Conservation Laws of Differential Equations beyond Lagragian Methods: Through Nonlocal Symmetries

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Introduction

If no invariant Lagrangian exists alternative methods of constructing lower order first integrals have been proposed by using use of the so called adjoint equations solutions. We shall call this the "adjoint equation method". The method is based on identities which link symmetries of the underlying differential equations, solutions of the adjoint equations and conservation laws. The purpose of this project is to extend the adjoint equation method and to look for new applications.
During our stay in the Department of Finance and Management Science, Norwegian School of Economics in Bergen toguether with R Kozlov

- We have derived first integrals of ODEs via nonlocal symmetries.
- We have considered a system which admit a nonlocal symmetry.
- We have constructed a formal Lagrangian and the adjoint system.

- From the adjoint system we have derived a first integral depending on the nonlocal variable.
- After obtaining two first integrals of the nonlocal variable we were able to derive proper first integrals eliminating the nonlocal variable.

- We have applied this approach to the second order Riccati equation.
- In order to do this study, three different programs have been designed with free software MAXIMA.
- These programs allow us to determine the adjoint equation, the conditions of self-adjointness and the first integrals of the Riccati equations.

References

[1] M.L. Gandarias, *Nonlocal symmetries and reductions for some ordinary differential equations M. L. Gandarias,* Theoretical and Mathematical Physics, 2009, 159, 779-786 .

[2] M. S. Bruzon, M. L. Gandarias, and M. Senthilvelan *Nonlocal symmetries of Riccati and Abel chains and their similarity reductions*J. Math. Phys. 53, 023512 (2012); .

[3] Vladimir Dorodnitsyn. and Roman Kozlov, *Invariance and first integrals of continuous and discrete Hamiltonian equations*,Journal of Engineering Mathematics 66,1-3, 253-270 (2010).

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First integrals of ODEs through nonlocal symmetries

We consider the system

$$\begin{aligned} F(x,u,u_x,u_{xx}) &= 0 \\ G = w_x - f(x,u) &= 0 \end{aligned} \tag{1}$$

which admits the nonlocal symmetry

$$X = e^w\bar{\eta}\frac{\partial}{\partial u} + e^w\bar{\zeta}\frac{\partial}{\partial w} + \ldots \tag{2}$$

Formal Lagrangian is

$$L = v_1F + v_2G$$

Adjoint system consists of the two equations

$$F_1^* = \frac{\delta}{\delta u}L = \left(\frac{\partial}{\partial u} - D\frac{\partial}{\partial u_x} + D^2\frac{\partial}{\partial u_{xx}}\right)L = 0$$

and

$$F_2^* = \frac{\delta}{\delta w}L = \left(\frac{\partial}{\partial w} - D\frac{\partial}{\partial w_x} + D^2\frac{\partial}{\partial w_{xx}}\right)L = -D(v_2) = 0$$

In this case we can obtain the first integral

$$I = \left[e^w\bar{\eta}\left(\frac{\partial}{\partial u_x} - D\frac{\partial}{\partial u_{xx}}\right) + [D(e^w\bar{\eta})]\frac{\partial}{\partial u_{xx}} + e^w\bar{\zeta}\frac{\partial}{\partial w_x}\right]L\Big|_{v_1,v_2} \text{ are substituted}.$$

The first integral follows from the identity

$$v_1X(F) + v_2X(G) = e^w\bar{\eta}F_1^* + e^w\bar{\zeta}F_2^* + D(I)$$

Problems: first integral depends on w.
Possibilities: if two first integrals are found, we can exclude e^w.

Riccati II

Equation

$$F = u_{xx} + 3uu_x + u^3 = 0 \tag{3}$$

is treated as system

$$\begin{aligned} u_{xx} + 3uu_x + u^3 &= 0 \\ w_x &= f(x,u) \end{aligned}$$

Associated to the multipliers, excluding w, we obtain first integrals i.e.

$$D(I_1) = \Lambda_1F$$

1. For multiplier

$$\Lambda_1 = -\frac{u}{(u_x+u^2)^2}$$

First integral

$$I_1 = \frac{u}{u_x+u^2} - x$$

2. For multiplier

$$\Lambda_2 = -\frac{xu-1}{(u_x+u^2)^2}$$

First integral

$$I_2 = \frac{xu-1}{u_x+u^2} - \frac{x^2}{2}$$

From these two independent first integrals I_1, I_2 we got the general solution in agreement with [2].

- The main benefit of the Project are the new results concerning the search of first integrals and solutions for ODEs,
- The main beneficiaries are the researchers which could use these new results and Universities envolved
- The expectation of this project is to extend this joint research of the adjoint equation method and to search for new applications.