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Game Theory and Centrality in Directed Social Networks.

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Abstract

In this paper we define a family of centrality measures for directed social networks from a game theoretical point of view. We follow the line started with our previous paper (Gómez et al. 2003) and besides the definition we obtain a characterization of the measures and an additive decomposition in three measures that can be interpreted in terms of emission, betweeness and reception centralities. Finally we apply the obtained results to rank the importance of players in a simplified version of a soccer game.

Key words: Social networks, game theory, centrality, Shapley value.
Classification code: C71.

1 Introduction

A social network is a set of nodes representing people, groups, organizations, enterprises, etc., that are connected by links showing relations or flows between them.

The social network analysis permits to understand patterns of behavior in a wide and variated range of situations. From the description of terrorist networks to the examination

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of useful patterns in clickstreams on the www or in e-mail flows, we can include the spread of HIV in a community, the network of innovators in the European regions or the vulnerability of an electrical network.

Network analysis study the implications of the restrictions of different actors in their communications and then in their opportunities of relation. The fewer constraints an actor face, the more opportunities he/she will have, and thus he will be in a more favorable position to bargain in exchanges and to intermediate in the bargains of others that need him, increasing his influence. Then, among other goals, network analysis try to obtain indices, as objective as possible, to measure hypothetic or not directly observable variables such that influence, opportunities, better position ...

Social networks analysts consider the closely related concepts of centrality and power as fundamental properties of individuals, that inform us about aspects as who is who in the network, who is a leader, who is an intermediary, who is almost isolated, who is central, who is peripheral... Under the network approach it is assumed that this power is inherently relational.

Social networks researchers have developed several centrality measures. Degree, Closeness and Betweenness centralities are without doubt the three most popular ones: Degree centrality (Shaw, 1954; Nieminen, 1974) focuses on the level of communication activity, identifying the centrality of a node with its degree. Closeness centrality (Beauchamp, 1965; Sabidussi, 1966) considers the sum of the geodesic distances between a given actor and the remaining as a decentrality measure in the sense that the lower this sum is, the greater the centrality. Closeness centrality is, then, a measure of independence in the communications, in the relations or in the bargaining, and thus, it measure the possibility to communicate with many others depending on a minimum number of intermediaries. Betweenness centrality (Bavelas, 1948; Freeman, 1977) emphasizes the value of the communication control: the possibility to intermediate in the relation of others. Under this approach all possible geodesic paths between pairs of nodes are considered. The centrality of each actor is the number of such paths in which it lies.

It is intuitively acceptable that the hub in a star is a node with a privileged position from a relational point of view. All previous measures give to this node the higher centrality as an actor in this position:

- can communicate directly with all the others,

- is maximally close to the remaining and
- intermediates in the communication of all pairs of nodes.

Stephenson and Zelen (1989) abandon the geodesic path as structural element in the definition of centrality, to introduce a measure based on the concept of information as it is used in the theory of statistical estimation. The defined measure uses a weighted combination of all paths between pairs of nodes, the weight of each path depending on the information contained in it.

Bonacich (1972, 1987) suggests another concept of centrality. He proposes to measure the centrality of different nodes using the eigenvector associated with the largest characteristic eigenvalue of the adjacent matrix. The ranking of web sites as they appear in the web search engine Google was created from this measure by Brin and Page (1998).

All previous approaches assume that the direct relation between two nodes (whenever it exists) is symmetrical. Nevertheless it is easy to find situations in which the connections are directed, having an specific sense: for example in the case of the network of citations in scientific papers or in the walks across the pages in the www. It seems, then, to be relevant to define measures of centrality (or to adapt the already existing ones) for these special situations that can be considered, in fact, more general than the not directed ones. Contributions in this direction can be found in White and Borgatti (1994), that generalize the Freeman's geodesic measures for betweenness in undirected graphs, Tutzauer (2007), who uses the entropy as a measure of centrality in networks characterized by path-transfer flow, and Pollner et al. (2008), that introduce an algorithm to calculate the centrality for cohesive subgroups in directed networks.

In this paper we propose a family of centrality measures for directed graphs using a game-theoretical point of view. The seminal work in applying game theory to the topic of centrality for nodes in graphs is due to Grofman and Owen (1982). They used the framework of games with restrictions in the communication introduced by Myerson (1977, 1980). In Gómez et al. (2003), we extend previous ideas to obtain a new family of centrality measures with some appealing properties and the corresponding calculation methods. Other contributions are closely related with the problem of centrality but ignore it focusing only in the definition and properties, including characterizations, of allocation rules for games with restrictions in the cooperation, these restrictions being given by graphs or digraphs. An excellent survey of the work on this topic can be found in Slikker and Van den Nouweland (2001). Other recent relevant contributions are debt to Amer et al. (2007), that define a family of measures for a concept they call accessibility in oriented networks, Van den Brink and Borm (2002) for a special type of digraphs representing competitions, González-Arangüena et al. (2008) where the classical Myerson value is generalized to games with restrictions in the communication given by digraphs and Kin and Jun (2008) on different types of connectivity in directed networks and associated characterizations of allocations rules.

The approach we present here assumes that actors in a directed network are simultaneously players in a TU game which model their economic interests. The restrictions in the communication generated by the digraph modify such game transforming it in a generalized TU one: the digraph restricted game. In these generalized TU games, as introduced by Nowak and Radzik (1994), the worth of a coalition depends not only on its members but also on the order in which they incorporate to that coalition. The centrality of each actor is then measured as the variation of his power from the game without restrictions to the digraph-restricted one. We will use as index of power for players in a TU game the Shapley value, and for generalized TU ones, a parametric family of index that include those characterized by Nowak and Radzik (1994) and Sánchez and Bergantiños (1997). Therefore a family of measures for each digraph is obtained, each member of this family corresponding to a particular election of the a priori economic interests (the game) and of the fixed power index.

The proposed approach is closely related with the ones in Gómez et al. (2003) and in Amer et al. (2007), specially in the technical framework. Moreover the introduced measures are characterized. This characterization, is based on two properties: component efficiency and α-directed fairness. The consideration of arcs (directed links) as units of relation, instead of the classical links of a graph, introduces an element of asymmetry in the bilateral relations and, as a consequence, a possible different bargaining power for both incident nodes. This is the meaning of the α-directed fairness, a closely related property with the α-hierarchical payoff property in Slikker et al. (2005).

The remaining of the paper is organized as follows. Section 2 contains the notation and some preliminary concepts. In Section 3 the definition and some properties (including a characterization) of the proposed family of centrality measures are given. In Section 4 each one of the measures is additively decomposed in three different ones. The obtained results are applied in Section 5 to obtain centralities in a simplified version of soccer. Final conclusions appear in Section 6.

2 Preliminaries

2.1 Games and Generalized Games

A game in characteristic function form (a coalitional game or a TU-game) is a pair (N, v) where v (the characteristic function) is a real function defined on 2^N, the set of all subsets of N (coalitions), that satisfies $v(\emptyset) = 0$. For each $S \in 2^N$, $v(S)$ represents the (transferable) utility that players in S can obtain if they decide to cooperate. Implicitly, it is supposed that, if the players in S form a coalition, members of S must talk together and achieve a binding agreement.

When there is no ambiguity with respect to the set of players N, we will identify a game (N, v) with its characteristic function v. We will denote by s the cardinality of the coalition $S \subset N$ and G^N will be the $2^n - 1$ dimensional vector space of TU-games with player set N. Its unanimity games basis $\{u_S\}_{\emptyset \neq S \subset N}$ is defined as follows:

$$\text{for all } S \subset N,\, S \neq \emptyset,\, u_S(T) = \begin{cases} 1 & S \subset T \\ 0 & \textit{otherwise.} \end{cases}$$

As a consequence, every TU game $v \in G^N$ is a linear combination of those games in the unanimity basis:

$$v = \sum_{\emptyset \neq S \subset N} \Delta_v(S) u_S.$$

The coordinates $\{\Delta_v(S)\}_{\emptyset \neq S \subset N}$ are known as the Harsany dividens, (Harsany 1963).

A game $v \in G^N$ is said to be superadditive if for all coalitions $S, T \subset N$ with $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$ holds.

A game $v \in G^N$ is said to be convex if for all coalitions $S, T \subset N$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ holds.

A game $v \in G^N$ is said to be zero-normalized if $v(\{i\}) = 0$ for all $i \in N$.

A game $v \in G^N$ is said to be symmetric if it exists a function $f : \{1, 2, \ldots, n\} \to \mathbb{R}$ such that for all coalition $S \subset N$, $v(S) = f(s)$. The subespace of G^N formed by symmetric games will be noted $\mathcal{S}^N$. $\mathcal{S}_0^N \subset \mathcal{S}^N$ will be the subespace of games that are also symmetric and zero-normalized.

A game $v \in G^N$ is said to be almost positive if all its Harsanyi dividens are nonegative. The family of all these games will be noted $\mathcal{AP}^N$.

A point solution for TU-games is a function which assigns a payoff vector $x \in \mathbb{R}^n$ in every TU-game in G^N. One of the most famous solutions is the *Shapley value* (Shapley (1953)), φ, which is given by:

$$\varphi_i(N, v) = \sum_{S \subset N \setminus \{i\}} \frac{(n - s - 1)!s!}{n!} (v(S \cup \{i\}) - v(S)), \qquad \text{for all } i \in N.$$

An alternative expression for the Shapley value is:

$$\varphi_i(N, v) = \sum_{S \subset N, i \in S} \frac{\Delta_v(S)}{s}, \text{ for all } i \in N.$$

Nevertheless, in many social or economic situations, the formation of coalitions is a process in which not only the members of the coalitions are important but also the order in which they appear. Taking this idea into account, Nowak and Radzik (1994) introduced the concept of game in generalized characteristic function form.

Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. For each $S \in 2^N \setminus \{\emptyset\}$, let us denote by $\pi(S)$ the set of all permutations or ordered coalitions of the players in S and, for notational convenience, $\pi(\emptyset) = \{\emptyset\}$. We will denote $\Omega(N) = \{T \in \pi(S) \mid S \subset N\}$ the set of all ordered coalitions with players in N.

Given an ordered coalition $T \in \Omega(N)$, there exists $S \subset N$ such that $T \in \pi(S)$. We will denote $H(T) = S$ for the set of players in the ordered coalition T, and $t = |H(T)|$.

A game in generalized characteristic function form is a pair (N, v), N being the players set and v a real function (the generalized characteristic function), defined on $\Omega(N)$ and satisfying $v(\emptyset) = 0$.

For each $S \subset N$, and for every ordered coalition $T \in \pi(S)$, $v(T)$ represents the economic possibilities of the players in S if the coalition is formed following the order given by T. When there is no ambiguity with respect to the set of players N, we will identify the (generalized) game (N, v) with its (generalized) characteristic function v.

We will denote by $\mathcal{G}^N$ the set of all generalized cooperative games with players set N. $\mathcal{G}^N$ is a vector space with dimension $|\Omega(N)| - 1$. Let us observe that there exists an isomorphism between the vector space G^N and the subspace of $\mathcal{G}^N$ consisting of all games for which $v(T) = v(R)$ if $H(T) = H(R)$ holds. Intuitively, for games in G^N, the order in which the coalitions are formed is irrelevant.

Taking into account the previous idea we will sometimes identify each game $v \in G^N$ with the (transformed) game $\hat{v} \in \mathcal{G}^N$ defined by:

$$\hat{v}(T) = v(H(T)) \text{ for all } T \in \Omega(N).$$

Each ordered coalition $T = (i_1, \ldots, i_t) \in \Omega(N)$ establishes a strict linear order $\prec_T$ in $H(T)$, defined as follows. For all $i, j \in H(T)$, $i \prec_T j$ (i precedes j in T) if and only if there exist $k, l \in \{1, \ldots, t\}$, $k < l$, such that $i = i_k$, $j = i_l$.

We base on this strict linear order to define an inclusion relation in $\Omega(N)$ in this way: for $A,\ B \in \Omega(N)$ we will say that A is included in B (noted $A \tilde{\subset} B$) if $H(A) \subset H(B)$ and for all $i, j \in H(A)$, and $i \prec_A j$ it holds $i \prec_B j$.

Given an ordered coalition $T = (i_1, i_2, \ldots, i_t) \in \Omega(N)$, we will note $i_j(T) = j$, $j = 1, 2, \ldots, t$, for the position of each player in that coalition. Moreover we will note $T(j) = i_j$, $j = 1, \ldots, t$, for the player that is in position j in the coalition.

In this paper, a special basis of $\mathcal{G}^N$, *the generalized unanimity basis*, consisting of the (generalized) unanimity games $\{w_T\}_{\emptyset \neq T \in \Omega(N)}$, will often be used. For any $T \in \Omega(N) \backslash \{\emptyset\}$, the generalized characteristic function w_T is defined as follows:

$$\text{for all } R \in \Omega(N),\ w_T(R) = \begin{cases} 1 & if \ \ T \tilde{\subset} R \\ 0 & otherwise. \end{cases}$$

The transformed games $\{\hat{u_S}\}_{\emptyset \neq S \subset N}$ of the classical unanimity games $\{u_S\}_{\emptyset \neq S \subset N}$ of G^N, can be easily expressed in terms of the $\{w_T\}_{\emptyset \neq T \in \Omega(N)}$ in the following way:

$$\hat{u}_S = \sum_{T \in \pi(S)} w_T, \qquad \text{for each } S \in 2^N \setminus \{\emptyset\}.$$

For a given $v \in \mathcal{G}^N$, $\{\Delta_v^*(T)\}_{\emptyset \neq T \in \Omega(N)}$ is the set of the generalized unanimity coefficients of v (the coordinates of v in the generalized unanimity basis). Sánchez and Bergantiños (1997) proved that, for all $T \in \Omega(N) \setminus \{\emptyset\}$:

$$\Delta_v^*(T) = \sum_{R \tilde{\subset} T} (-1)^{t-r} v(R).$$

In their seminal paper on games in generalized characteristic function form, Nowak and Radzik (1994), define and characterize a value Ψ^{NR} for these games that generalizes the Shapley value for TU-games. For each $v \in G^N$ and all $i \in N$ this value is given by:

$$\Psi_i^{NR}(N, v) = \sum_{S \subset N \setminus \{i\}} \sum_{T=(i_1, i_2, \ldots, i_t) \in \pi(S)} \frac{(n-t-1)!}{n!} (v(i_1, i_2, \ldots, i_t, i) - v(T)).$$

An alternative expression for this value based on the generalized unanimity coeficients of v is:

$$\Psi_i^{NR}(N, v) = \sum_{T \in \Omega(N), i(T)=t} \frac{\Delta_v^*(T)}{t!}.$$

Later, Sánchez and Bergantiños (1997) define and study another generalization, Ψ^{SB} of the Shapley value for TU-games to this class of generalized games, differing from the former in null player and symmetry axioms. This value can be obtainined from the two alternative equivalent following expressions:

$$\Psi_i^{SB}(N, v) = \sum_{S \subset N \setminus \{i\}} \sum_{T=(i_1, i_2, \ldots, i_t) \in \pi(S)} \frac{(n-t-1)!}{n!(t+1)} \sum_{l=1}^{t+1} (v(i_1, \ldots, i_{l-1}, i, i_l, \ldots, i_t) - v(T)).$$

$$\Psi_i^{SB}(N, v) = \sum_{T \in \Omega(N), i \in H(T)} \frac{\Delta_v^*(T)}{t!t}.$$

In this paper we use a parametric family of functions defined on $\mathcal{G}^N$, $\{\Psi^\alpha\}_{\alpha \in [0,1]}$. Each one of them can be considered as a particular point solution. They are defined, for each generalized TU-game $(N, v) \in \mathcal{G}^N$ and all $i \in N$ by:

$$\Psi_i^\alpha(N, v) = \sum_{T \in \Omega(N), i \in H(T)} \Delta_v^*(T) \frac{\alpha^{t-i(T)}}{t! \sum_{l=0}^{t-1} \alpha^l}, \; \alpha \in [0, 1]. \tag{1}$$

The defined family includes the point solutions Ψ^{NR} and Ψ^{SB}. In particular $\Psi^{NR} = \Psi^0$ (it is implicit in (1) the abuse of notation $0^0 = 1$) and $\Psi^{SB} = \Psi^1$.

The idea behind this point solution is that, in a unanimity game w_T we share the reward among the non dummy players, no equitatively (as Sánchez and Bergantiños does) nor all for the last one (as Nowak and Radzik does) but proportionaly to the position of the player in T.

2.2 Graphs and Directed Graphs

A graph is a pair (N, γ), $N = \{1, 2, \ldots, n\}$ being a finite set of nodes and γ a collection of *links* (edges or ties), that is, unordered pairs $\{i, j\}$ with $i, j \in N$, $i \neq j$. When there is no ambiguity with respect to N, we will refer to the graph (N, γ) as γ.

If $\{i, j\} \in \gamma$, we will say that i and j are *directly connected* in γ. We will say that i and j are *connected* in γ if it is possible to join them by a sequence of edges from γ.

Given a graph (N, γ), the notion of connectivity induces a partition of N in connected components. Two nodes i and j, $i \neq j$, are in the same *connected component* if and only if they are connected. By connected component we mean what is also known as a maximal connected subset. N/γ denotes the set of all connected components in γ.

A directed graph or digraph is a pair (N, d), $N = \{1, 2, \ldots n\}$ being a set of nodes and d a subset of the collection of all ordered pairs (i, j), $i \neq j$, of elements of N. Each pair $(i, j) \in d$ is called an *arc*. In the following, if there is no ambiguity with respect to N, we will refer to the digraph (N, d) as d. We will denote D^N for the set of all possible digraphs with nodes set N.

Given a digraph (N, d), if $(i, j) \in d$, we will say that i is *directly connected* with j. Obviously, if i is directly connected with j, the reverse is not necessarily true. If i is not directly connected with j in the digraph, it may still be possible to connect them, provided that there are other nodes through which we can do so. We will say that i is *connected* with j in the digraph (N, d) if there is a directed path connecting them, i.e., if there exists an ordered sequence of nodes in N, $(i_1, i_2, \ldots, i_s)$, such that $i_1 = i$, $i_s = j$ and $(i_l, i_{l+1}) \in d$ for all $l \in \{1, 2, \ldots, s-1\}$.

We will say that an ordered set $T = (i_1, i_2, \ldots, i_t) \in \Omega(N)$ is *connected* in the digraph (N, d) if, for all $l = 1, \ldots, t-1$, i_l is directly connected with i_{l+1} in the digraph (N, d).[2]

[2]This concept of connectedness coincides with the one in Amer, Giménez and Magaña (2007) but differs from the used in González-Arangüena et al. (2008) that was introduced basically with the idea of

Given $(N, d) \in D^N$, $\mathcal{C}_d^N$ is defined as:

$$\mathcal{C}_d^N = \{T \in \Omega(N) | T \text{ is connected in } (N, d)\},$$

and we will assume $\emptyset \notin \mathcal{C}_d^N$.

Given a digraph $(N, d) \in D^N$, we can define the *induced graph* $(N, \gamma(d))$ as follows:

$$\gamma(d) = \{\{i, j\} \mid i, j \in N \text{ and } (i, j) \in d \text{ or } (j, i) \in d\}.$$

A (not ordered) set $C \subset N$ is a component in the digraph (N, d) if $C \in N/\gamma(d)$, i.e., if C is a connected component in the graph $(N, \gamma(d))$. So, given a digraph $(N, d) \in D^N$ we will establish a partition of N in components. We will denote by N/d the set of all the components of the directed graph (N, d). Obviously, $N/d = N/\gamma(d)$.

Let us observe that, given a component $C \in N/d$ and $T \in \Omega(C)$, it is possible that T can be not connected ordered set in (N, d).

Then the connection concept for components we use is clearly weaker than the one used for ordered sets. We will say that the digraph $d \in D^N$ is (weakly) connected if $|N/d| = 1$.

We will use, for short, (N, d^{ij}) instead of $(N, d\backslash\{(i, j)\})$ for the digraph obtained when the arc (i, j) is removed from (N, d).

We will use, also, $L_i(N, d)$ to note the subset of d consisting of all the arcs incident on i.

2.3 Digraph Communication Situations

A directed communication situation[3] is a triplet (N, v, d) where (N, v) is a TU-game in G^N and (N, d) a digraph in D^N. We will note $\mathcal{DC}^N$ for the family of all directed communication situations with nodes-players set N. The subset of $\mathcal{DC}^N$ corresponding to

generalizing the Myerson value to the case of games restricted to digraphs. The definition of connectedness in digraphs is not so obvious as in the case of graphs. In Kun and Jim (2008) several alternative concept of connection in digraphs are used.

[3]This denomination is used in Slikker and Van den Nouweland (2001) with a different meaning. Nevertheless we have preferred to maintain it in order to emphasize the generalization of the classical concept of communication situation to this new setting of directed graphs.

directed communication situations in which the game is a symmetric one will be noted as $\mathcal{DC}_S^N$, whereas $\mathcal{DC}_{APS}^N$ will correspond to directed communication situations in which the game is simultaneously almost positive and symmetric. The respective subsets of $\mathcal{DC}_S^N$ and $\mathcal{DC}_{APS}^N$ formed by directed communication situations in which the game is also a 0-normalized one will be noted $\mathcal{DC}_{S_0}^N$ and $\mathcal{DC}_{APS_0}^N$.

3 A family of centrality measures for digraph communication situations

In order to define centrality measures for digraphs using a game theoretical approach, let us consider $(N, u_S, d) \in \mathcal{DC}^N$, $S \subset N$, where $(N, u_S) \in G^N$ is a unanimity game and $(N, d) \in D^N$ a digraph. The restrictions in the communication modeled by a digraph affect to the worth of the several coalitions and thus a new game arises to take into account this constrains. This is the classical approach to study the problem of games with resctrictions in the communications as in Myerson (1977,1980).

Given the directed communication situation (N, u_S, d), the digraph-restricted game (N, u_S^d) can be interpreted as the game of connect S in d, and in our approach, we propose to consider all the connected ordered coalitions $T \in \pi(S)$. This leads to define the game (N, u_S^d) as the generalized TU one with (generalized) characteristic function given by

$$u_S^d = \sum_{T \in \pi(S) \cap \mathcal{C}_d^N} w_T.$$

We will extend previous definition to $\mathcal{DC}^N$ by linearity and thus, given $(N, v, d) \in \mathcal{DC}^N$, we define the digraph-restricted game $(N, v^d) \in \mathcal{G}^N$ as the one with generalized characteristic function:

$$v^d = \sum_{\emptyset \neq S \subset N} \Delta_v(S) u_S^d.$$

In the next proposition we will give an expression for v^d in terms of v.

Proposition 3.1 *Given $(N, v, d) \in \mathcal{DC}^N$, the generalized characteristic function of the digraph restricted game (N, v^d) is given by:*

$$v^d(T) = \sum_{\emptyset \neq R \tilde{\subset} T} \lambda_v^d(R) v(H(R)),$$

being $\lambda_v^d(R) = \sum_{R \tilde{\subset} K \tilde{\subset} T, K \in \mathcal{C}_d^N} (-1)^{k-r}$.

Proof:

$$v^d(T) = \sum_{\emptyset \neq S \subset N} \Delta_v(S) u_S^d(T) = \sum_{\emptyset \neq S \subset N} \Delta_v(S) \sum_{K \in \pi(S) \cap \mathcal{C}_d^N} w_K(T) =$$
$$= \sum_{K \tilde{\subset} T,\ K \in \mathcal{C}_d^N} \Delta_v(H(K)) = \sum_{K \tilde{\subset} T,\ K \in \mathcal{C}_d^N} \sum_{\emptyset \neq L \subset H(K)} (-1)^{k-l} v(L) =$$
$$= \sum_{\emptyset \neq R \tilde{\subset} T} \sum_{R \tilde{\subset} K \tilde{\subset} T,\ K \in \mathcal{C}_d^N} (-1)^{k-r} v(H(R)),$$

last equality holding because there exists a unique permutation of R, L, $(L = H(R))$ such that $R \tilde{\subset} K \tilde{\subset} T$. ■

Once the previous framework is obtained we pass to define a family of centrality measures.

Definition 3.1 *Given* $(N, v, d) \in \mathcal{DC}^N$ *the centrality of node* $i \in N$*, noted* $K_i^\alpha(N, v, d)$ *is defined as:*

$$K_i^\alpha(N, v, d) = \Psi_i^\alpha(N, v^d) - \varphi_i(N, v).$$

As it is obvious from that definition, we are assuming that the centrality of a given node can be measured as the difference in its allocation when the restrictions in the communication given by the digraph are taking into account and the corresponding allocation when the restrictions do not exist.

Interpreting Shapley value and Ψ^α, $\alpha \in [0, 1]$ as indices of power in G^N and $\mathcal{G}^N$ respectively, the defined measure of centrality for a given node can be viewed as the variation in its power due to the position in the digraph.

In order to avoid *a priori* differences among players given by different status in the original game, that can contaminate the obtained centrality, we propose to use symmetric games (N, v). As a consequence, the term $\varphi_i(N, v)$ is equal for all players and then, removing it, only a shift transformation is produced. Another shift transformation (that will permit us to associate null centrality to isolated nodes) is obtained when replacing each symmetric game by its zero-normalized version. Then when defining the centrality we will restrict ourselves to symmetric and zero-normalized games.

From now on we will use this alternative definition of centrality:

Definition 3.2 *Given* $(N, v, d) \in \mathcal{DC}^N_{S_0}$, $\alpha \in [0,1]$ *the centrality of node* $i \in N$ *is defined as:*

$$\kappa_i^\alpha(N, v, d) = \Psi_i^\alpha(N, v^d).$$

$\alpha \in [0,1]$ evaluates the assymmetry of the arcs impact and can be view as a discount factor of the importance of the iniciator node versus the receiver one. Of course, chossing a value for α, is a critical aspect of the defined family of measures. This value can be obtained from empirical information, from the relative importance of to be initiator in a relation... Obviously similar results to those proposed here are obtained if we change α by $\frac{1}{\alpha}$ assuming that the initiator node is in a better position than the receiver node.

As a consequence, the special case $\alpha = 1$ can be interpreted as the one in which both incident nodes in an arc play symmetrical roles. Then it is natural to assume that swiching these roles for all players, the respective centralities are not affected. This result is stated in next proposition.

Proposition 3.2 *For* $\alpha = 1$, $\kappa^1(N, v, d) = \kappa^1(N, v, \overline{d})$ *and so when the measure considers symmetrically both nodes incident in an arc, the obtained centralities coincide with those obtained changing the sense of the arcs.*

Proof: For each $S \subset N$ we have $u_S^d = \sum_{T \in \pi(S) \cap \mathcal{C}_d^N} w_T$. For $T = (i_1, i_2 \ldots, i_{r-1}, i_r)$ let us note $\overline{T} = (i_r, i_{r-1}, \ldots, i_2, i_1)$. Then $T \in \mathcal{C}_d^N$ if and only if $\overline{T} \in \mathcal{C}_{\overline{d}}^N$, and thus $u_S^{\overline{d}} = \sum_{T \in \pi(S) \cap \mathcal{C}_d^N} w_{\overline{T}}$.

Using that $\Psi_i^1(N, w_T)$ does not depend on i for all $i \in T$ (and 0 if $i \notin T$) we have that $\Psi_i^1(N, w_T) = \Psi_i^1(N, w_{\overline{T}})$ for all $i \in N$. Then, by the linearity of the centrality measure, we have

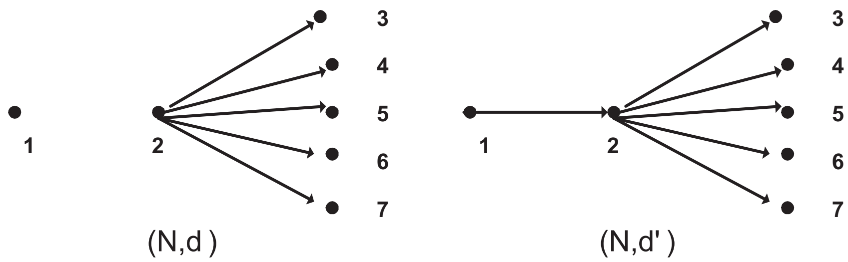
$$\kappa_i^1(N, v, d) = \kappa_i^1(N, v, \overline{d})$$

$\blacksquare$

In a sociological context it is assumed that, adding an arc between two given nodes increases (or at least does not decreas) their centralities. In allocation rules for communication situations, this property is known as stability. So, it is reasonable to explore the extent to which this property is verified by the proposed family of centrality measures.

Unfortunately, in order to satisfy this property, we must restrict ourselves to use games in a subset of G^N. Next example illustrates this point.

Example 3.1 *Consider the directed communication situation* (N,v,d) *with* $N = \{1,2,3,4,5,6,7\}$, v *the symmetric and superadditive, (and even convex) game* $v(S)=\sum_{k=2}^{7}\binom{s}{k}(-1)^k$ *and* $d=\{(2,3),(2,4),(2,5),(2,6),(2,7)\}$. *Consider also the directed communication situation* (N,v,d') *with* $d'=d\cup\{(1,2)\}$.



As $v=\sum_{S\subset N,\, s\geq 2}(-1)^s u_S$ *and using the definition of the restricted game, we have:*

$$v^d = w_{(2,3)} + w_{(2,4)} + w_{(2,5)} + w_{(2,6)} + w_{(2,7)},$$

$$v^{d'} = w_{(1,2)} + w_{(2,3)} + w_{(2,4)} + w_{(2,5)} + w_{(2,6)} + w_{(2,7)} -$$

$$-w_{(1,2,3)} - w_{(1,2,4)} - w_{(1,2,5)} - w_{(1,2,6)} - w_{(1,2,7)}.$$

and thus, for $\alpha\in[0,1]$,

$$\kappa_1^\alpha(N,v,d)=0, \text{ and } \kappa_1^\alpha(N,v,d')=\frac{\alpha}{2!(1+\alpha)}-\frac{5\alpha^2}{3!(1+\alpha+\alpha^2)}.$$

It is easy to see that, for $\alpha>\frac{\sqrt{7}-1}{2}$, $\kappa_1^\alpha(N,v,d)>\kappa_1^\alpha(N,v,d')$.

Of course, previous example points out an undesirable aspect of the proposed meaures. This behaviour can be avoided if we restrict ourselves to the family $\mathcal{AP}^N\subset G^N$, that will permit us to guarantee the stability. Even this restriction can be viewed as a weakness of the defined measures, on the other hand, it must be pointed out that this type of games incentivate the formation of coalitions in a stronger way than, for example superadditive games or convex games, giving a positive dividend to each coalition. So, from now on, we will suppose that the economical interests of players satisfy this strong tendency to cooperate. Nevertheless, in order to obtain the maximum degree of generality possible about the properties of $\Psi^\alpha(N,v^d)$, the results will be written assuming the lesser restrictive set of hypothesis.

Several of the games in this family have an intuitive interpretation as communication games:

(a) The game (N, v_k), $k = 2, \ldots, n$ with characteristic function,

$$v_k(S) = \begin{cases} k!\binom{s}{k} & k = 1, \ldots, s \\ 0 & k = s+1, \ldots, n, \end{cases}$$

represents, for each subset $S \subset N$, the number of ordered subcoalitions of S with size k that can be formed.

(b) The zero-normalized conferences game is $v = \sum_{k=2}^{n} v_k$.

(c) And more generally any linear positive combination $v = \sum_{k=2}^{n} \mu_k v_k$ of v_k games with $\mu_i \geq 0, \ \ i = 2, \ldots, n$.

Proposition 3.3 *Given a directed communication situation* $(N, v, d) \in \mathcal{DC}^N_{APS_0}$ *and* $(i, j) \in d$*:*

$$\kappa^\alpha_l(N, v, d) \geq \kappa^\alpha_l(N, v, d^{ij}) \textit{ for } l \in N \textit{ and } \alpha \in [0, 1].$$

Proof: Consider first the unanimity games (N, u_S), $S \subset N$, $s \geq 2$. Obviously $(N, u_S) \in \mathcal{AP}^N \cap \mathcal{S}^N_0$. Then, for all $l \in N$ and all $\alpha \in [0, 1]$,

$$\kappa^\alpha_l(N, u_S, d) = \sum_{T \in \pi(S) \cap \mathcal{C}^N_d} \Psi^\alpha_l(N, w_T) \geq \sum_{T \in \pi(S) \cap \mathcal{C}^N_{d^{ij}}} \Psi^\alpha_l(N, w_T) = \kappa^\alpha_l(N, u_S, d^{ij}),$$

the inequality holding because $\mathcal{C}^N_{d^{ij}} \subset \mathcal{C}^N_d$ and $\Psi^\alpha_l(N, w_T) \geq 0$. The result for a more general directed communication situation (N, v, d) with $(N, v) \in \mathcal{AP}^N \cap \mathcal{S}^N_0$ is deduced using the linearity of the measure and the fact that all dividends are nonegative. ∎

As a straightforward consequence we obtain the stability of the defined measures.

Using previous proposition is easy to see that the centrality of a given node $i \in N$ is minimal when i is an isolated node.

Proposition 3.4 *Given* $(N, v, d) \in \mathcal{DC}^N_{APS_0}$ *and* $i \in N$ *such that* $L_i(N, d) = \emptyset$ *(i.e.* i *is an isolated node) then:*

$$0 = \kappa_i^\alpha(N, v, d) \leq \kappa_i^\alpha(N, v, d'),$$

for all $(N, v, d') \in \mathcal{DC}^N_{APS_0}$ *and all* $\alpha \in [0, 1]$.

Proof: Given $(N, v, d) \in \mathcal{DC}^N_{APS_0}$, $i \in N$ and $\alpha \in [0, 1]$, using previous proposition sequentially:

$$\kappa_i^\alpha(N, v, d') \geq \kappa_i^\alpha(N, v, d' \setminus L_i(N, d')) = v(\{i\}) = 0 = \kappa_i^\alpha(N, v, d), \text{ for all } \alpha \in [0, 1].$$

■

Next property stablishes that the centrality of a given node depends only on the component to which it belongs. So, we can calculate centralities in a local way.

Proposition 3.5 *Let* $(N, v, d) \in \mathcal{DC}^N_{APS_0}$ *and let* $N_1,\ N_2 \subset N$ *with* $N_1 \cup N_2 = N$ *and* $N_1 \cap N_2 = \emptyset$. *Let* $d_l = \{(i, j) \in d$ *such that* $(i, j) \in N_l\}$, $\ l = 1, 2$ *and suppose* d *verifies* $d = d_1 \cup d_2$. *Then, for* $i \in N_l$, $\ l = 1, 2$ *and* $\alpha \in [0, 1]$*:*

$$\kappa_i^\alpha(N, v, d) = \kappa_i^\alpha(N_l, v_{|N_l}, d_l).$$

Proof: The characteristic function v^d satisfies for all $T \subset N$, $v^d(T) = v^d(T \cap N_1)+$ $v^d(T \cap N_2) = v^{d_1}(T \cap N_1) + v^{d_2}(T \cap N_2) = (v_{|N_1})^{d_1}(T) + (v_{|N_2})^{d_2}(T)$, and thus,

$$v^d = (v_{|N_1})^{d_1} + (v_{|N_2})^{d_2}.$$

Without lost of generality, suppose $i \in N_1$, therefore:

$$\Psi_i^\alpha(N, v^d) = \Psi_i^\alpha(N_1, (v_{|N_1})^{d_1}) = \kappa_i^\alpha(N_1, v_{|N_1}, d_1),\ \alpha \in [0, 1].$$

■

In order to compare centralities of nodes belonging to different digraphs, it would be interesting to know the total centrality in a given digraph, i.e., the sum of the centralities of different actors in the network.

Next proposition stablishes that the defined measures are efficient in connected digraphs. In this case we can drop the hypothesis of being the game almost positive.

Proposition 3.6 *For all* $\alpha \in [0,1]$, κ^α *is efficient in digraphs, i.e.: given* $(N,v,d) \in \mathcal{DC}^N_{S_0}$, (N,d) *connected, we have:*

$$\sum_{i\in N} \kappa_i^\alpha(N,v,d) = \sum_{T\in\pi(N)} \frac{v^d(T)}{n!}.$$

Proof: For $\alpha \in [0,1]$, $\sum_{i\in N} \kappa_i^\alpha(N,v,d) = \sum_{i\in N} \Psi_i^\alpha(N,v^d)$.

So, all that is left to prove is the average efficiency of Ψ^α. In fact, for all $(N,w) \in \mathcal{G}^N$:

$$\sum_{i\in N} \Psi_i^\alpha(N,w) = \sum_{i\in N} \Psi_i^\alpha(N, \sum_{\emptyset\neq T\in\Omega(N)} \Delta_w^*(T) w_T) =$$

$$= \sum_{i\in N} \sum_{\emptyset\neq T\in\Omega(N)} \Delta_w^*(T)\Psi_i^\alpha(N,w_T) = \sum_{\emptyset\neq T\in\Omega(N)} \Delta_w^*(T) \sum_{i\in N} \Psi_i^\alpha(N,w_T) =$$

$$= \sum_{\emptyset\neq T\in\Omega(N)} \Delta_w^*(T)\frac{1}{t!} = \sum_{\emptyset\neq T\in\Omega(N)} \frac{1}{t!} \sum_{\emptyset\neq R\tilde{\subset} T} (-1)^{t-r} w(R) = \sum_{\emptyset\neq T\in\Omega(N)} \left(\sum_{\emptyset\neq R\tilde{\subset} T} \frac{(-1)^{t-r}}{t!} \right) w(R).$$

Moreover, taking $l = t - r$ we have:

$$\sum_{\emptyset\neq R\tilde{\subset} T} \frac{(-1)^{t-r}}{t!} = \sum_{l=0}^{n-r} \binom{r+l}{l} \binom{n-r}{l} l! \frac{(-1)^l}{(r+l)!} =$$

$$= \frac{1}{r!} \sum_{l=0}^{n-r} \binom{n-r}{l} (-1)^l = \begin{cases} \frac{1}{n!} & \text{if } r = n \\ 0 & \text{otherwise} \end{cases}$$

and therefore,

$$\sum_{i\in N} \Psi_i^\alpha(N,w) = \sum_{R\in\pi(N)} \frac{w(R)}{n!}.$$

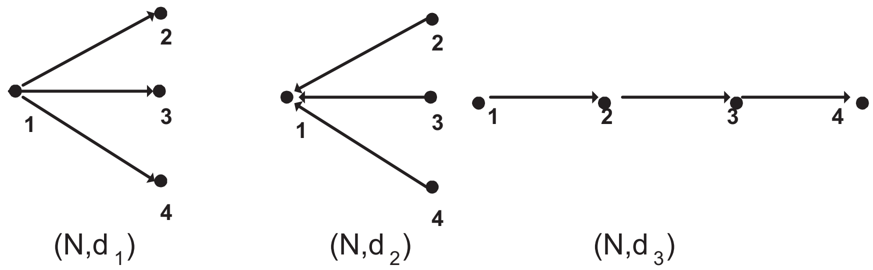
And taking $w = v^d$, the result is proved.

Next corollary (which proof is straightforward) extends to the general case the result in Proposition 3.6.

Corollary 3.1 *For* $\alpha \in [0,1]$, κ^α *satisfy components efficiency i.e.: given* $(N,v,d) \in \mathcal{DC}^N_{S_0}$ *, if* $N/d = \{C_1, C_2, \ldots, C_k\}$, *for* $j = 1, 2, \ldots, k$,

$$\sum_{i\in C_j} \kappa_i^\alpha(N,v,d) = \sum_{T\in\pi(C_j)} \frac{v^d(T)}{c_j!}.$$

Example 3.2 *Consider the following directed communication situations* (N, v, d_1), (N, v, d_2), *and* (N, v, d_3) *where* (N, d_1), (N, d_2), (N, d_3) *are the out-star, the in-star and the oriented left-right chain with four nodes, respectively, i.e.,* $d_1 = \{(1, 2), (1, 3), (1, 4)\}$, $d_2 = \{(2, 1), (3, 1), (4, 1)\}$ *and* $d_3 = \{(1, 2), (2, 3), (3, 4)\}$, *and the game* $v = v_2 + v_3 + v_4$. *Then:*



$$v^{d_1} = 2!(w_{(1,2)} + w_{(1,3)} + w_{(1,4)}),$$

$$v^{d_2} = 2!(w_{(2,1)} + w_{(3,1)} + w_{(4,1)}),$$

$$v^{d_3} = 2!(w_{(1,2)} + w_{(2,3)} + w_{(3,4)}) + 3!(w_{(1,2,3)} + w_{(2,3,4)}) + 4!w_{(1,2,3,4)},$$

$$\kappa^\alpha(N, v, d_1) = \left(\frac{3\alpha}{1+\alpha}, \frac{1}{1+\alpha}, \frac{1}{1+\alpha}, \frac{1}{1+\alpha}\right), \ \alpha \in [0, 1].$$

$$\kappa^\alpha(N, v, d_2) = \left(\frac{3}{1+\alpha}, \frac{\alpha}{1+\alpha}, \frac{\alpha}{1+\alpha}, \frac{\alpha}{1+\alpha}\right), \ \alpha \in [0, 1].$$

$$\kappa^\alpha(N, v, d_3) = \left(\frac{\alpha}{1+\alpha} + \frac{\alpha^2}{1+\alpha+\alpha^2} + \frac{\alpha^3}{1+\alpha+\alpha^2+\alpha^3}, 1 + \frac{\alpha^2+\alpha}{1+\alpha+\alpha^2} + \frac{\alpha^2}{1+\alpha+\alpha^2+\alpha^3},\right.$$
$$\left. 1 + \frac{1+\alpha}{1+\alpha+\alpha^2} + \frac{\alpha}{1+\alpha+\alpha^2+\alpha^3}, \frac{1}{1+\alpha} + \frac{1}{1+\alpha+\alpha^2} + \frac{1}{1+\alpha+\alpha^2+\alpha^3}\right), \ \alpha \in [0, 1].$$

The total centralities, which do not depend on α *as Corollary 3.1 shows, are*

$$\sum_{i=1}^{4} \kappa_i^\alpha(N, v, d_1) = \sum_{i=1}^{4} \kappa_i^\alpha(N, v, d_2) = 3 \text{ and } \sum_{i=1}^{4} \kappa_i^\alpha(N, v, d_3) = 6.$$

As it is obvious, with these measures, the centrality of the hub in an in-star is greater than the corresponding one in the out-star and conversely for satellites. In this case, any normalization (based on the sum) can be avoided as the total sum coincides in both stars.

In the chain d_3, centrality increasaes from node 1 to 3, for all $\alpha \in [0, 1]$. When comparing node 3 and 4 the ranking depends on the value of α. These results, that seem to be not very appealling, will be interpreted when we consider the decomposition of the measures in next section. On the other hand, for $\alpha \in [0,1]$ $\frac{1}{6}\kappa_i^\alpha(N,v,d_3) \leq \frac{1}{3}\kappa_1^\alpha(N,v,d_2)$, $i = 1,2,3,4$ and then the normalized centrality of each node in the four-nodes oriented chain is lesser than the corresponding normalized centrality for the hub of a four-nodes in-star.

Finally, when comparing $\frac{1}{3}\kappa_1^\alpha(N,v,d_1)$ with $\frac{1}{6}\kappa_i^\alpha(N,v,d_3)$, the ranking depends on the value of α.

Another question of interest, from sociological point of view, is the impact that has in a pair of individuals to remove a directed relation between them. The defined family of measures covers the possibility of an asymmetrical impact on the two individuals that break their directed relation. Both change their centralities, but the initiator-node one only in a proportion $\alpha \in [0,1]$ that the receiver one. We will refer to this property as α-directed fairness (for $\alpha \in [0,1]$).

Proposition 3.7 *κ^α satisfies the α-directed fairness property, i.e.: given $(N,v,d) \in \mathcal{DC}_{S_0}^N$ and $i,j \in N$ such that $(i,j) \in d$, for all $\alpha \in [0,1]$,*

$$\kappa_i^\alpha(N,v,d) - \kappa_i^\alpha(N,v,d^{ij}) = \alpha[\kappa_j^\alpha(N,v,d) - \kappa_j^\alpha(N,v,d^{ij})].$$

Proof: Let us first consider the unanimity game (N,u_S) , $S \subset N$ with $s \geq 2$. Then,

$$u_S^d - u_S^{d^{ij}} = \sum_{T \in \pi(S) \cap \mathcal{C}_d^N, T(j)=T(i)+1} w_T,$$

and therefore

$$\kappa_i^\alpha(N,u_S,d) - \kappa_i^\alpha(N,u_S,d^{ij}) = \sum_{T \in \pi(S) \cap \mathcal{C}_d^N, T(j)=T(i)+1} \frac{\alpha^{t-T(i)}}{t! \sum_{r=0}^{t-1} \alpha^r} =$$

$$= \alpha \sum_{T \in \pi(S) \cap \mathcal{C}_d^N, T(j)=T(i)+1} \frac{\alpha^{t-(T(i)+1)}}{t! \sum_{r=0}^{t-1} \alpha^r} = \alpha[\kappa_j^\alpha(N,u_S,d) - \kappa_j^\alpha(N,u_S,d^{ij})].$$

As κ^α is linear in v, the result is proved ■

Again, if we assume that removing an arc the centrality of incident nodes must decrease or at least not increase, we must use almost positive games. In this case the impact is nonegative for both players but minor for the iniciator one than for the receiver one. If the game is not almost positive the centrality of both can increase, but in this case the corresponding variation of the reveiver-node more.

Previous properties are interesting by themselves as they reflect appelling aspects for a measure to be considered as a centrality one. But it is even more important from a theorical point of view is the fact that the last two characterize the defined family of centrality measures when $\alpha \neq 0$. The proof is given in the following theorem.

Theorem 3.1 *For each $\alpha \in (0,1]$, $\kappa^\alpha : \mathcal{DC}_{S_0}^N \to \mathbb{R}^n$ is the unique function defined on $\mathcal{DC}_S^N$ satisfying components efficiency and α-directed fairness.*

Proof: As it is already proved, $\kappa^\alpha : \mathcal{DC}_{S_0}^N \to \mathbb{R}^n$ satisfies components efficiency and α-directed fairness. Conversely, suppose $\alpha \in (0,1]$ and $\xi^\alpha : \mathcal{DC}_{S_0}^N \to \mathbb{R}^n$ is a function satisfying these two properties. We will prove, by induction on the number of arcs, $|d|$, in (N,d), that $\xi^\alpha(N,v,d) = \kappa^\alpha(N,v,d)$ for all $(N,v,d) \in \mathcal{DC}_{S_0}^N$ and all $\alpha \in (0,1]$.

If $|d| = 0$, we have $N/d = \{\{1\},\ldots,\{n\}\}$ and thus by components efficiency $\xi_i^\alpha(N,v,d) = v(\{i\}) = 0 = \kappa_i^\alpha(N,v,d)$, $i \in N$.

Suppose, then, that $\xi^\alpha = \kappa^\alpha$ for all $\alpha \in (0,1]$ and all $(N,v,d) \in \mathcal{DC}_{S_0}^N$ with $|d| < m$ and consider $(N,v,d) \in \mathcal{DC}_S^N$ with $|d| = m$. For $(h,k) \in d$, using the induction hypothesis and the fact that both functions ξ^α and κ^α satisfy α-directed fairness we have:

$$\xi_h^\alpha(N,v,d) - \alpha\xi_k^\alpha(N,v,d) = \xi_h^\alpha(N,v,d^{hk}) - \alpha\xi_k^\alpha(N,v,d^{hk}) =$$

$$= \kappa_h^\alpha(N,v,d^{hk}) - \alpha\kappa_k^\alpha(N,v,d^{hk}) = \kappa_h^\alpha(N,v,d) - \alpha\kappa_k^\alpha(N,v,d),$$

and thus

$$\xi_h^\alpha(N,v,d) - \kappa_h^\alpha(N,v,d) = \alpha[\xi_k^\alpha(N,v,d) - \kappa_k^\alpha(N,v,d)]. \tag{2}$$

Consider $i \in N$ and $C \in N/d$ the component to which i belongs. If $|C| = 1$ it is trivial by components efficiency that $\xi_i^\alpha = \kappa_i^\alpha$ for $\alpha \in (0,1]$. So, suppose there is $j \in C$, $j \neq i$. It exists a sequence (not necessarily unique) $i_0 = i, i_1, \ldots, i_{l-1}, i_l = j$, such that for $t = 0,1,\ldots,l-1$, $(i_t, i_{t+1}) \in d$ or $(i_{t+1}, i_t) \in d$, or both possibilities.

Let us define $r_t(i,j) = \begin{cases} 1 & \text{if } (i_t, i_{t+1}) \in d \\ -1 & \text{otherwise .} \end{cases}$

Obviously $r_t(i,j)$ depends on the considered sequence $i_0 = i, i_1, \ldots, i_l = j$ but this fact is notationaly ignored. Then, using (2) sequentially:

$$\xi_i^\alpha(N,v,d) - \kappa_i^\alpha(N,v,d) = \alpha^{\sum_{t=0}^{l-1} r_t(i,j)}[\xi_j^\alpha(N,v,d) - \kappa_j^\alpha(N,v,d)].$$

Therefore,

$$\sum_{j\in C}[\xi_j^\alpha(N,v,d)-\kappa_j^\alpha(N,v,d)] = \left[1+\left(\sum_{j\in C,\, j\neq i}\alpha^{-\sum_{k=0}^{l-1} r_t(i,j)}\right)\right][\xi_i^\alpha(N,v,d)-\kappa_i^\alpha(N,v,d)]. \tag{3}$$

But using the components efficiency, the left hand term in (3) is zero. As $1+\sum_{j\in C, j\neq i}\alpha^{-\sum_{t=0}^{l-1} r_t(i,j)} \neq 0$ for all $\alpha\in(0,1]$, we conclude that for $i\in N$:

$$\xi_i^\alpha(N,v,d)=\kappa_i^\alpha(N,v,d), \quad \text{for } \alpha\in[0,1],$$

which completes the proof. ∎

A similar proof to the given in previous proposition shows us that the restriction of the defined measures to $\mathcal{DC}^N_{APS_0}$ can be characterized in terms of the same properties. Next example proves that both previous properties are not sufficient to guarantee the unicity of κ^0.

Example 3.3 *Given* $(N,v,d)\in\mathcal{DC}^N_{S_0}$, *let* $N/d=\{C_1,\ldots,C_r\}$, *consider the digraphs* $(C_k,d^{l_k}), l_k=1,2,\ldots,c_k$ *where* $d^{l_k}=\{(i_{l_k},j),\ j\neq i_{l_k},\ j\in C_k\}$ *for each* $i_{l_k}\in C_k$.

Let us define the following function ξ *on* $\mathcal{DC}^N_{S_0}$. *Given* $(N,v,d)\in\mathcal{DC}^N_{S_0}$, *for* $i\in N$, *let be* C_k *the component of* (N,d) *to which* i *belongs.*

$$\xi_i(N,v,d)=\kappa_i^0(N,v,d)+\sum_{d^{l_k}\subset d}\left[\xi_i(N,v,d^{l_k})-\kappa_i^0(N,v,d^{l_k})\right],$$

where:

$$\xi_i(N,v,d^{l_k})=\begin{cases}0 & \text{if } i=i_{l_k}\\ b_i & \text{if } i\neq i_{l_k},\end{cases} \tag{4}$$

with $b_i\in\mathbb{R}$, *and* $\sum_{i\neq i_{l_k}} b_i=\sum_{T\in\pi(C_k)}\frac{v^d(T)}{c_k!}$.

Let us recall $\kappa_i^0(N,v,d^i)=v(\{i\})=0$ *for the hub of an out-star as we can sequentially remove all the arcs without change of the i-values (because of the* 0*-directed fairness) and thus* $\kappa_i^0(N,v,d^i)=\kappa_i^0(N,v,\emptyset)$. *Then by components efficiency* $\kappa_i^0(N,v,\emptyset)=v(\{i\})=0$. *Choosing the values* $b_i,\ i\neq i_{l_k}$ *of (4) in an appropiate way (which is always possible when there exist* k *with* $c_k\geq 3$*), we have that* ξ *differs from* κ^0.

Let us prove that this new function, as κ^0 *does, satisfies components efficiency and* 0*-directed fairness properties.*

The efficiency is given by the fact that:

$$\sum_{i\in C_k}\xi_i(N,v,d)=\sum_{i\in C_k}\kappa_i^0(N,v,d)+\sum_{i\in C_k}\sum_{d^{l_k}\subset d}\left[\xi_i(N,v,d^{l_k})-\kappa_i^0(N,v,d^{l_k})\right]=$$

$$=\sum_{i\in C_k}\kappa_i^0(N,v,d)+\sum_{d^{l_k}\subset d}\sum_{i\in C_k}\left[\xi_i(N,v,d^{l_k})-\kappa_i^0(N,v,d^{l_k})\right]=\frac{\sum_{T\in\pi(C_k)}v^d(T)}{c_k!},$$

this last equallity holding as the double summatory vanishes taking into account the efficiency in out-stars of allocaions rules κ^0 *and* ξ.

In order to prove that ξ *satisfies* 0*-directed fairness, consider first the case* $(i,j)\in d$, $i\in C_k$, $(i,j)\notin d^{l_k}$ *for each* $d^{l_k}\subset d$. *Then,*

$$\xi_i(N,v,d)-\xi_i(N,v,d^{ij})=\kappa_i^0(N,v,d)+\sum_{d^{l_k}\subset d}\left[\xi_i(N,v,d^{l_k})-\kappa_i^0(N,v,d^{l_k})\right]-$$

$$-\kappa_i^0(N,v,d^{ij})-\sum_{d^{l_k}\subset d^{ij}}\left[\xi_i(N,v,d^{l_k})-\kappa_i^0(N,v,d^{l_k})\right]=\kappa_i^0(N,v,d)-\kappa_i^0(N,v,d^{ij})$$

as taking into account that $(i,j)\notin d^{l_k}$ *for each* $d^{l_k}\subset d$, *we have* $d^{l_k}\subset d^{ij}$ *if and only if* $d^{l_k}\subset d$. *Finally, because of the* 0*-directed fairness of* κ^0, $\kappa_i^0(N,v,d)-\kappa_i^0(N,v,d^{ij})=0$.

Consider, then, the case in which $(i,j)\in d$, $i\in C_k$ *and it exists* $d^i\subset d$ *(which will be necessarily unique) such that* $(i,j)\in d^i$. *Then*

$$\xi_i(N,v,d)-\xi_i(N,v,d^{ij})=\kappa_i^0(N,v,d)+\sum_{d^{l_k}\subset d}\left[\xi_i(N,v,d^{l_k})-\kappa_i^0(N,v,d^{l_k})\right]-$$

$$-\kappa_i^0(N,v,d^{ij})-\sum_{d^{l_k}\subset d^{ij}}\left[\xi_i(N,v,d^{l_k})-\kappa_i^0(N,v,d^{l_k})\right]=$$

$$=\kappa_i^0(N,v,d)-\kappa_i^0(N,v,d^{ij})+\xi_i(N,v,d^i)-\kappa_i^0(N,v,d^i)=0+v(\{i\})-v(\{i\})=0.$$

Then both different functions ξ *and* κ^0 *satisfy efficiency and* 0*-directed fairness, and thus these two properties not characerize* κ^0.

Nevertheless, we have emphasize the utility of using symmetric games where measuring the centrality of nodes in a digraph. In next result we prove that κ^0 can be characterized in the family $\mathcal{DC}_{S_0}^N$ adding to the two previous ones the symmetry axiom.

Definition 3.3 *We will say that nodes* i *and* j *are symmetric in the digraph* (N,d) *if it exists a permutation* $\pi:N\to N$ *such that* $\pi(i)=j$, $\pi(j)=i$ *and* $(k,l)\in d$ *if and only if* $(\pi(k),\pi(l))\in d$.

Definition 3.4 *A function $\xi : \mathcal{DC}^N_{S_0} \rightarrow \mathrm{I\!R}^n$ satisfies the symmetry axiom if, for all $(N, v, d) \in \mathcal{DC}^N_{S_0}$ and all pair i, j of symmetric nodes in (N, d), $\xi_i(N, v, d) = \xi_j(N, v, d)$ holds.*

Proposition 3.8 *$\kappa^0 : \mathcal{DC}^N_{S_0} \rightarrow \mathrm{I\!R}^n$ is the unique function defined in $\mathcal{DC}^N_{S_0}$ that satisfies components efficiency, 0-directed fairness and symmetry.*

Proof: It is obvious that κ^0 satisfies symmetry and it is already proved that it satisfies 0-directed fairness and components efficiency. Consider $\xi : \mathcal{DC}^N_{S_0} \rightarrow \mathrm{I\!R}^n$, satisfying these three properties. We will prove, by backward induction on the cardiality of d, that $\xi(N, v, d) = \kappa^0(N, v, d)$ for all $(N, v, d) \in \mathcal{DC}^N_{S_0}$.

If $d = K_N$ then, by symmetry and efficiency, $\xi_i(N, v, K_N) = \frac{v(N)}{n} = \kappa^0(N, v, K_N),\ i = 1, \ldots, n$.

Suppose that ξ coincides with κ^0 for all (N, v, d) with $|d| \geq k$ and consider (N, v, d) with $|d| = k - 1$. Then for $i \in N$ and any $(i, j) \in K_n \setminus d$:

$$\xi_i(N, v, d \cup \{(i, j)\}) - \xi_i(N, v, d) = 0,$$

as ξ satisfies 0-directed fairnesss. Then:

$$\xi_i(N, v, d) = \xi_i(N, v, d \cup \{(i, j)\}) = \kappa^0_i(N, v, d \cup \{(i, j)\}),$$

last equality holding due to the induction hypothesis. As κ^0 satifies 0-directed fairness, $\kappa^0_i(N, v, d \cup \{(i, j)\}) = \kappa^0_i(N, v, d)$ and thus $\xi_i(N, v, d) = \kappa^0_i(N, v, d)$, and the result is proved.

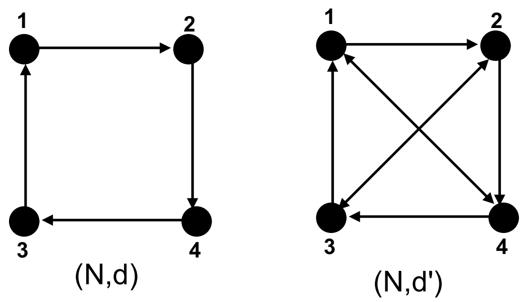
■

Following the steps of previous proposition we can prove that the restriction of κ^0 to the $\mathcal{DC}^N_{APS_0}$ is also characterized by efficiency, zero-directed fairness and symmetry.

4 A decomposition of the centrality measures

Besides the appealing properties analyzed in previous section, it must be pointed out some miopic behaviour of the defined measures. Next example shows that, when comparing centralities of two nodes in different digraphs and with a different connectivity (but the same game), it is possible to obtain for these two nodes equal proportions of the total centrality. Obviously this can be interpreted as a weakness of the defined measures that are not able to higlight all the differences in the connectivity between the considered nodes.

Example 4.1 *Given the directed communication situations (N, v, d) and (N, v, d') where $N = \{1, 2, 3, 4\}$ $v = v_2 + v_3$ (v_2 and v_3 being the previously defined communication games), $d = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$ and $d' = d \cup \{(1, 3), (3, 1), (2, 4), (4, 2)\}$. A representation of this two digraphs is given in Figure 4.1:*



It is straightforward to see that, for each $\alpha \in [0, 1]$: $\kappa_i^\alpha(N, v_2 + v_3, d) = 2$, for each $i = 1, 2, 3, 4$, and thus:

$$\frac{\kappa_i^\alpha(N, v_2 + v_3, d)}{\sum_{j=1}^{4} \kappa_i^\alpha(N, v_2 + v_3, d)} = \frac{1}{4}, \quad i = 1, 2, 3, 4.$$

Similarly, $\kappa_i^\alpha(N, v_2 + v_3, d') = 5$, for each $i = 1, 2, 3, 4$ and thus

$$\frac{\kappa_i^\alpha(N, v_2 + v_3, d')}{\sum_{j=1}^{4} \kappa_i^\alpha(N, v_2 + v_3, d')} = \frac{1}{4}, \quad i = 1, 2, 3, 4.$$

Consequently the relative centrality for each node in $(N, v_2 + v_3, d)$ coincides with its corresponding one in $(N, v_2 + v_3, d')$. But it is obvious there exist several differences between these digraphs (density, incident arcs, etc). This example inspires the idea that, perhaps, the centrality is a vector measure and not a scalar one. In other words, using a unique number to measure the centrality of a node is excesively reduccionist. Taking this idea into account, we eill try to split each one of the defined measures in three different ones, which can be considered as alternative centrality measures by themselves. One of them will be interpreted as an emission measure, the second has a meaning of betweeness and the third one will measure the reception centrality.

This three-dimensional measure will permit us to highlight some differences in the centrality that a single measure is unable to point out.

Definition 4.1 *Given a directed communication situation $(N, v, d) \in \mathcal{DC}_{S_0}^N$, we define, for each $\alpha \in [0, 1]$, the emission centrality of node i, $\epsilon_i^\alpha(N, v, d)$ as:*

$$\epsilon_i^\alpha(N, v, d) = \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), i(T)=1} \Psi_i^\alpha(N, w_T) =$$

$$= \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), i(T)=1} \frac{\alpha^{t-1}}{t! \sum_{k=0}^{t-1} \alpha^k}.$$

Definition 4.2 *Given a directed communication situation $(N, v, d) \in \mathcal{DC}_{S_0}^N$, we define, for each $\alpha \in [0, 1]$, the betweeness centrality of node i, $\beta_i^\alpha(N, v, d)$ as:*

$$\beta_i^\alpha(N, v, d) = \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), 1 < i(T) < t} \Psi_i^\alpha(N, w_T) =$$

$$= \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d^N(S), 1 < i(T) < t} \frac{\alpha^{t-1}}{t! \sum_{k=0}^{t-1} \alpha^k}.$$

Definition 4.3 *Given a directed communication situation $(N, v, d) \in \mathcal{DC}_{S_0}^N$, for $\alpha \in [0, 1]$, the reception centrality of node i, $\rho_i^\alpha(N, v, d)$ is given by*

$$\rho_i^\alpha(N, v, d) = \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d^N(S), i(T)=t} \Psi_i^\alpha(N, w_T) =$$

$$= \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), i(T)=t} \frac{1}{t! \sum_{k=0}^{t-1} \alpha^k}.$$

The given definitions of $\epsilon_i^\alpha(N, v, d)$, $\beta_i^\alpha(N, v, d)$ and $\rho_i^\alpha(N, v, d)$ split the total centrality of node i, $\kappa_i^\alpha(N, v, d)$, in three quantities. Let us note for the case $\alpha = 0$ $\kappa^0(N, v, d) = \rho^0(N, v, d)$ and then the decomposition is inessential. They are obtained from the value assigned to i by its position in ordered connected sets and taking into account that this position can be the first one, the last one, or an intermediate one. Te first position in an ordered connected set permits to initiate the communication, the intermediate positions are clearly associated with betweeness and finally, to be at the end of each connected chain transforms a node in a receiver one. These ideas justify the chosen notation and its interpretation.

The first result stablishes that this new measures give us an additive decomposition of κ^α.

Proposition 4.1 *For each* $(N, v, d) \in \mathcal{DC}_{S_0}^N$ *and* $\alpha \in [0, 1]$ *we have,*

$$\kappa^\alpha(N, v, d) = \epsilon^\alpha(N, v, d) + \beta^\alpha(N, v, d) + \rho^\alpha(N, v, d).$$

Proof: If $(N, v, d) \in \mathcal{DC}_{S_0}^N$, $\alpha \in [0, 1]$ and $i \in N$,

$$\kappa_i^\alpha(N, v, d) = \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d(S)} \Psi_i^\alpha(N, w_T) =$$

$$= \sum_{S \subset N} \Delta_v(S) \left[\sum_{T \in \pi(S) \cap \mathcal{C}_d(S), i(T)=1} \Psi_i^\alpha(N, w_T) + \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), 1 < i(T) < t} \Psi_i^\alpha(N, w_T) + \right.$$

$$\left. + \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), i(T)=t} \Psi_i^\alpha(N, w_T) \right] = \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), i(T)=1} \Psi_i^\alpha(N, w_T) +$$

$$+ \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), 1 < i(T) < t} \Psi_{i(T)}^\alpha(N, w_T) + \sum_{S \subset N} \Delta_v(S) \sum_{T \in \pi(S) \cap \mathcal{C}_d(S), i(T)=t} \Psi_t^\alpha(N, w_T) =$$

$$= \epsilon_i^\alpha(N, v, d) + \beta_i^\alpha(N, v, d) + \rho_i^\alpha(N, v, d)$$

and thus the result is proved. ■

Let us consider again the directed communication situations given in the Example 4.1. Previous decomposition permits us to illustrate some differences between nodes of different digraphs having equal relative centrality.

Example 4.2 *Given the directed communication situations* $(N, v_2 + v_3, d)$ *and* $(N, v_2 + v_3, d')$ *as in Example 4.1 we have for* $i = 1, 2, 3, 4$,

$$\epsilon_i^\alpha(N, v_2 + v_3, d) = \frac{\alpha}{1 + \alpha} + \frac{\alpha^2}{1 + \alpha + \alpha^2},$$

$$\rho_i^\alpha(N, v_2 + v_3, d) = \frac{1}{1 + \alpha} + \frac{1}{1 + \alpha + \alpha^2},$$

$$\beta_i^\alpha(N, v_2 + v_3, d) = \frac{\alpha}{1 + \alpha + \alpha^2},$$

$$\epsilon_i^\alpha(N, v_2 + v_3, d') = \frac{2\alpha}{1 + \alpha} + \frac{3\alpha^2}{1 + \alpha + \alpha^2},$$

$$\rho_i^\alpha(N, v_2 + v_3, d') = \frac{2}{1 + \alpha} + \frac{3}{1 + \alpha + \alpha^2},$$

$$\beta_i^\alpha(N, v_2 + v_3, d') = \frac{3\alpha}{1 + \alpha + \alpha^2}.$$

As different nodes have the same centrality inside each communication situation and the same proportion of the total centrality when comparing both communication situations, in order to highlight some differences, we will use the new defined measures. Let us compare the proportion of the centrality of each node due to the emission, reception and betweeness in both digraphs to explain the existing differences.

We have for $i = 1, 2, 3, 4$,

$$\frac{\epsilon_i^\alpha(N, v_2 + v_3, d)}{\kappa_i^\alpha(N, v_1 + v_2, d)} - \frac{\epsilon_i^\alpha(N, v_2 + v_3, d')}{\kappa_i^\alpha(N, v_1 + v_2, d')} = \frac{\alpha}{10(1+\alpha)(1+\alpha+\alpha^2)} \geq 0,$$

i.e. for $\alpha \in (0, 1]$, *the proportion of the centrality of each node due to emission is greater in* $(N, v_2 + v_3, d)$ *than in* $(N, v_2 + v_3, d')$. *In the particular case* $\alpha = 0$ *as the measure* κ^0 *coincides with* ρ^0, *this proportion is zero for each node in both difraphs.*

If we look at the proportion of the centrality due to reception, for $i = 1, 2, 3, 4$,

$$\frac{\rho_i^\alpha(N, v_2 + v_3, d)}{\kappa_i^\alpha(N, v_1 + v_2, d)} - \frac{\rho_i^\alpha(N, v_2 + v_3, d')}{\kappa_i^\alpha(N, v_1 + v_2, d')} = \frac{\alpha^2}{10(1+\alpha)(1+\alpha+\alpha^2)},$$

and thus, we see that it is greater too in $(N, v_2 + v_3, d)$ *for* $\alpha \in (0, 1]$ *and unchanged for* $\alpha = 0$. *Finally as it was expected, the proportion os the centrality due to betweeness is higher (except for the case* $\alpha = 0$*) for nodes in* $(N, v_2 + v_3, d')$ *than for nodes in* $(N, v_2 + v_3, d)$*:*

$$\frac{\beta_i^\alpha(N, v_2 + v_3, d)}{\kappa_i^\alpha(N, v_1 + v_2, d)} - \frac{\beta_i^\alpha(N, v_2 + v_3, d')}{\kappa_i^\alpha(N, v_1 + v_2, d')} = \frac{-\alpha}{10(1+\alpha+\alpha^2)} \leq 0.$$

Let us turn now our the attention to some appealling (because of their intuitiveness) properties of these new measures.

First proposition shows that the hub of an in-star is the node in which the proportion of the reception centrality (with respect to the total reception centrality) is maximized.

Proposition 4.2 *Given* $(N, v, d^{IS}) \in \mathcal{DC}^N_{APS_0}$ *and* $d^{IS} = \{(i, 1), i \in N \setminus \{1\}\}$, *for all* $(N, d) \in D^N$ *and* $\alpha \in [0, 1]$*:*

$$\frac{\rho_1^\alpha(N, v, d^{IS})}{\kappa_1^\alpha(N, v, d^{IS})} \geq \frac{\rho_i^\alpha(N, v, d)}{\kappa_i^\alpha(N, v, d)}, \ i = 1, 2, \ldots, n. \tag{5}$$

Proof:

It is obvious that for $(N,v) \in \mathcal{AP}^N \cap S_0^N$, $\rho_1^\alpha(N,v,d^{IS}) = \kappa_1^\alpha(N,v,d^{IS})$, and the left hand side in (5) is equal to 1. Being the game almost positive $\rho_i^\alpha(N,v,d) \leq \kappa_i^\alpha(N,v,d)$ for all $i \in N$ and thus the result is proved. ∎

Symmetrically we have the next result for emission centrality.

Proposition 4.3 *Given* $(N,v,d^{OS}) \in \mathcal{DC}^N_{APS_0}$ *and* $d^{OS} = \{(1,i), i \in N \setminus \{1\}\}$*, for all* $(N,d) \in D^N$ *and* $\alpha \in [0,1]$.

$$\frac{\epsilon_1^\alpha(N,v,d^{OS})}{\kappa_1^\alpha(N,v,d^{OS})} \geq \frac{\epsilon_i^\alpha(N,v,d)}{\kappa_i^\alpha(N,v,d)},\ i = 1,2,\ldots,n. \tag{6}$$

In next proposition it is proved that in a oriented chain, for almost positive and symmetrical games, the emission centrality is maximal in the inital node whereas reception centrality is maximal in the last node. Moreover betweeness centrality increases from the first node to the median one. We will suppose in order to simplify the proof, that the chain has an odd number of nodes.

Proposition 4.4 *Given* $(N,v,d) \in \mathcal{DC}^N_{APS_0}$ *and* $d = \{(1,2),(2,3),\ldots,(n-1,n)\}$ *(n odd) an oriented chain, for all* $\alpha \in [0,1]$,

$$\epsilon_1^\alpha(N,v,d) \geq \epsilon_i^\alpha(N,v,d),\quad i = 1,2,\ldots,n, \tag{7}$$

$$\rho_n^\alpha(N,v,d) \geq \rho_i^\alpha(N,v,d),\quad i = 1,2,\ldots,n, \tag{8}$$

$$\beta_m^\alpha(N,v,d) \geq \beta_i^\alpha(N,v,d),\quad i = 1,2,\ldots,m-1,\ m \textit{ being the median node.} \tag{9}$$

Proof: As for every $(N,v) \in \mathcal{AP}_0^N$, v is a linear combination of $\{v_2,\ldots,v_n\}$ with non negatives scalars, it is sufficient to prove the results for each v_k, $\ k = 2,\ldots,n$.

Then (7) holds because for $k = 2,3,\ldots,n$ and $\alpha \in [0,1]$

$$\epsilon_i^\alpha(N,v_k,d) = \begin{cases} \frac{\alpha^{k-1}}{\sum_{r=0}^{k-1}\alpha^r} & \text{for } i = 1,2,\ldots,n-k+1 \\ 0 & i = n-k,\ldots,n. \end{cases}$$

Analogously (8) is satisfied as for $\alpha \in [0,1]$,

$$\rho_i^\alpha(N,v_k,d) = \begin{cases} \frac{1}{\sum_{r=0}^{k-1}\alpha^r} & i = k,\ldots,n \\ 0 & i = 1,2,\ldots,k-1. \end{cases}$$

Finally, in order to prove (9), consider $\alpha \in [0,1]$ and $i \in N, \quad i \leq m$. An ordered set of size k, with $3 \leq k \leq n$, containing i, connected in (N,d) and where i has a non extreme position is:

$$T(i,r,k) = (i-r+1, i-r, \ldots, i-1, i, i+1, \ldots, i+k-r)$$

with $max\{2, i+k-n\} \leq r \leq min\{i, k-1\}$.

Then, for $3 \leq k \leq n$,

$$\beta_i^\alpha(N, v_k, d) = \sum_{r=max\{2,i+k-n\}}^{min\{i,k-1\}} k!\Psi_i^\alpha(N, w_{T(i,r,k)}) =$$

$$= \sum_{r=max\{2,i+k-n\}}^{min\{i,k-1,i+k-m-1\}} k!\Psi_i^\alpha(N, w_{T(i,r,k)}) + \sum_{r=i+k-m}^{min\{i,k-1\}} k!\Psi_i^\alpha(N, w_{T(i,r,k)}) \leq$$

$$\leq \sum_{r=max\{2,i+k-n\}}^{min\{i,k-1,i+k-m-1\}} k!\Psi_m^\alpha(N, w_{T(i,r,k)}) + \sum_{r=i+k-m}^{min\{i,k-1\}} k!\Psi_i^\alpha(N, w_{T(i,r,k)}),$$

last inequallity holding because for $r \leq i+k-m-1$ we have $m \leq i+k-r-1$ and thus $m \in T(i,r,k)$ and

$$\Psi_i^\alpha(N, w_{T(i,r,k)}) = \frac{\alpha^{k-r}}{\sum_{l=0}^{k-1} \alpha^l} \leq \Psi_m^\alpha(N, w_{T(i,r,k)}).$$

as $m(T(i,r,k)) \geq r$.

On the other hand, (because m is the median node), for each $r = i+k-m, \ldots, min\{i, k-1\}$,

$$\Psi_i^\alpha(N, w_{T(i,r,k)}) = \frac{\alpha^{k-r}}{\sum_{l=0}^{k-1} \alpha^l},$$

and it exists $T(m,r,k)$ such that $\Psi_m^\alpha(N, w_{T(m,r,k)}) = \Psi_i^\alpha(N, w_{T(i,r,k)})$. Thus,

$$\beta_i^\alpha(N, v_k, d) \leq \sum_{r=max\{2,i+k-n\}}^{min\{i,k-1,i+k-m-1\}} k!\Psi_m^\alpha(N, w_{T(i,r,k)}) +$$

$$+ \sum_{r=i+k-m}^{min\{i,k-1\}} k!\Psi_m^\alpha(N, w_{T(m,r,k)}) \leq \beta_m^\alpha(N, v_k, d),$$

this last inequality holding because $\beta_m^\alpha(N, v_k, d)$ contains at least all the terms in both summatories of the left hand side but possibly additional non negative terms.

■

The proof of last proposition suggests some additional properties in the special case in which the measure does not distinguish the positions of iniciator or receiver in the communication. That is, the case $\alpha = 1$.

Corollary 4.1 *Given* $(N, v, d) \in \mathcal{DC}^N_{APS_0}$ *and* $d = \{(1,2),(2,3),\ldots,(n-1,n)\}$ *(n odd) an oriented chain, we have*

$$\epsilon^1_i(N, v, d) = \rho^1_{n-i+1}(N, v, d), \;\; i = 1, 2, \ldots, n,$$

$$\beta^1_m(N, v, d) \geq \beta^1_i(N, v, d), \;\; i = 1, 2, \ldots, n.$$

5 An example: A model of simplified soccer

Suppose a simplified game of soccer played by two teams with five players each: the goal keeper, two defenders, two middles and one forward. We can model with an arc (i, j) the fact that player i pass the ball to player j. If this is the last arc of a sequence of passes, two possibilities are opened: j shots to the goal or he looses the ball which is intercepted by a rival. Of course, a match consists of many plays (sequences of passes) but in the following example, adding another symplification, we will suppose that the two teams A and A' have completed only three plays which are given by digraphs d_1, d_2, d_3 and d'_1, d'_2, d'_3 respectivly.

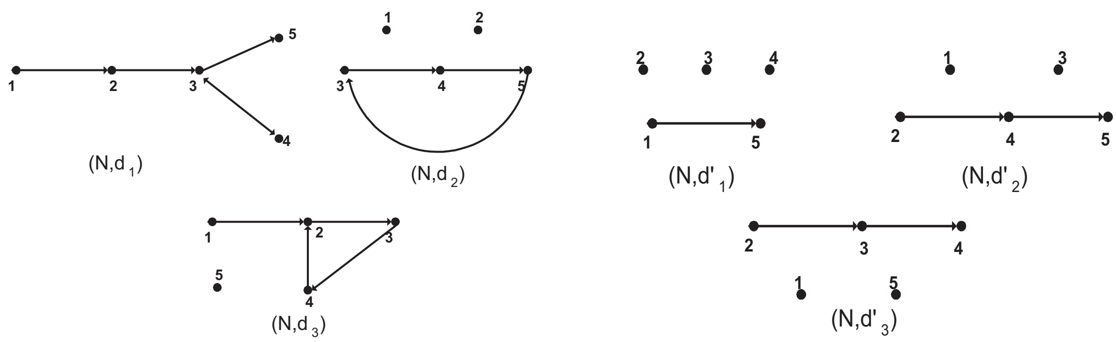


Figure 1: Teams A and A'

Assuming d_1 finishes in the unique goal of the match, we obtain the centrality (and its decomposition) of each player in each digraph (play) and the expected centrality in the match. This expected value assumes for team A a weight of 0.5 for the play of the goal and equal weights for the other two plays, while for team A' it is supposed an equal weight for the three plays. And thus we will use $\overline{\kappa^\alpha A} = \frac{1}{2}\kappa^\alpha(N, v, d_1) + \frac{1}{4}\kappa^\alpha(N, v, d_2) + \frac{1}{4}\kappa^\alpha(N, v, d_3)$, and $\overline{\kappa^\alpha A'} = \frac{1}{3}\kappa^\alpha(N, v, d'_1) + \frac{1}{3}\kappa^\alpha(N, v, d'_2) + \frac{1}{3}\kappa^\alpha(N, v, d'_3)$ for the expected centralities in the match for teams A and A' respectively. Analogously $\overline{\epsilon^\alpha A}$, $\overline{\beta^\alpha A}$, $\overline{\rho^\alpha A}$, and $\overline{\epsilon^\alpha A'}$, $\overline{\beta^\alpha A'}$, $\overline{\rho^\alpha A'}$ will represent the expected values for teams A and A' for the three components of centrality.

As it is obvious, the emission centrality of each player can be viewed as an index of his participation in the game as initiator of plays (tipically this centrality must be higher for goal keeper and defenders) while his betweenness centrality measures the ability to intermediate in the plays (greater for middles) and finally his reception centrality informs us about his shots to the goal and his lost balls (in general greater for forwards).

We will use the conferences game and several values of $\alpha \in [0, 1]$. In particular, 0, 0.5 and 1. These values can be considered as a measure of the relative difficulty when comparing to pass the ball versus to control it. As in the special case of $\alpha = 0$ the total centrality in each play coincides with the reception one, we will omit the remaining components.

Moreover the total centralty of each team when comparing with others teams can give an idea about the differences in the style when playing or in the strategy used. The more total centrality, the more elaborated its game is. The corresponding results are summarized in the following tables.

Tables for team A

$\alpha = 0$	$\kappa_i^\alpha(N,v,d_1)$	$\rho_i^\alpha(N,v,d_1)$	$\kappa_i^\alpha(N,v,d_2)$	$\rho_i^\alpha(N,v,d_2)$	$\kappa_i^\alpha(N,v,d_3)$	$\rho_i^\alpha(N,v,d_3)$	$\overline{\kappa_i^\alpha}$	$\overline{\rho_i^\alpha}$
$i = 1$	**0**	0	**0**	0	**0**	0	**0**	0
$i = 2$	**1**	1	**0**	0	**3**	3	**1.25**	1.25
$i = 3$	**3**	3	**2**	2	**3**	3	**2.75**	2.75
$i = 4$	**3**	3	**2**	2	**3**	3	**2.75**	2.75
$i = 5$	**4**	4	**2**	2	**0**	0	**2.5**	2.5

$\alpha = 1$	$\kappa_i^\alpha(N,v,d_1)$	$\epsilon_i^\alpha(N,v,d_1)$ $\beta_i^\alpha(N,v,d_1)$ $\rho_i^\alpha(N,v,d_1)$	$\kappa_i^\alpha(N,v,d_2)$	$\epsilon_i^\alpha(N,v,d_2)$ $\beta_i^\alpha(N,v,d_2)$ $\rho_i^\alpha(N,v,d_2)$	$\kappa_i^\alpha(N,v,d_3)$	$\epsilon_i^\alpha(N,v,d_3)$ $\beta_i^\alpha(N,v,d_3)$ $\rho_i^\alpha(N,v,d_3)$	$\overline{\kappa_i^\alpha}$	$\overline{\epsilon_i^\alpha}$ $\overline{\beta_i^\alpha}$ $\overline{\rho_i^\alpha}$
$i = 1$	**1.333**	1.333 0 0	**0**	0 0 0	**1.083**	1.083 0 0	**0.937**	0.937 0 0
$i = 2$	**2.5**	1.166 0.833 0.5	**0**	0 0 0	**3.083**	0.833 0.916 1.333	**2.021**	0.792 0.646 0.583
$i = 3$	**3.833**	1 1.5 1.333	**2**	0.833 0.333 0.833	**2.583**	0.833 0.583 1.166	**3.062**	0.916 0.979 1.166
$i = 4$	**1.916**	0.833 0 1.083	**2**	0.833 0.333 0.833	**2.25**	0.833 0.333 1.083	**2.021**	0.833 0.166 1.021
$i = 5$	**1.416**	0 0 1.146	**2**	0.833 0.333 0.833	**0**	0 0 0	**1.208**	0.025 0.083 0.916

$\alpha = 0.5$	$\kappa_i^\alpha(N, v, d_1)$	$\epsilon_i^\alpha(N, v, d_1)$ $\beta_i^\alpha(N, v, d_1)$ $\rho_i^\alpha(N, v, d_1)$	$\kappa_i^\alpha(N, v, d_2)$	$\epsilon_i^\alpha(N, v, d_2)$ $\beta_i^\alpha(N, v, d_2)$ $\rho_i^\alpha(N, v, d_2)$	$\kappa_i^\alpha(N, v, d_3)$	$\epsilon_i^\alpha(N, v, d_3)$ $\beta_i^\alpha(N, v, d_3)$ $\rho_i^\alpha(N, v, d_3)$	$\overline{\kappa_i^\alpha}$	$\overline{\epsilon_i^\alpha}$ $\overline{\beta_i^\alpha}$ $\overline{\rho_i^\alpha}$
$i = 1$	**0.609**	0.609 0 0	**0**	0 0 0	**0.543**	0.543 0 0	**0.440**	0.440 0 0
$i = 2$	**1.838**	0.619 0.552 0.666	**0**	0 0 0	**3.086**	0.476 0.704 1.904	**1.690**	0, 428 0.452 0.809
$i = 3$	**3.962**	0.666 1.390 1.905	**2**	0.476 0.285 1.238	**2.838**	0.476 0.552 1.809	**3.190**	0.571 0.904 1.714
$i = 4$	**2.247**	0.476 0 1.771	**2**	0.476 0.285 1.238	**2.533**	0.476 0.285 1.638	**2.257**	0.476 0.143 1.638
$i = 5$	**2.343**	0 0 2.343	**2**	0.476 0.285 1.238	**0**	0 0 0	**1.671**	0.119 0.071 1.481

Analogously the centralities and their decomposition for players in team A' are:

$\alpha = 0$	$\kappa_i^\alpha(N, v, d'_1)$	$\rho_i^\alpha(N, v, d'_1)$	$\kappa_i^\alpha(N, v, d'_2)$	$\rho_i^\alpha(N, v, d'_2)$	$\kappa_i^\alpha(N, v, d'_3)$	$\rho_i^\alpha(N, v, d'_3)$	$\overline{\kappa_i^\alpha}$	$\overline{\rho_i^\alpha}$
$i = 1$	**0**	0	**0**	0	**0**	0	**0**	0
$i = 2$	**0**	0	**0**	0	**0**	0	**0**	0
$i = 3$	**0**	0	**1**	1	**1**	1	**0.333**	0.333
$i = 4$	**0**	0	**2**	2	**2**	2	**1**	1
$i = 5$	**1**	1	**0**	0	**0**	0	**1**	1

$\alpha = 1$	$\kappa_i^\alpha(N,v,d'_1)$	$\epsilon_i^\alpha(N,v,d'_1)$ $\beta_i^\alpha(N,v,d'_1)$ $\rho_i^\alpha(N,v,d'_1)$	$\kappa_i^\alpha(N,v,d'_2)$	$\epsilon_i^\alpha(N,v,d'_2)$ $\beta_i^\alpha(N,v,d'_2)$ $\rho_i^\alpha(N,v,d'_2)$	$\kappa_i^\alpha(N,v,d'_3)$	$\epsilon_i^\alpha(N,v,d'_3)$ $\beta_i^\alpha(N,v,d'_3)$ $\rho_i^\alpha(N,v,d'_3)$	$\overline{\kappa_i^\alpha}$	$\overline{\epsilon_i^\alpha}$ $\overline{\beta_i^\alpha}$ $\overline{\rho_i^\alpha}$
$i = 1$	**0.5**	0.5 0 0	**0**	0 0 0	**0**	0 0 0	**0.166**	0.166 0 0
$i = 2$	**0**	0 0 0	**0.833**	0.833 0 0	**0.833**	0.833 0 0	**0.555**	0.555 0 0
$i = 3$	**0**	0 0 0	**0**	0 0 0	**1.333**	0.5 0.333 0.5	**0.444**	0.166 0.111 0.166
$i = 4$	**0**	0 0 0	**1.333**	0.5 0.333 0.5	**0.833**	0 0 0.833	**0.722**	0.166 0.111 0.444
$i = 5$	**0.5**	0 0 0.5	**0.833**	0 0 0.833	**0**	0 0 0	**0.444**	0 0 0.444

$\alpha = 0.5$	$\kappa_i^\alpha(N,v,d'_1)$	$\epsilon_i^\alpha(N,v,d'_1)$ $\beta_i^\alpha(N,v,d'_1)$ $\rho_i^\alpha(N,v,d'_1)$	$\kappa_i^\alpha(N,v,d'_2)$	$\epsilon_i^\alpha(N,v,d'_2)$ $\beta_i^\alpha(N,v,d'_2)$ $\rho_i^\alpha(N,v,d'_2)$	$\kappa_i^\alpha(N,v,d'_3)$	$\epsilon_i^\alpha(N,v,d'_3)$ $\beta_i^\alpha(N,v,d'_3)$ $\rho_i^\alpha(N,v,d'_3)$	$\overline{\kappa_i^\alpha}$	$\overline{\epsilon_i^\alpha}$ $\overline{\beta_i^\alpha}$ $\overline{\rho_i^\alpha}$
$i = 1$	**0.333**	0.333 0 0	**0**	0 0 0	**0**	0 0 0	**0.111**	0.111 0 0
$i = 2$	**0**	0 0 0	**0.476**	0.476 0 0	**0.476**	0.476 0 0	**0.317**	0.317 0 0
$i = 3$	**0**	0 0 0	**0**	0 0 0	**1.285**	0.333 0.285 0.666	**0.428**	0.111 0.095 0.222
$i = 4$	**0**	0 0 0	**1.285**	0.333 0.285 0.666	**1.238**	0 0 1.238	**0.828**	0.111 0.095 0.635
$i = 5$	**0.666**	0 0 0.666	**1.238**	0 0 1.238	**0**	0 0 0	**0.635**	0 0 0.635

Finally let us compare some of the previous centralities with other defined ranks for nodes in digraphs.

With a similar framework to the proposed here, Amer el al. (2007) define for a digraph communication situation (N, v, d) a new digraph restricted game v_d (a generalized TU one) as $v_d(T) = \sum_{R \in \mathcal{MC}_d^T} v(H(R))$ for all $T \in \Omega(N)$, where $\mathcal{MC}_d^T = \{R \widetilde{\subset} T$ such that R is maximal connected set in $(N, d)\}$. The accessibility of each given node i is defined by:

$$\alpha_i(N, v, d) = \Psi^0(N, v_d), \quad \text{for } i \in N.$$

Then, the normalized accesibilities of players of team A in the different plays:

$$\alpha(N, v, d_1) = (0, 0.082, 0.246, 0.294, 0.376),$$
$$\alpha(N, v, d_2) = (0, 0, 0.333, 0.333, 0.333),$$
$$\alpha(N, v, d_3) = (0, 0.313, 0.313, 0.373, 0),$$

are similar and with the same order for diferent nodes, to the corresponding centralities for $\alpha = 0$.

The centrality based on the eigenvector corresponding to the dominant eigenvalue of the adjacency matrix is, for d_1, d_2, and d_3 respectively.

$$(0.25, 0.25, 0.25, 0.25, 0),$$
$$(0, 0, 0.333, 0.333, 0.333),$$
$$(0.25, 0.25, 0.25, 0.25, 0).$$

In this case the differences with respect the ones we propose here are obvious. We think that at least for these examples the eigenvalue measure does not emphasize all the existing differences in the nodes connections.

In van den Brink and Borm (2002) authors introduce a model in which they summarize the results of matches in sports competitions between the teams in a given set N by means of a digraph competition. In this model, the competition digraph includes an arc (i, j) if and only if j did not lose the match it played agains i. The model we propose in this work is not suitable for this type of situations. The use of almost positive games implies the stability of the associated allocation rule (for any $\alpha \in [0, 1]$) and thus when modeling a draw as two victories (one for each of the teams in the match) the draws will be excesively rewarded.

6 Conclusions

This paper proposes a new instrument to measure centrality of nodes in directed networks following our previous work (Gómez et al. 2003). Actors in the network are simultaneously players in a cooperative game. This game represents the interests that motivate the interactions among actors, whereas directed network imposes restrictions in the cooperation. Given the (directed) network and the game, a new (generalized) restricted game is obtained.

In the literature there are many centrality measures, but they usually assume the links are undirected, i.e., both agents connected by a link equally benefit from its existence. In the present paper, we will consider the directed case.

In our proposal, the centrality of nodes is measured as the difference between:

(i) the Shapley value (this is assumed to be the node's payoff/power when no digraph is binding the interaction among agents), and

(ii) a new value for generalized TU games belonging to a parametric family that contains the two existing ones defined by Nowak and Radzik (1994) and Sánchez and Bergantiños (1997). This new index is interpreted as the nodes' payoff/power considering the restrictions imposed by the directed network on the interaction possibilities among agents. In particular, this index weights the specific ranking among agents derived from the directed network structure.

The paper discuses several properties of these measures. For each $\alpha \in (0, 1]$, the corresponding measure is characterized in terms of efficiency and α-directed fairness properties. In the case $\alpha = 0$, a symmetry axiom is also needed.

This characterization must be interpreted in a double sense: first, it fixes the range of variation of our measure and second, it highlights the fact that the initiator and the receiver do not necessarily play symmetrical roles in the type of analyzed relations. Of course, other measures can be obtained with different properties and with alternative characterizations. Even, the proposed measures probably admit characterizations based on alternative properties.

In this paper we also point to some problems concerning the fact that nodes of which centralities should be clearly different in two different networks have the same outcome of the measure. This inspires us the idea that the proposed measures are a sort of a specific module of the centrality. This centrality perhaps can be thought as a vector measure instead of a scalar one. With this idea in mind, we additively decompose each one of our measures in three different ones, that can be seen as components or factors of the centrality in the dimensions of emission, reception and betweenness.

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