \widehat{v}_a^*)B^*$ *and* $c_4 > 0$ $\iff B^*\widehat{v}_f^* > |A^*|\,(V - \widehat{v}_f^*)$.

Proof. *See Appendix.* ■

Given theorem (6), we can take derivatives in (19) and (20), we get the following corollary.

Corollary 8

$$\frac{\partial \widehat{x}^{*M}}{\partial \alpha} = \frac{\widehat{U}^*}{\widehat{U}^* - (\widehat{U}^* - \widehat{v}_a^*)\varepsilon_{x,\widehat{v}_a}} \frac{\partial \widehat{x}^M}{\partial \alpha} \tag{40}$$

$$\frac{\partial \widehat{\theta}_u^{*M}}{\partial \alpha} = \frac{\partial \widehat{\theta}_u^M}{\partial \alpha} + \frac{\partial \widehat{\theta}_u^M}{\partial \theta_u^m} \frac{\partial \theta_u^m}{\partial w} \frac{\partial \widehat{w}^{*M}}{\partial \alpha} \tag{41}$$

where the first term on the right hand side of (40) is always positive. The term $\varepsilon_{x,\widehat{v}_a}$ *stands for the elasticity of* $\widehat{x}^M(\widehat{v}_a, \widehat{U}, \alpha)$ *with respect to* $\widehat{v}_a$*.*■

Assume that, when the value of α changes, it does not affect the speed of job contracting, i.e. $\partial \widehat{x}^M / \partial \alpha = 0$. Given (36), (38) and (39), a higher number of advertised jobs is accompanied by higher unemployment and shorter jobs. A policy leading to an improvement in the distribution of relative qualifications among the unemployed is adequate if it clears the glut of unoccupied jobs but it worsens matters if it leads to a situation with higher unemployment, shorter jobs and a congestion of advertised vacancies. To see which of the two outcomes occurs and how salaries change when α varies, we assume that the distribution $F(\theta_u; \alpha)$ belongs to the *Beta* family, $B(\alpha, 2)$, $\alpha > 0$.

Lemma 9 *For a* $B(\alpha, 2)$ *distribution,* $\partial \widehat{\theta}_u^M / \partial \alpha > 0$ *and* $\partial^2 \widehat{\theta}_u^M / \partial w \partial \alpha > 0$.

Proof. See Appendix. ■

Assume $\theta_u \sim B(\alpha, 2)$. For $\partial^2 \widehat{\theta}_u^M / \partial w \partial \alpha$ large and $Y_{\widehat{\theta}_u} \left| \partial^2 \widehat{\theta}_u^M / \partial w^2 \right|$ small, an increase in α leads to higher salaries, shorter jobs and higher unemployment although the number of advertised vacancies goes up. On the other hand if $Y_{\widehat{\theta}_u}\left|\partial^2\widehat{\theta}_u^M/\partial w^2\right|$ were to be relatively large, salaries and job durations could go up with α, while unemployment and advertised vacancies come down. Given corollary (8), there is an improvement in the expected relative qualification levels hired. Hence the labor market is flooded with job searchers able to occupy vacant highly qualified jobs. The glut of high level advertised vacancies is cleared. Job searchers are lured to high skill jobs by offers of high salaries thought to be justified by their output contribution. Note that it is quite probable that these results depend on the particular family of distributions $F(\theta_u, \alpha)$ belongs to. If we were to choose a different family the signs of the derivatives in the previous proposition would very likely no longer be independent of the particular values of θ_v taken as given in the analysis.

Assume an increase in the proportion of highly qualified labor speeds up job contracting, i.e. $\partial\widehat{x}^M/\partial\alpha < 0$. If this term is large, and the production function not too concave in $\widehat{\theta}_u$, the third term in (36), (38) and (39) may have a higher value than the other two terms put together. In that case an increase in α leads to fewer advertised vacancies, lower unemployment and shorter jobs. This is a case where an improvement in labor qualifications eases up the existing congestion of advertised vacancies but it also contributes to a faster rate of job destruction. The net result is a diminution in total unemployment. This is something that cannot happen in Aghion and Howitt (1994).

Note that, if $\partial\widehat{x}^M/\partial\alpha \leq 0$, a change in α cannot induce a simultaneous increase in employment and in the number of advertised vacancies. It is easy to see that, given proposition (7) and corollary (8),

$$\frac{(\widehat{v}_f^*)^2}{\lambda(\widehat{S}^{*M})^2}\frac{\partial\widehat{x}^{*M}}{\partial\alpha} = (V - \ \widehat{v}_f^*)\frac{\partial\widehat{v}_a^*}{\partial\alpha} + \widehat{v}_a^*\frac{\partial\widehat{v}_f^*}{\partial\alpha}. \tag{42}$$

Hence in this case, a simultaneous increase in the number of advertised and filled vacancies must be accompanied by a slowing down in the speed of contracting. If α changes and the number of advertised vacancies goes up, unemployment can only diminish if the job contracting process is simultaneously lengthened. This is always true when the rate at which new technologies are incorporated by advertising firms is independent of α.

If we were to allow $(\widehat{T}^{*M}, \widehat{\theta}_v^{*M})$ to depend on α, we could observe that as a consequence of an increase in α, employment and the number of advertised vacancies both go up without any simultaneous lengthening in the process of job contracting. This is so because, in that case,

$$\frac{\partial \widehat{x}^{*M}}{\partial \alpha} = \frac{\widehat{x}^{*M}}{\widehat{T}^{*M}} \left[\frac{\widehat{S}^{*M}(V - \ \widehat{v}_f^*)}{\widehat{v}_a^* \ \widehat{v}_f^*} \frac{\partial \widehat{v}_a^*}{\partial \alpha} + \frac{\widehat{S}^{*M}}{\widehat{v}_f^*} \frac{\partial \widehat{v}_f^*}{\partial \alpha} + \frac{\partial \widehat{T}^{*M}}{\partial \alpha} \right]. \tag{43}$$

If a higher α induces new advertising firms to incorporate new technologies, unemployment can come down even if job contracting takes up less time. This occurs when $\partial \widehat{T}^{*M} / \partial \alpha < 0$ and also $\left| \partial \widehat{T}^{*M} / \partial \alpha \right|$ is large enough. This case is compatible with Acemoglu (1999) for whom an improvement in relative skill levels induces innovation and with it the creation of firms. On the other hand, Laing, Palivos and Wang (1995) maintain that educational policies, which lead to higher salaries, induce an acceleration of the matching process. In our case the change in salaries following an increase in the proportion of the highly qualified is not straightforward. An acceleration in the matching process is not necessarily a good idea unless the educational policies are accompanied by measures designed to induce the adoption of new technologies that accommodate the better educated unemployed population. This is consistent with Acemoglu (1997).

Finally, it is worth noting that changes in expected and actual unemployment following a variation in α may be of opposite signs. Using proposition (7) and taking derivatives in (29) and in (30) we get,

$$\partial Var(\widetilde{v}_f^*) / \partial \alpha = (V - 2\widehat{v}_f^*) \partial \widehat{v}_f^* / \partial \alpha. \tag{44}$$

$$\partial Cov(\widetilde{v}_a^*, \widetilde{v}_f^*) / \partial \alpha = -V^{-1} (\widehat{v}_a^* (\partial \widehat{v}_f^* / \partial \alpha) + \widehat{v}_f^* (\partial \widehat{v}_a^* / \partial \alpha)) \tag{45}$$

If $\partial\widehat{v}_f^*/\partial\alpha < 0$ and $V < 2\widehat{v}_f^* \Leftrightarrow x < \widehat{S}^{*M}$, then $\partial Var(\widetilde{v}_f^*)/\partial\alpha > 0$. The expected value and variance of unemployment change with α. An educational policy that leads to an increase in α makes the observation of lower unemployment more likely. Note also that if, as a consequence of the increase in α, $\partial\widehat{v}_f^*/\partial\alpha > 0$ and $\partial\widehat{v}_a^*/\partial\alpha > 0$, we also get $\left|\partial Cov(\widetilde{v}_a^*, \widetilde{v}_f^*)/\partial\alpha\right| > 0$. In this case the adopted policy is quite sound but it has become more likely to observe an increase in unemployment. This points to the convenience of having a stochastic theory of growth and unemployment. Otherwise steps may be taken to correct what is only due to the inherent randomness of events. The effect of such measure may be undesirable in the long run.

A more general model may consider firms taking current or past actual values of unemployment and advertised vacancies as measures of labor tightness instead of their stationary mean values. The interrelationship among moments of different order would then become more complex and allow for a situation where lower expected unemployment is accompanied by more volatile values.

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6 CONCLUSION

In this article we have characterized the state of the economy as a stochastic equilibrium. Individual agents are ex ante and ex post heterogeneous. Job searchers of different abilities find employment in firms using a wide spectrum of technologies. Each firm chooses optimally the type and time of adoption of the technology it uses, the salary it pays for the labor qualification it hires and the operating length of business. These choices depend on the distribution of relative qualifications among job searchers and on the degree of friction in the labor market. Friction is expressed in terms of the time it takes firms to hire workers of different abilities. We have integrated the choices of all firms within a queueing Markovian network whose transition functions are partly determined by the optimal timing choices of firms. We characterize the resulting equilibrium by its probability distribution. This is an entirely novel approach. Its use seems very natural since queueing systems deal with situations of systemic waiting. The creation of new firms and the search for labor partners are obvious cases. To depict an equilibrium as a probability distribution permits a distinction between expected and actual equilibrium values that is not due to error terms but to the inherent randomness of the system. In the last section we have determined the effect on the equilibrium solution of an educational policy that changes the relative distribution of abilities across the unemployed population when exogenous shocks determine the implementation of technologies. Our approach provides an explanation of various facts generally reported in the literature.

Some extensions seem immediate. The simplest one is to analyze the consequences of changes in the shape of the distribution of initial expertise among firms. It is also possible to incorporate on-the-job search. This would require a different modelling of job contracting but the queueing network we have used could easily encompass it. The use of an open network would allow for the ex novo creation of firms or for the arrival of immigrant workers. Finally, the possibility of discriminating between job applicants on the basis of how long they have remained unemployed, or of ordering the flow out of any node, could be accomplished by the use of non Markovian networks.

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APPENDIX

Proof of theorem (2). Given assumptions (A1), (A2) and (A3), the set where the profit function is strictly positive is a compact set in $\mathbb{R}^4_+$. The result follows from Weierstrass theorem.■

Proof of theorem (3). Given theorem (2), there exists a solution to (9). Since $\theta_v, w \in (0,1)$, it lies in the interior of the set. Given (A4), it is also unique and it satisfies the first order conditions of the problem. The last part of the theorem follows from the implicit function theorem.■

Proof of proposition (4). We can characterize this solution more closely using the Chapman-Kolmogorov equations (from now onwards CKE). They establish that, in equilibrium, the probabilities of getting in and out of any given state $(v_a, v_f) \in \mathsf{E}$ are the same. See Asmussen (1987). The CKE can be easily obtained using the transition diagram given below. For a given state (v_a, v_f, v_d), $1 < v_a, v_f, v_d < V$, $v_d = V - v_a - v_f$, the possible transitions in and out of this state are,

$$\begin{array}{lcl} (v_a, v_f+1, v_d-1) \longrightarrow (v_f+1)/S \longrightarrow & & \longrightarrow v_f/S \longrightarrow (v_a, v_f-1, v_d+1) \\ (v_a-1, v_f, v_d+1) \longrightarrow (v_d+1)\lambda \longrightarrow & (v_a, v_f, v_d) & \longrightarrow v_d\lambda \longrightarrow (v_a+1, v_f, v_d-1) \\ (v_a+1, v_f-1, v_d) \longrightarrow (v_a+1)\mu \longrightarrow & & \longrightarrow v_a\mu \longrightarrow (v_a-1, v_f+1, v_d) \end{array}$$

The transition functions are zero if the state the system comes out of (or it arrives into) does not belong to $\mathcal{E}$. we identify $(v_a, v_f, V - v_a - v_f)$ with (v_a, v_f). we have to treat separately the two cases, $L \geq V$ and $L < V$.

Assume $L \geq V$. For every $(v_a, v_f) \in \mathsf{E}$, $L \geq V$, the CKE are

$$\begin{aligned} (\mu v_a + (V - v_a - v_f)\lambda + \gamma v_f) p_{v_a v_f} &= (v_f + 1)\gamma p_{v_a, v_f+1} + \\ + (v_a + 1)\mu p_{v_a+1, v_f-1} &+ (V - v_a - v_f + 1)\lambda p_{v_a-1, v_f}. \end{aligned} \tag{46}$$

Note that $p_{v_a v_f} = 0$ if $v_a, v_f < 0$ or $v_a, v_f > V$. The linear system in (46) together with

$$\sum_{v_a=0}^{V} \sum_{v_f=0}^{V-v_a} p_{v_a v_f} = 1$$

uniquely determine the equilibrium probabilities $\{p_{v_a v_f}\}$. With the help of the generating function,

$$P(z, y) = \sum_{v_a=0}^{V} \sum_{v_f=0}^{V-v_a} p_{v_a v_f} z^{v_a} y^{v_f}, \text{ for } z, y \in [0, 1] \tag{47}$$

We can rewrite (46) as

$$\begin{aligned} 0 &= \lambda V(1-z)P(z,y) + \left((\mu-\lambda)\,z + \lambda z^2 - \mu y\right) P_z'(z,y) + \\ &\quad + \left((y-1)\,\gamma + \lambda y\,(z-1)\right) P_y'(z,y) \end{aligned}$$

which is the equation (25) given in Proposition (4). The solution $P(z,y)$ identifies the unique probability distribution $\{p_{v_a v_f}\}$.

We now prove that the moments of the distribution p_{v_a,v_f} can be found by differentiation of (25). Taking derivatives in (25) with respect to z and y, at $z=y=1$, we get,

$$\mu S P_z'(1,1) = P_y'(1,1)$$

$$\lambda V P(1,1) - (\lambda z + \mu) P_z'(1,1) - \lambda P_y'(1,1) = 0.$$

Note that $P(1,1) = 1$, $P_z'(1,1) = \widehat{v}_a$ and $P_y'(1,1) = \widehat{v}_f$. Given this, and solving the previous two equations, we get the first order moments. Differentiating (25) twice in zz, yy and zy, at $z=y=1$, we get

$$\lambda P_z'(1,1) + (\lambda+\mu) P_{zz}''(1,1) + \lambda P_{zy}''(1,1) = \lambda V P_z'(1,1)$$

$$S^{-1} P_{yy}''(1,1) = \mu P_{zy}''(1,1)$$

$$(\lambda+\mu) P_{zy}''(1,1) + \lambda P_y'(1,1) + \left(\lambda + S^{-1}\right) P_{yy}''(1,1) = \mu P_{zz}''(1,1) + \lambda V P_y'(1,1).$$

Note that, $P_{zz}''(1,1) = E[\widetilde{v}_a^2] - \widehat{v}_a$, $P_{yy}''(1,1) = E[\widetilde{v}_f^2] - \widehat{v}_f$ and $P_{zy}''(1,1) = E[\widetilde{v}_a \widetilde{v}_f]$. Given this and the previous three equations, we get the second order moments and the correlation coefficient.■

Proof of proposition (5). As in the previous proposition, the first part also follows from the ergodic theorem for finite Markov processes. In the exponential case, when $L < V$, the CKE equations coincide with (46) above, except when $v_f = L$. In this case we have to add the term

$$-\mu v_a p_{v_a L}, \tag{48}$$

on the left hand side of (46). This additional term represents the probability of zero unemployment (or of its feasible minimum level). when $L < V$, the stationary matching function is

$$m(U, v_a) = \mu v_a, \text{ if } U > 0 \tag{49}$$

and zero otherwise. Using the generating function

$$\begin{aligned} Q(z,y) &= \sum\nolimits_{v_a=0}^{v_a=V-L} \sum\nolimits_{v_f=0}^{v_f=L} p_{v_a v_f} z^{v_a} y^{v_f} + \\ &+ \sum\nolimits_{v_a=V-L+1}^{v_a=V} \sum\nolimits_{v_f=0}^{v_f=V-v_a} p_{v_a v_f} z^{v_a} y^{v_f}. \end{aligned}$$

the CKE equations become

$$\begin{aligned} &\lambda V Q(z,y) + \left((y-1) S^{-1} + \lambda y (z-1) \right) Q'_y(z,y) + (\lambda z (z-1) + \mu (z-y)) Q'_z(z,y) = \\ &= \lambda V z^V (z-1) p_{V0} + \mu y^L (z-y) \sum\nolimits_{v_a=1}^{v_a=V-L} v_a p_{v_a,L} z^{v_a - 1} \end{aligned} \tag{50}$$

Proceeding as we did in proposition **(4)**, the results follow.■

Proof of proposition (7). Since $\widehat{T}^{*M}$ and $\widehat{\theta}_v^{*M}$ are independent of α, the accumulated profits given in (9) can be written as $\Pi = YQ(S) - w\exp(gx)P(S)$. Since all latent firms are identical, mean market values coincide with individual values. Hence the first order conditions for each firm are also satisfied by the mean market equilibrium values $\left(\widehat{w}^{*M}, \widehat{S}^{*M}\right)$. Therefore,

$$\Pi_w = Y_{\widehat{\theta}_u} \frac{\partial \widehat{\theta}_u}{\partial w} Q(S) - \exp(gx) P(S) = 0$$

$$\Pi_S = (Y - w \exp\left(g\left(x+S\right)\right)) \exp(-rS) = 0.$$

From proposition **(4)**,

$$\left(\lambda\left(x+S\right)+1\right)\widehat{v}_a - \lambda x V = 0$$

$$\left(\lambda\left(x+S\right)+1\right)\widehat{v}_f - \lambda S V = 0.$$

Differentiating the previous four equations with respect to α we get

$$\begin{pmatrix} \Pi^*_{S^2} & \Pi^*_{Sw} & \Pi^*_{S\widehat{v}_a} & \Pi^*_{S\widehat{v}_f} \\ \Pi^*_{wS} & \Pi^*_{w^2} & \Pi^*_{w\widehat{v}_a} & \Pi^*_{w\widehat{v}_f} \\ H^*_S & 0 & H^*_{\widehat{v}_a} & H^*_{\widehat{v}_f} \\ G^*_S & 0 & G^*_{\widehat{v}_a} & G^*_{\widehat{v}_f} \end{pmatrix} \begin{pmatrix} \partial \widehat{S}^{*M}/\partial\alpha \\ \partial \widehat{w}^{*M}/\partial\alpha \\ \partial \widehat{v}^*_a/\partial\alpha \\ \partial \widehat{v}^*_f/\partial\alpha \end{pmatrix} = - \begin{pmatrix} \Pi^*_{S\alpha} \\ \Pi^*_{w\alpha} \\ H^*_\alpha \\ G^*_\alpha \end{pmatrix}$$

where

$$\Pi^*_{w^2} = \frac{\partial^2 Y^*}{\partial w^2} Q(\widehat{S}^{*M}) < 0, \quad \Pi^*_{S^2} = -gY^* \exp(-r\widehat{S}^{*M}) < 0,$$

$$\Pi^*_{Sw} = \Pi^*_{wS} = \left(Y^*_{\widehat{\theta}_u} \frac{\partial \widehat{\theta}^{*M}_u}{\partial w} - \frac{Y^*}{\widehat{w}^{*M}} \right) \exp(-r\widehat{S}^{*M}),$$

$$\Pi^*_{S\widehat{v}_a} = -gY^* \exp(-r\widehat{S}^{*M}) \frac{\partial \widehat{x}^M}{\partial \widehat{v}_a}, \qquad \Pi^*_{S\widehat{v}_f} = -gY^* \exp(-r\widehat{S}^{*M}) \frac{\partial \widehat{x}^M}{\partial \widehat{v}_f},$$

$$\Pi^*_{S\alpha} = \left(-gY^* \frac{\partial \widehat{x}^M}{\partial \alpha} + Y^*_{\widehat{\theta}_u} \frac{\partial \widehat{\theta}^M_u}{\partial \alpha} \right) \exp(-r\widehat{S}^{*M}),$$

$$\Pi^*_{w\widehat{v}_a} = -gY^*_{\widehat{\theta}_u} \frac{\partial \widehat{\theta}^M_u}{\partial w} Q(\widehat{S}^{*M}) \frac{\partial \widehat{x}^M}{\partial \widehat{v}_a}, \qquad \Pi^*_{w\widehat{v}_f} = -gY^*_{\widehat{\theta}_u} \frac{\partial \widehat{\theta}^M_u}{\partial w} Q(\widehat{S}^{*M}) \frac{\partial \widehat{x}^M}{\partial \widehat{v}_f}$$

$$\Pi^*_{w\alpha} = \left(Y^*_{\widehat{\theta}^2_u} \frac{\partial \widehat{\theta}^M_u}{\partial w} \frac{\partial \widehat{\theta}^M_u}{\partial \alpha} + Y^*_{\widehat{\theta}_u} \frac{\partial^2 \widehat{\theta}^M_u}{\partial w \partial \alpha} \right) Q(\widehat{S}^{*M}) - gY^*_{\widehat{\theta}_u} \frac{\partial \widehat{\theta}^M_u}{\partial w} Q(\widehat{S}^{*M}) \frac{\partial \widehat{x}^M}{\partial \alpha}$$

$$H^*_{\widehat{v}_a} = \lambda \left(\widehat{x}^{*M} + \widehat{S}^{*M} \right) + 1 - \lambda \left(V - \widehat{v}^*_a\right) \frac{\partial \widehat{x}^M}{\partial \widehat{v}_a}, \quad H^*_{\widehat{v}_f} = -\lambda \left(V - \widehat{v}^*_a\right) \frac{\partial \widehat{x}^M}{\partial \widehat{v}_f}$$

$$G^*_{\widehat{v}_a} = \lambda \widehat{v}^*_f \frac{\partial \widehat{x}^M}{\partial \widehat{v}_a}, \quad G^*_{\widehat{v}_f} = \lambda \left(\widehat{x}^{*M} + \widehat{S}^{*M} \right) + 1 + \lambda \widehat{v}^*_f \frac{\partial \widehat{x}^M}{\partial \widehat{v}_f}$$

$$H^*_\alpha = -\lambda \left(V - \widehat{v}^*_a\right) \frac{\partial \widehat{x}^M}{\partial \alpha}, \quad H^*_S = \lambda \widehat{v}^*_a, \quad G^*_\alpha = \lambda \widehat{v}^*_f \frac{\partial \widehat{x}^M}{\partial \alpha}, \quad G^*_S = -\lambda \left(V - \widehat{v}^*_f\right).$$

The solution to the previous linear system is,

$$\frac{\partial \widehat{S}^{*M}}{\partial \alpha} = \frac{1}{N^*}\left\{R^*\varepsilon_1\frac{\partial \widehat{\theta}_u^M}{\partial \alpha} + K^*\varepsilon_1\frac{\partial^2 \widehat{\theta}_u^M}{\partial w \partial \alpha} - \left(\lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1\right)A^*\frac{\partial \widehat{x}^M}{\partial \alpha}\right\}$$

$$\frac{\partial \widehat{w}^{*M}}{\partial \alpha} = \frac{1}{N^*}\left\{D^*\frac{\partial \widehat{\theta}_u^M}{\partial \alpha} + gY^*Y^*_{\widehat{\theta}_u}\varepsilon_3\frac{\partial^2 \widehat{\theta}_u^M}{\partial w \partial \alpha} + J^*\frac{\partial \widehat{x}^M}{\partial \alpha}\right\}$$

$$\frac{\partial \widehat{v}_a^*}{\partial \alpha} = \frac{1}{\left(\lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1\right)N^*}\left\{-R^*\varepsilon_4\frac{\partial \widehat{\theta}_u^M}{\partial \alpha} - K^*\varepsilon_4\frac{\partial^2 \widehat{\theta}_u^M}{\partial w \partial \alpha} + M^*\frac{\partial \widehat{x}^M}{\partial \alpha}\right\}$$

$$\frac{\partial \widehat{v}_f^*}{\partial \alpha} = \frac{1}{\left(\lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1\right)N^*}\left\{R^*\varepsilon_5\frac{\partial \widehat{\theta}_u^M}{\partial \alpha} + K^*\varepsilon_5\frac{\partial^2 \widehat{\theta}_u^M}{\partial w \partial \alpha} + P^*\frac{\partial \widehat{x}^M}{\partial \alpha}\right\}$$

where

$$\varepsilon_1 = \lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1 + \lambda\widehat{v}_f^*\frac{\partial \widehat{x}^M}{\partial \widehat{v}_f} - \lambda\left(V - \widehat{v}_a^*\right)\frac{\partial \widehat{x}^M}{\partial \widehat{v}_a},$$

$$\varepsilon_2 = \lambda\widehat{v}_a^*\frac{\partial \widehat{x}^M}{\partial \widehat{v}_a} - \lambda\left(V - \widehat{v}_f^*\right)\frac{\partial \widehat{x}^M}{\partial \widehat{v}_f},$$

$$\varepsilon_3 = \lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1 + \lambda V\left(\frac{\partial \widehat{x}^M}{\partial \widehat{v}_f} - \frac{\partial \widehat{x}^M}{\partial \widehat{v}_a}\right),$$

$$\varepsilon_4 = \lambda\widehat{v}_a^*\left(\lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1\right) - \lambda^2 V\left(V - \widehat{v}_a^* - \widehat{v}_f^*\right)\frac{\partial \widehat{x}^M}{\partial \widehat{v}_f},$$

$$\varepsilon_5 = \lambda\left(V - \widehat{v}_f^*\right)\left(\lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1\right) - \lambda^2 V\left(V - \widehat{v}_a^* - \widehat{v}_f^*\right)\frac{\partial \widehat{x}^M}{\partial \widehat{v}_a},$$

$$N^* = \varepsilon_1 B^* - \varepsilon_2 A^*, \quad B^* = -gY^*\frac{\partial^2 Y}{\partial w^2} - \frac{\left(\Pi^*_{Sw}\right)^2}{Q(\widehat{S}^{*M})}\exp(r\widehat{S}^{*M}),$$

$$A^* = -gY^*\frac{\partial^2 Y}{\partial w^2} + g\frac{\partial Y}{\partial w}\Pi^*_{Sw}\exp(r\widehat{S}^{*M}),$$

$$R^* = Y^*_{\widehat{\theta}_u^2}\frac{\partial \widehat{\theta}_u^M}{\partial w}\frac{\Pi^*_{Sw}}{Q(\widehat{S}^{*M})}\exp(r\widehat{S}^{*M}) - Y^*_{\widehat{\theta}_u}\frac{\partial^2 Y}{\partial w^2},$$

$$J^* = -gY^*\left(\lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1\right)\left(gY^*_{\widehat{\theta}_u}\frac{\partial \widehat{\theta}_u^M}{\partial w} + \frac{\Pi^*_{Sw}}{Q(\widehat{S}^{*M})}\right),$$

$$D^* = gY^*Y^*_{\widehat{\theta}_u^2}\varepsilon_3\frac{\partial \widehat{\theta}_u^M}{\partial w} + g(Y^*_{\widehat{\theta}_u})^2\varepsilon_2\frac{\partial \widehat{\theta}_u^M}{\partial w} + \frac{\Pi^*_{Sw}}{Q(\widehat{S}^{*M})}Y^*_{\widehat{\theta}_u}\varepsilon_1,$$

$$K^* = \Pi^*_{Sw}Y^*_{\widehat{\theta}_u}\exp(r\widehat{S}^{*M}), \; M^* = \lambda\left(\lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1\right)\left(B^*\left(V - \widehat{v}_a^*\right) + A^*\widehat{v}_a^*\right),$$

and

$$P^* = -\lambda\left(\lambda\left(\widehat{x}^{*M} + \widehat{S}^{*M}\right) + 1\right)\left(A^*\left(V - \widehat{v}_f^*\right) + B^*\widehat{v}_f^*\right).$$

All functions are evaluated at the equilibrium solution. Note that, given the second order condition, $B^* > 0$. Using the first order conditions we get $\Pi^*_{Sw} < 0$ and $B^* > A^*$. This and the second order conditions imply $R^* > 0$, $K^* < 0$ and $J^* < 0$. (Note that $\partial^2 Y/\partial w^2 < 0$). If x is homogenous of degree zero, we have

$$\varepsilon_{x\widehat{v}_a} = \frac{\widehat{U}^*}{\widehat{v}^*_f}\varepsilon_{x\widehat{v}_f}.$$

when $L \geq V$ and $L \sim V$, $\varepsilon_2 \sim 0$. Then $N^* \sim \varepsilon_1 B^* > 0 \Longleftrightarrow \varepsilon_1 > 0 \Longleftrightarrow \varepsilon_{x\widehat{v}_a} < \widehat{U}^*/(\widehat{U}^* - \widehat{v}^*_a) \Longleftrightarrow$ $\varepsilon_3 > 0$, $\varepsilon_4 > 0$ and $\varepsilon_5 > 0$. Since $m(\widehat{U}, \widehat{v}_a; \alpha)$ is non decreasing in $\widehat{v}_a$, we have that $\varepsilon_{x\widehat{v}_a} < 1$. Hence, $\varepsilon_{x\widehat{v}_a} < \widehat{U}^*/(\widehat{U}^* - \widehat{v}^*_a)$ holds. Therefore the coefficients a_i and b_i, $i = 1, 2, 3, 4$, are all positive. (Note that $D^* < 0$).

Note that $c_1 > 0 \Longleftrightarrow A^* > 0$. The coefficient c_2 does not depend on A^*. Since $\Pi_{Sw} < 0$ and $B^* > A^*$, it is easy to prove that $c_2 > 0$. If $A^* > 0$, we have that $M^* > 0$ and $P^* < 0$. Then $c_3 > 0$ and $c_4 > 0$. If $A^* < 0$, we have that $\widehat{v}^*_f > 0 \Longleftrightarrow |A^*|\,\widehat{v}^*_a < B^*\,(V - \widehat{v}^*_a)$ and $P^* > 0$ $\Longleftrightarrow B^*\widehat{v}^*_f < |A^*|\left(V - \widehat{v}^*_f\right)$. Hence the proposition follows.■

Proof of lemma (9). We find (5) for $\theta_u \sim \beta(\alpha, 2)$. This is,

$$\widehat{\theta}_u = \frac{\alpha(\alpha+2)\left((\theta^m_u)^{\alpha+1} - (\theta_v)^{\alpha+1}\right) - \alpha(\alpha+1)\left((\theta^m_u)^{\alpha+2} - (\theta_v)^{\alpha+2}\right)}{(\alpha+1)(\alpha+2)\left((\theta^m_u)^{\alpha} - (\theta_v)^{\alpha}\right) - \alpha(\alpha+2)\left((\theta^m_u)^{\alpha+1} - (\theta_v)^{\alpha+1}\right)}. \tag{51}$$

Therefore $\partial\widehat{\theta}_u/\partial\alpha > 0$ and $\partial^2\widehat{\theta}_u/\partial\theta^m_u\partial\alpha > 0$.

To determine the sign of $\partial^2\widehat{\theta}_u/\partial w\partial\alpha$, note that $\partial\widehat{\theta}_u/\partial w = (\partial\widehat{\theta}_u/\partial\theta^m_u)(\partial\theta^m_u/\partial w)$, where $\partial\theta^m_u/\partial w > 0$ and θ^m_u is independent of α. Hence, $\partial^2\widehat{\theta}_u/\partial\theta^m_u\partial\alpha$ and $\partial^2\widehat{\theta}_u/\partial w\partial\alpha$ have the same sign.■