Paraxial Optics

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INTRODUCTION

Paraxial optics is sometimes known as the Gaussian optics. It is the simplest framework where optical systems are described. A great variety of textbooks in optics include specific chapters to the paraxial approximation of the geometrical optics. Other textbooks focused in geometrical optics propose that the paraxial treatment should be included as part of the first approach to the subject. The paraxial approximation explains how light propagates along an optical system when rays are close to the optical axis. Roughly speaking, paraxial optics applies when the transversal size is small compared with the longitudinal size of the objects, images, and constructive parameters of the optical systems. For an image-forming system, it corresponds with the ideal status where the system can be considered as perfect. Geometrical optics uses the advantages of the paraxial approach to provide a first-order description of the behavior of an optical system. Because of the intensive use of the paraxial approach in geometrical optics, we are sometimes tempted to identify both concepts. However, geometrical optics goes beyond the paraxial approach, and the paraxial approximation is not only applicable into the geometrical optics. The main conclusions of this contribution are summarized in the “Conclusion.”

PARAXIAL REGIME

The most common way to introduce the paraxial approximation is by approximations of the trigonometric functions.

It is well known that sine, cosine, and tangent functions can be expanded as power series around \( z = 0 \). These expansions are:

\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots
\]

\[
\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots
\]

\[
\tan z = z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots
\]

Then the paraxial approach means to take only those terms until linear dependence. The paraxial values of the trigonometric functions are given by

\[\sin z \approx z\]
\[\cos z \approx 1\]
\[\tan z \approx z\]

where the angle, \( z \), needs to be given in radians. In Fig. 1, we have plotted the value of the trigonometric functions along with their paraxial approach in dashed line. In Fig. 2, we have calculated the relative error between the actual and the paraxial values for an angular range from \( 0^\circ \) to \( 20^\circ \). A positive relative error means that the paraxial approach is under evaluated. A negative relative error means that the paraxial approach is larger than the actual value of the function.
These figures present the departure of the paraxial approach with respect to the actual behavior.

**GEOMETRICAL PARAXIAL OPTICS**

Geometrical optics is based on a few axioms: the existence of the incidence plane containing the normal to the interface, the incident, the reflected, and the refracted rays; the Snell law and the reflection law, easily derived from the Fermat’s Principle; the Fermat’s Principle itself; and the reversibility of the optical paths. Although no one of these axioms involves the paraxial approach, the paraxial optics is traditionally linked with the basis of geometrical optics. These axioms are the actual foundations of the calculation of light trajectories—the goal of the geometrical optics. The first axiom, defining the incidence plane, has important consequences for the simplification of the treatment. It allows to treat rotationally symmetric optical systems, after neglecting the effect of the skew rays, only by analyzing the ray trajectories in a meridian plane that contains the optical axis of the system. This axiom is implicitly used to draw in a two-dimensional plot the ray tracing of a three-dimensional optical system.

More specifically, paraxial optics appears as the regime where the concept of perfect optical system applies. There are three conditions for an optical system to be considered as perfect: Every object point corresponds with an image point, then every ray departing from an object point arrives to the corresponding, conjugated, image point; every plane in the object space is imaged onto another plane in the image space; and a figure located in an object plane produces an image having a size proportional to the object figure, with a constant ratio between its length dimensions. These three conditions are the ultimate goal of an image-forming optical system. Optical designers bend the surfaces and make a customized layout of the system to meet the specifications requirements and converge to the perfect optical system behavior. In the very first stage of the design, paraxial optics may play an important role because it is able to produce a first-order output of the characteristic of the system, neglecting ray aberrations.

**Snell Law in Paraxial Optics**

When the paraxial approach is applied to the Snell law,

\[ n \sin \epsilon = n_0 \sin \epsilon_0 \]

it produces the following paraxial relation:

\[ n \epsilon = n_0 \epsilon_0 \]

This equation makes it possible to treat refraction as a linear transformation of angles. The reflection law

\[ \epsilon'' = -\epsilon \]

is linear already. This linearization is formalized within the scope of matrix optics, where a ray within an optical system is characterized in terms of the height with respect to the optical axis and its slope.

**Correspondence Equations and Image-Forming Systems**

The definition of a perfect optical system is based on the stigmatism concept. This can be formulated as the constancy of the optical path along any light trajectory...
from the object point to the image point. The optical path is defined as the following integral

\[ L = \int_C n(\vec{r}) \, d\vec{r} \]

where \( n(\vec{r}) \) represents the index of refraction at a point \( \vec{r} \) that belongs to the light trajectory, \( C \). The light trajectory may travel along different materials. The index of refraction characterizes the propagation properties of the light, and if these materials are linear, homogeneous, and isotropic, then \( n(\vec{r}) \) is constant along the light trajectory inside the same material. In this case, the optical path is merely the product of the index of refraction times the geometrical path along the light trajectory in the media.

A general case of stigmatism can be considered from Fig. 3, where a curved diopter is the interface between two linear, homogeneous, and isotropic media having index \( n \) and \( n' \).

The astigmatism condition is written as the constancy of optical path for any arbitrary light trajectory from \( O \) to \( O' \). This condition is:

\[ L = n \rho + n' \rho' = K \]

being \( \rho \) and \( \rho' \), the geometrical paths of the light incident on the interface. The invariance of this equation with respect to the actual trajectory provides the following equation:

\[ L = n \sqrt{y^2 + z^2} + n' \sqrt{y^2 + (l - z)^2} = K \]

This equation can be used to customize a surface having stigmatism behavior for a given pair of conjugated object and image points. The Maxwell’s eye-fish and conicoids are academic examples for selected pairs of conjugate points (object and image points). Also, the sphere for the Weierstrass points shows stigmatism. However, the extension of the stigmatism to wider spatial regions fails. The conditions need to be relaxed and the stigmatism concept is replaced by the concept of isoplanatism, where all the regions of the image plane are equivalent although not perfect.

The shape of the optical surfaces is usually a sphere due to its easy manufacture and testing. When the diopter is a spherical surface having a radius of curvature of \( r \), the geometry is the same with that of the previous figure. The optical path in terms of the frontal distances \( s \) and \( s' \), \( r \), and the angle \( \varphi \) becomes

\[ L = n \sqrt{r^2 + (r - s)^2} - 2r(r - s) \cos \varphi + n' \sqrt{r^2 + (s' - r)^2} - 2r(s' - r) \cos \varphi = K \]

When the stigmatism condition about the constancy of the optical path is applied, the result is

\[ \frac{n}{\rho} + \frac{n'}{\rho'} = \frac{1}{r} \left( \frac{ns}{\rho} + \frac{n's'}{\rho'} \right) \]

where \( \rho \) and \( \rho' \) are the object and image optical paths reaching the diopter at the incidence point. In this formula, it is possible to apply the paraxial approach to the trigonometric functions included in the expressions of \( \rho \) and \( \rho' \) to obtain the following correspondence equation:

\[ \frac{-n}{s} + \frac{n'}{s'} = \frac{n' - n}{r} \]

where we have applied the sign convention that considers light propagation from left to right.

The previous object–image equation, along with the equation relating the distances referred to two consecutive dioplers, that is written as

\[ s_2 = s'_1 - d \]

can be used to obtain properly the transformation of an object through a combination of paraxial optical systems. One of the first concepts that appear when combining dioplers is the optical axis of the combination. If the dioplers are spherical surfaces, the optical axis is defined as the line containing the centers of curvature of the diopter surfaces. If the system contains a rotational axis of symmetry, the optical axis corresponds with this symmetry axis.

**Sign convention**

Assuming that the propagation of the light is coming from the left and going to the right, the frontal distances are positive when they follow the direction of propagation of the light. These frontal distances are measured from the vertex of the diopter, and they are considered positive when the associated point is to the right of the vertex. The radius of curvature also is positive when the center is located to the right of the vertex, i.e., for a convex surface. The angles with respect to the optical axis, \( \sigma \) and \( \sigma' \), are positive if rotating the ray to reach the optical axis by the shortest way, the rotation is made in counterclockwise...
direction. The angles of incidence are positive when rotating the rays to reach the perpendicular to the incidence point, the rotation is in the clockwise direction. Fig. 4 shows a situation where all the variables are positive.

Mirrors

A simple and useful optical system is formed by a reflecting surface. The reflection law was previously described, and it is the same in the paraxial regime than in the exact one. However, the corresponding equation relating the object and image location can be adapted to the case of a reflecting surface by assuming that the index of refraction of the image space is \( n' = \frac{1}{n} \). Then the object–image equation becomes

\[
\frac{1}{s} + \frac{1}{s'} = \frac{2}{r}
\]

where \( r \) is the radius of curvature of the surface of the mirror. The radius is measured from the vertex of the diopter, i.e., the intersection of the diopter with the optical axis, to the center of curvature. Following the sign convention, a convex mirror has a positive radius and the radius of a concave mirror is negative.

Cardinal Points

The combination of diopters into an optical system needs a more detailed characterization in terms of the cardinal points. These cardinal points describe the behavior of the optical system by summarizing the effects of the combination of the individual diopters. The cardinal points are: the focal points, the principal points, and the nodal points. The focal points are those points conjugated with the infinity. The object focal point defines the location of an object that produces an image at the infinity. The image focal point is the image of the infinity given by the optical system. The transversal planes containing the focal points are the focal planes. They are the conjugate planes of the infinity. The points of the focal planes correspond with bundles of parallel rays at different angles with respect to the optical axis. These situations are shown in Fig. 5.

The principal planes are defined in terms of the value of the lateral magnification of the system, \( \beta' \). This lateral magnification is given as the ratio between the transversal size of the image with respect to the transversal size of the object

\[
\beta' = \frac{y'}{y}
\]

Then the principal points define two planes, the principal planes conjugate to each other (one is the image and the other is the object through the optical system) showing a...
lateral magnification equal to $\beta' = +1$. In practice, it means that a ray incident on the object principal plane at a given height produces a point in the image principal plane at the same height. The principal planes of an optical system can be obtained as it follows (Fig. 6). Let us take a ray parallel to the optical axis and coming from the infinity in the object space. The only thing we know is that this ray will reach the image focal point at the output. If the system is a black box and we are not allowed to get into its components, we still can extend the incoming and the outgoing rays and intersect them. The intersection point belongs to the image principal plane. To obtain the object principal plane, we proceed in the same way but, now, we are using a ray departing from the object focal point and reaching the infinity parallel to the optical axis. The intersection of the extended input and output rays produces a point that belongs to the principal object plane. The actual composition of the optical system is only needed to properly establish the character real or virtual of the object and image.

After locating the principal planes, it is possible to define the focal length and the object and image focal lengths as the distances between the principal plane and the focal point, both for the object and the image. One of the main parameters characterizing the optical system is the value of the image focal distance, the focal that is given by the distance from the image principal point and the image focal point.

The last pair of cardinal points are the nodal points. They are defined as those showing an angular magnification, $\gamma'$, of +1. The angular magnification is defined as the ratio between the angles of the output and input rays with respect to the optical axis of the system

$$\gamma' = \sigma'/\sigma$$

For those systems having the same index of refraction at the object space and the image space, the nodal points coincide with the principal points.

![Fig. 6](image) Graphical method to obtain the principal planes.

Correspondence equations referred to the cardinal planes

These cardinal points, specially the principal and the focal points, are used to estimate the location and the size of the image. In Fig. 7, we have plotted a situation where the location and the size of the image are obtained by ray tracing some special trajectories, showing a very well-defined behavior. The relation between the object and the image distances, $a$ and $a'$, measured from the principal planes is

$$-\frac{n}{a} + \frac{n'}{a'} = \frac{n'}{f'}$$

where $f'$ is the image focal distance, and $n$, $n'$, are the indices of refraction of the object and image space, respectively. When the system is immersed in air, or in general, when $n = n'$, the previous equation becomes the well-known conjugate relation

$$-\frac{1}{a} + \frac{1}{a'} = \frac{1}{f}$$

We should recall that $a$ is negative when the object is located to the left of the object principal plane. The correspondence equation using the distances from the focal planes is given as

$$\z' / f = f'/f$$

where $f$ is the object focal distance that is related with the image focal distance by means of

$$f = -\frac{n}{n'}f'$$

that becomes $f = -f'$ when $n = n'$. In this case, the corresponding equation becomes the usual form of the Newton equation of correspondence

$$\z' = -f'^2$$
The lateral magnification can be obtained also from Fig. 7 as the following set of equivalent equations

\[ \beta' = \frac{-f}{z} = \frac{-f}{a-f} = \frac{-z'}{f'} = \frac{a' - f'}{f'} \]

This lateral magnification can be written also in terms of the object and image distances as

\[ \beta' = \frac{n's'}{n's} \]

**Combining Paraxial Elements Within the Paraxial Approach**

Paraxial optics is able to deal with a combination of diophters to find out what are the characteristic parameters of the combination. The goal is to obtain the location of the focal and principal planes of a system formed by a combination of diophters or other optical subsystems. The first step in the process is to know how to combine two optical systems, each one having their own principal and focal planes. In Fig. 8, we have presented the graphical solution to this problem. To obtain it, we have traced a ray parallel to the optical axis, coming from infinity and reaching the focal point of the compounded system, \( F' \), after passing through the two optical systems. The intersection of the input and the output rays provides a point belonging to the image principal plane of the total system, \( H' \). To obtain the object focal and principal planes, \( F \) and \( H \), we have traced a ray coming from the right and passing through the object focal point, \( F \). The analytical solution relates the involved magnitudes to provide the following results. The image focal distance is given by

\[ \frac{1}{f'} = \frac{n_2}{n_1 f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \]

**Fig. 8** Combination of two optical systems. The principal and the focal points are found by the same general method used to define them.

![Diagram of two optical systems](image)

**Fig. 9** A lens is a combination of two curved diopters with radius, \( r_1 \) and \( r_2 \), thickness (\( t \)), and fabricated with a material showing an index of refraction of \( n \).

where \( f_1 \) and \( f_2 \) are the image focal distances of the individual systems, \( n_2 \) is the index of refraction of the medium between the systems, \( n_2' \) is the index of refraction of the image space of the combined system, and \( t \) is the distance between the image principal plane of the first system and the object principal plane of the second system.

The location of the principal planes of the combined system is given by

\[ H' = \frac{f' t}{f_1} \]

\[ H = \frac{f t}{f_2} \]

One of the most useful applications of this method is to find the characteristics of a thick lens. A lens is a combination of two diopters separated by a distance equal to the central thickness and having an internal index of refraction due to the material that the lens is made of (Fig. 9). The principal planes for the subsystems are located on the vertex of the refracting surfaces. Then if we assume that the lens is immersed in air, the value of the focal length is given as

\[ \frac{1}{f} = (n-1) \left( \frac{1}{r_1} - \frac{1}{r_2} \right) + \frac{(n-1)^2 t}{n r_1 r_2} \]

If the central thickness can be considered negligible, then the first term provides the focal length of the lens in the thin lens approximation. The previous equation and the thin lens approximations are sometimes known as the lens-maker formula.

**Prisms**

Prisms are the combination of two plane interfaces forming a given angle, \( \alpha \), the angle of the prism. When the angle of the prism and the incidence angle are small to be inside the paraxial approach, it is possible to reduce...
the general expression of the deviation of a prism to its paraxial version.

The deviation of a prism depends on the incidence angle, \( e_1 \), the angle of the prism, \( \alpha \), and the index of refraction of the material, \( n \). The situation is presented in Fig. 10, where \( e_1, e_1' \) are the positive angles, and \( e_2, e_2' \) are the negative angles. The angle of the prism in this figure is also positive. A prism angle is considered positive when rotating the input surface to reach the output surface by the shortest way, the rotation is made in the counterclockwise direction. The expression for the deviation of the prism is as follows

\[
\delta = e_1 - e_2 - \alpha
\]

where the dependence with the index of refraction appears when calculating the angle of refraction at the output interface of the prism, \( e_2' \). By using the paraxial Snell law, it is possible to relate \( e_1 \) with \( e_1' \) and \( e_2 \) with \( e_2' \). If this is done in the previous equation and using the relation with the angle of the prism \( \alpha = e_1' - e_2 \), the previous equation becomes

\[
\delta = (n - 1) \alpha
\]

which is the paraxial form of the deviation of a prism immersed in air.

**Matrix Optics and Paraxial Optics**

The first equation of paraxial optics that we found in this contribution was the linearization of the Snell law. This linearization can be extended and fully completed when using the matricial formulation of paraxial optics. In this sense, a given optical system can be seen as a transformer that changes linearly the characteristic parameters of a given light trajectory (Fig. 11). These characteristic parameters are the height and slope of the ray with respect to the optical axis of the optical system. This matricial relation is written as

\[
\begin{pmatrix}
  h' \\
  \omega'
\end{pmatrix} =
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\begin{pmatrix}
  h \\
  \omega
\end{pmatrix}
\]

A derivation of the matricial relation provides also the object and image distance relation. This relation is known also as the ABCD law

\[
s' = \frac{As - B}{Cs + D}
\]

It is important to note that due to the convention sign used in the definition of the previous parameters, the slope of the ray, \( \omega \), and the angle with respect to the axis, \( \sigma \), have opposite signs. In Fig. 11, \( h, h' \), and \( \omega \), are positive and \( \omega' \) is negative. An illustration of the application of the matrix optics is shown in Fig. 12 for the case of a spherical surface having a radius of curvature, \( r \), that provides the following ABCD matrix

\[
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  -\frac{n' - n}{n'r} & \frac{n}{n'}
\end{pmatrix}
\]

When the elements of the matrix for a curved diopter are replaced in the ABCD law, it is possible to obtain the correspondence paraxial equation for a curved diopter.

The main advantage of matrix optics is that the combination of optical elements is easily done by the matrix multiplication of the individual matrices of the optical elements. This modularization, along with some extensions applied to laser beam propagation and array optics, makes matrix optics a powerful tool for the paraxial analysis of optical systems.

**Fig. 10** Diagram of the ray tracing for a prism.

**Fig. 11** The ABCD matrix describes the transformation between the height and the slope of a ray at the output and the input planes of an optical system.

**Fig. 12** Diagram to find the elements of the ABCD matrix for a curved diopter.
Paraxial Ray Tracing

A common application of the paraxial optics deals with the rules for ray tracing. When practicing with ray tracing, we sometimes need to draw rays having large angles with respect to the axis. These angles are sometimes well beyond paraxial approach. However, the ray tracing is still valid and the predictions of it produce the location and the size of the image. Then is this case in contradiction with the paraxial approach? The answer to this question can be found by analyzing the situation of the refraction through a curved diopter. The actual trajectory of light reaches the actual refracting surface before it is transformed by the Snell law. When using paraxial approach, the refracting surface, independently on the value of the radius of curvature, is represented by its principal planes that are tangent to the vertex of the surface. Then the ray is not reaching the surface anymore. The geometrical meaning of this approach is clear in this case, it neglects the sagita of the curved surface at the incidence point. Then the relation between the parameters is given in terms of the tangents of the angles with respect to the axis, or as a proportion between frontal and transversal distances. This tangent keeps the proportionality factors and allows dealing with large transversal distances and, therefore, large angles, without losing the proportionality that permits to maintain the perfect optical system behavior. This can be stated in the relation between the angles of the rays with respect to the axis, the Lange formula. The typical formulation of this expression is given as

\[ n'\theta' - n\theta = h \frac{n' - n}{r} \]

But it is still valid when \( \theta \) and \( \theta' \) are replaced by the tangents of the angles. Actually, the expression with the tangents is that it is valid in the extreme situations sometimes encountered in ray tracing. Obviously, the paraxial approach applied to the tangent produces the classical Lange expression.

**F# and Paraxial Optics**

Moreover, the tangent calculation previously derived is used also when calculating, within the paraxial approach, the location and the size of pupils and windows, stops, and apertures.

The aperture number, F#, is a ratio between the focal distance and the diameter of the lens (Fig. 13). If this ratio is interpreted as a tangent, we can easily find the following relation

\[ F# = \frac{f'}{\phi} = \frac{1}{2 \tan \theta'} \]

and the numerical aperture

\[ \text{N.A.} = 2n' \sin \theta' \]

The concept of aberration is mostly related with the departure of an optical system with respect to its paraxial behavior. However, it is possible to take into account the transversal and the longitudinal chromatic aberrations keeping the paraxial approach. These aberrations are produced by the dependence of the index of refraction with respect to the wavelength of the light propagation along the optical system. The most known parameter describing this variation is the Abbe number that is defined as

\[ v = \frac{n_d - 1}{n_F - n_C} \]

where \( n_F, n_d, \) and \( n_C, \) are the values of the index of refraction for three selected wavelengths in the blue, yellow, and red portions of the spectrum, respectively. These wavelengths are: \( \lambda_d = 587.6 \) nm, \( \lambda_F = 486.1 \) nm, and \( \lambda_C = 656.3 \) nm. The calculation of the longitudinal chromatic aberration defines it as the distance between the image for \( \lambda_d \) and \( \lambda_F \). The transversal chromatic aberration will be given as the difference in the transversal size of the image for \( \lambda_F \) and \( \lambda_C \). These values can be obtained by replacing the value of the index of refraction with the value \( n_F \) and \( n_C \), respectively, and keeping the paraxial form for the calculation.

Then the compensation of the chromatic aberration can be calculated within the paraxial scope. Let us take the case of a given thin lens that should be represented by its image focal distance, \( f' \). The variation of \( f' \) when the index changes can be related with the Abbe number as

\[ \frac{df'}{f'} = \frac{1}{v} \]

By combining properly two elements with focal \( f_1' \) and \( f_2' \) and fabricated with materials having \( v_1 \) and \( v_2 \) Abbe numbers, it is possible to obtain an achromatic cemented doublet if the following relation is fulfilled

\[ \frac{1}{v_1 f_1'} + \frac{1}{v_2 f_2'} = 0 \]
PARAXIAL APPROACH IN PHYSICAL OPTICS

Usually paraxial optics is linked with geometrical optics where the wave nature of radiation is neglected. However, some regimes of physical optics use the paraxial approach to describe phenomena involving small angles where the trigonometric functions can be paraxially treated.

Reflection and Refraction at Normal Incidence

The paraxial approach of the Snell law is applied to obtain the normal incidence version of the Fresnel equations, relating optical fields along a plane interface. The coefficients of reflection and transmission are decoupled for the parallel and perpendicular states of polarization. When reaching small angles of incidence, this distinction becomes less and less noticeable. In this case, the trigonometric functions in the transmission and the reflection coefficients involve the use of small angles. Then the paraxial approach can be applied to write the equations in terms of the incidence and the refraction angle. On the other hand, these angles are related by the Snell law. The paraxial approach of the Snell law can be used to finally derive the reflectance and the transmittance for the normal incidence regime

\[
R = \left( \frac{n' - n}{n' + n} \right)^2
\]

and

\[
T = \frac{4nn'}{(n + n')^2}
\]

Fig. 14 shows the relative error in the reflectance and the transmittance in the paraxial range for an air-dielectric incidence with indices \( n = 1 \), \( n' = 1.5 \), as a function of the angle of incidence.

Paraxial Wave Equation

The scalar form of the wave equation for optical fields is usually written as (e.g., Ref. [24])

\[
\left[ \nabla^2 + \left( \frac{2\pi}{\lambda} \right)^2 \right] \Psi(x, y, z) = 0
\]

where \( \nabla^2 \) is the Laplacian operator, and \( \Psi(x, y, z) \) is the optical field. When propagating this optical field across an optical system, we are mainly interested in the analysis of the transversal distribution in successive planes along the optical axis. If the axis is aligned along \( Z \) and we are not close to focalization points, the dependence associated with the propagation along \( Z \) can be factorized out and the field is written as

\[
\Psi(x, y, z) = \psi(x, y, z) \exp\left( -i \frac{2\pi}{\lambda} \frac{z}{\lambda} \right)
\]

On the other hand, it is very common to assume that the changes along \( z \) are slowly enough to neglect the contribution of the second partial derivative with respect to \( z \) in the calculation of the wave equation. This assumption is called the paraxial approach in wave optics. It provides the following form of the paraxial wave equation

\[
\nabla^2 \Psi(x, y, z) = -i \frac{\pi}{\lambda} \frac{\partial \psi(x, y, z)}{\partial z}
\]

where \( \nabla^2 \) is the two-dimensional Laplacian operator in a transverse plane perpendicular to \( z \).

The range of validity of this paraxial wave equation is established by using the plane-wave spectrum of the optical field under analysis. When the contribution of the optical field is inside a cone having a half-angle of about 30°, then it is possible to use the paraxial wave equation and its solutions as a first-order approach to the exact behavior of the optical field. This is the case for most of the optical beams usually encountered in most of the optical systems. If this condition is not fulfilled, then the paraxial wave optics cannot be applied, and the exact calculation should be carried out.
Huygens–Fresnel Integral in the Paraxial Approximation

Scalar diffraction theory is usually divided in two regimes: the far-field Fraunhofer diffraction and the Fresnel diffraction. Both of them can be obtained from the well-established Huygens–Fresnel principle that provides the amplitude of the electric field of a light beam by means of an integration over all the sources along the aperture size

$$\Psi'(x', y') = \int \int K(x, y, x', y') \Psi(x, y) \, dx \, dy'$$

where the kernel of such integration is given by

$$K(x, y, x', y') = \frac{1}{i \lambda z} \exp\left(i \frac{2\pi}{\lambda z} r\right) \cos(\hat{u} \cdot \hat{p})$$

where $\lambda$ is the wavelength, $\rho$ is the distance between the point O with coordinates $(x, y)$ at the aperture plane and the point $O'$ with coordinates $(x', y')$ at the observation plane, $\hat{u}$ is a unitary vector normal to the aperture plane, and $\hat{p}$ is a unitary vector along the line between $O$, $O'$ (Fig. 15). The cosine term is identified as an obliquity factor for the Fresnel and the Fraunhofer regimes that approaches to 1. This is a clear application of the paraxial regime. Moreover, the calculation of the phase term needs the value of $\rho$ that can be given as

$$\rho = \sqrt{z^2 + (x' - x)^2 + (y' - y)^2}$$

This square root can be properly expanded in powers of $x$ and $x'$ until second order in $x'$ and $y'$.

$$\rho \approx z \left[1 + \frac{1}{2} \left(\frac{x - x'}{z}\right)^2 + \frac{1}{2} \left(\frac{y - y'}{z}\right)^2\right]$$

The paraxial approach applied here is related with the degree of this expansion. When only linear terms in $x'$ and $y'$ are left, the kernel of the integrand becomes

$$K(x, y, x', y') = \frac{1}{i \lambda z} \exp(i k z) \exp\left[i \frac{\pi}{\lambda z} \left(x^2 + y^2\right)\right]$$

$$\times \exp\left[-i \frac{2\pi}{\lambda z} \left(xx' + yy'\right)\right]$$

where the last exponential function is responsible for the Fourier transform properties of the Fraunhofer diffraction. This regime applies when the quadratic dependence with $x'$ and $y'$ can be neglected. This means that

$$\frac{\pi}{\lambda} (x'^2 + y'^2) \ll z$$

This previous equation can be written in terms of the angles subtended from the center of the aperture by the observation point at $(x', y')$. These angles are $\alpha_x = x'/z$, $\alpha_y = y'/z$. Then the previous equation becomes

$$(\alpha_x^2 + \alpha_y^2) \ll \frac{\lambda}{\pi}$$

The corresponding values of the angles are very close to zero; therefore, the paraxial regime is properly fulfilled.

The previous reasoning can be derived also in terms of matrix optics. We have seen that $\rho$ represents the optical path between two points in the input and the output planes. This optical path is written in terms of the elements of the ABCD matrix for a rotationally symmetric system in the XZ meridional plane as [23]

$$\rho = z + \frac{1}{2z} \left(Ax^2 + 2x + Dx^2\right)$$

Then the Huygens–Fresnel integral is given as

$$\Psi'(x', y') = -\frac{i}{IB} \exp\left(i \frac{2\pi}{\lambda z}\right) \int \int \Psi(x, y)$$

$$\times \exp\left\{i \frac{2\pi}{IB} \left[A(x^2 + y^2) - 2(xy + yy') + D(x'^2 + y'^2)\right]\right\} \, dx' \, dy'$$

This equation clearly shows the link between the paraxial geometrical optics and the paraxial wave optics.

Paraxial Approach for an Spheric Wavefront

Typically, spherical wavefronts are defined in terms of its radius of curvature. A general expression for a spherical wavefront is

$$E(r) = \frac{1}{r} \exp\left(i \frac{2\pi}{\lambda} r\right)$$
where \( r \) is the radius of curvature. The term \( 1/r \) is responsible for the attenuation of the amplitude after propagating from the source. The phase term varies very fast along the propagation and configures the spherical shape of the wavefront.

A sphere in a meridian plane is expressed as

\[
(x^2 + z^2) = r^2
\]

Then the phase term expressed in Cartesian coordinates contains a square root. Usually, we are interested in the spherical waves that propagate along coordinate, \( z \), being \( x \) the transversal coordinate. Mostly, the propagation distance from the source is larger than the transversal dimension of the optical layout. Then the square root can be written as

\[
r(z) = r(z) = z\sqrt{1 + \frac{x^2}{z^2}}
\]

that can be expanded also in powers until second order

\[
r(z) \approx z \left(1 + \frac{x^2}{2z^2}\right)
\]

If we only take the first term, the spherical dependence is lost and the phase behaves as a plane wave. This approach can be valid for the \( 1/r \) term of the amplitude. Then the very first approximation to the spherical wavefront is given by the following form

\[
E(x, z) = \frac{1}{z} \exp \left(\frac{2\pi x^2}{2z}\right)
\]

This equation represents a spherical wavefront having its center at a distance \( z \) from the observation plane. It is interesting to notice that this previous equation has replaced the spherical wavefront by a parabolic wavefront. This approximation will be related with the paraxial approach previously presented in geometrical optics.

**Paraxial Thin Lens as a Phase Screen**

As we have seen previously, the most of the optical systems use spherical surfaces as dioptries. When representing these dioptries, we assume that the principal planes replace the curved surface and the rays change their directions at the principal planes. It seems like the sagita does not need to be accounted for. Actually, the paraxial approach is a little more refined.

The sagita of a curved surface is given within the meridian plane as

\[
sag = r - r\sqrt{1 - \frac{x^2}{r^2}}
\]

where the square root can be expanded into powers until several orders. The first order provides a value of 1 and then the sagita is totally neglected. When the next term of the expansion is taken into account, it provides the following expression for the sagita

\[
sag = \frac{x^2}{2r}
\]

A thick lens can be seen as a phase plate introducing a phase change variable along the transversal direction.\(^{[23]}\) Then any incoming wavefront will change its phase accordingly to the value of this phase screen. To account for the dependence with the transversal coordinate, \( x \), for a thick lens, we need to calculate the optical paths at any height from the input plane (tangent to the vertex of the first surface of the lens) to the output plane (tangent to the vertex of the second surface of the lens). This calculation needs the values of the sagita for both surfaces to build up the thickness function as

\[
t(x) = t_0 + \frac{x^2}{2r_1} - \frac{x^2}{2r_2}
\]

The phase screen is obtained as \((n - 1)t(x)\), where \( n \) is the index of refraction of the material of the lens, assuming it is immersed in air (Fig. 16). This product contains the following term

\[
(n - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \frac{x^2}{2} = \frac{x^2}{2f'}
\]

where we have made use of the lens-maker formula to write down the focal distance of a thin lens, \( f' \), within the paraxial approach.

**BEYOND PARAXIAL APPROACH**

The departure of the paraxial approach may be interpreted as a departure from a perfect optical system. This departure is known as aberration. The analysis of the optical aberrations is made in terms of their third-order approach.
This analysis can be made in terms of ray tracing and also in terms of the wavefront aberration. Both types of analysis are widely used in the fine optimization routines for designing optical systems.

CONCLUSION

Paraxial optics is based on the paraxial approach. This approach is a linearization of the trigonometric functions used in the description of the optical systems and their related phenomenology.

Geometrical optics uses very intensively the paraxial approach to provide a first approximation to the behavior of light inside optical systems. Paraxial optics allows making paraxial ray tracing. It defines and uses the concepts of cardinal points and planes: focal, principal, and nodal. Paraxial optics is very well adapted to describe an optical system as a perfect optical system. Moreover, the departure of the paraxial behavior can be easily treated inside the paraxial approach. Some other important concepts, such as the aperture and the field of an optical system, can be understood also inside the paraxial framework. The matrix optics is a complete formalization of the paraxial approach.

The paraxial approach also applies to physical optics concepts. Then the reflectance and the transmittance of light for small angles are obtained by using paraxial concepts. There exists a paraxial wave equation that describes the propagation of an optical field when its plane-wave spectrum is within a cone of about 30° of half-angle. This is the case for most of the light beams actually propagating along optical systems. The paraxial form of the kernel of the Huygens–Fresnel integral yields the Fresnel and the Fraunhofer diffraction equations. Moreover, when the elements of the ABCD matrix are included in the paraxial form of the kernel of the Huygens–Fresnel integral, we finally find the strong links between geometrical and physical paraxial optics. Finally, as another example of this close relation, the description of a thin lens as a phase screen is shown using the paraxial approach in the definition of the thickness function of the lens.

The extensions of the paraxial optics correspond with the analysis of the optical aberrations. This study can be carried out with the use of a geometrical point of view by means of the exact ray-tracing calculation, or by the formalism dealing with the propagation of optical fields beyond paraxial approach.

REFERENCES