On Organizing Principles of Discrete Differential Geometry. Geometry of Spheres

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Plan of the talk

Based on joint work with A. Bobenko:
“Discrete Differential Geometry. Consistency as Integrability” (math.DG/0504358), and

- Discrete Differential Geometry
- Multidimensional Consistency Principle
- Discrete Erlangen Program
- New results on discrete curvature line parametrized surfaces in Lie, Laguerre and Möbius geometry.
Discrete Differential Geometry

- **Aim:** Development of discrete equivalents of the geometric notions and methods of differential geometry. The latter appears then as a limit of refinements of the discretization.

- **Current conclusions:** Discretizations which preserve fundamental properties of the smooth theory turn out to be also good for applications: they represent smooth shapes by discrete ones with just few elements (approximation often better then one might expect).

- **By-product:** New geometric understanding of integrability as consistency.
Surfaces and transformations

Surfaces (special classes: constant curvature, isothermic, etc.) and their transformations (Bianchi, Bäcklund, Darboux) with permutability properties: smooth vs. discrete.
Basic features of the discrete theory

- Defining geometric properties of discrete surfaces are identical with those of their transformations. Discrete master theory: surfaces and their transformations are just different coordinate slices of multidimensional nets.

- Complicated differential-geometric notions and properties are reduced to “elementary” incidence theorems.
Consistency of 2D equations

- 2D equation
  (for $x : \mathbb{Z}^2 \to X$)

- 3D consistency

$Q(x, x_1, x_{12}, x_2) = 0$
Consistency of 2D equations

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Consistency of 2D equations

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\[ Q(x, x_1, x_{12}, x_2) = 0 \]
Message: “Consistency ⇒ Integrability” or “Consistency is Integrability” or even “Integrability is Consistency” (depending on your taste and discreteness). Why?

- Consistency ⇒ Darboux transformations
- Consistency ⇒ Lax (zero curvature) representation [Bobenko, Suris ’02], [Nijhoff ’02]
- Consistency ⇒ hierarchy of commuting equations (in smooth limit).

And:

- Consistency ⇒ Classification of integrable systems (within certain ansatz) [Adler, Bobenko, Suris ’02].

Ansatz: $Q(x, u, y, v) = 0$, $Q$ affine with respect to all variables, square symmetry.
Classification of discrete integrable systems: 2D

\[ \begin{align*}
(Q1) \quad & \alpha(x - v)(u - y) - \beta(x - u)(v - y) + \delta^2 \alpha \beta (\alpha - \beta) = 0, \\
(Q2) \quad & \alpha(x - v)(u - y) - \beta(x - u)(v - y) + \alpha \beta(\alpha - \beta)(x + y + u + v) \\
& - \alpha \beta(\alpha - \beta)(\alpha^2 - \alpha \beta + \beta^2) = 0, \\
(Q3) \quad & \sin(\alpha)(xu + vy) - \sin(\beta)(xv + uy) - \sin(\alpha - \beta)(xy + uv) \\
& + \delta^2 \sin(\alpha - \beta) \sin(\alpha) \sin(\beta) = 0, \\
(Q4) \quad & \sinh(\alpha)(xu + vy) - \sinh(\beta)(xv + uy) - \sinh(\alpha - \beta)(xy + uv) \\
& + \sinh(\alpha - \beta) \sinh(\alpha) \sinh(\beta)(1 + k^2 xyuv) = 0, \\
(H1) \quad & (x - y)(u - v) + \beta - \alpha = 0, \\
(H2) \quad & (x - y)(u - v) + (\beta - \alpha)(x + y + u + v) + \beta^2 - \alpha^2 = 0, \\
(H3) \quad & \alpha(xu + vy) - \beta(xv + uy) + \delta(\alpha^2 - \beta^2) = 0.
\end{align*} \]
Consistency of 3D equations

▶ 3D equation
(for $f : \mathbb{Z}^3 \to \mathcal{X}$)

▶ 4D consistency

Example of a fundamental importance: Q-nets, with all planar quadrilaterals [Doliwa, Santini '97].
Discretization Principles

- **Transformation Group Principle.** Smooth geometric objects and their discretizations belong to the same geometry, i.e. are invariant with respect to the same transformation group (discrete Klein’s Erlangen Program)

- **Consistency Principle.** Discretizations of smooth parametrized geometries can be extended to multidimensional consistent nets (Integrability)
Consistency principle can be imposed for discretization of classical geometries (Möbius, Laguerre, Lie,...):

- transformation groups of various geometries (Möbius, Laguerre, Lie,...) are subgroups of the projective transformation group preserving absolute (distinguished quadric),

- multidimensional Q-nets (projective geometry) can be restricted to an arbitrary quadric [Doliwa ’99].
Discretization of curvature lines 1: circular nets

Martin, de Pont, Sharrock, Nutbourne ['86], Bobenko ['96], Cieslinski, Doliwa, Santini ['97], Konopelchenko, Schief ['98], Akhmetishin, Krivchever, Volvovski ['99], ...

three “coordinate nets” of a discrete orthogonal coordinate system

underlying 3D system on an elementary cube → Miquel theorem
Definition. Neighboring quads touch a common cone of revolution (in particular intersect at the tip of the cone)

Conical net $\Leftrightarrow$ circular Gauss map

Normal shift

Consistency
Curvature lines through spheres

Pencil of touching spheres

Principal directions are invariant with respect to:
- Möbius transformations
- normal shift

Curvature lines belong to Lie geometry.
Objects: oriented spheres (including points and planes)

Contact element: family of oriented spheres in oriented contact at a point; can be described by pairs \((x, P)\) with \(x \in P\)

Lie sphere transformations: map oriented spheres to oriented spheres preserving the oriented contact of sphere pairs (thus map contact elements to contact elements)
Möbius geometry:

- points are distinguished among spheres
- surfaces are considered as consisting of points:
  \[ x : \mathbb{R}^2, \text{ resp. } \mathbb{Z}^2 \rightarrow \{\text{points of } \mathbb{R}^3\} . \]
- transformations: generated by inversions in spheres

Laguerre geometry:

- planes are distinguished among spheres
- surfaces are considered as envelopes of their tangent planes:
  \[ P : \mathbb{R}^2, \text{ resp. } \mathbb{Z}^2 \rightarrow \{\text{planes of } \mathbb{R}^3\} . \]
- transformations: normal shift of all planes (resp. changing signed radii of all spheres) by a fixed \( c \in \mathbb{R} \), ...
Lie geometry:

- no distinguished elements
- surfaces are described by their contact elements:
  \[(x, P) : \mathbb{R}^2, \text{resp. } \mathbb{Z}^2 \rightarrow \{\text{contact elements of } \mathbb{R}^3\}\]

- transformations: generated by Möbius and Laguerre transformations

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“Die Krümmungsstreifen sind in Lieinvarianter Weise dadurch ausgezeichnet, daß zwei konsekutive ihrer Flächenelemente eine Kugel gemeinsam haben”,

or

“Curvature lines are characterized in a Lie-invariant fashion by the property that two their consecutive contact elements share a sphere”.
Blaschke means two infinitesimally close contact elements, but in the Discrete Differential Geometry this can be taken literally:
**Definition.** Discrete curvature line parametrized surface is a map

\[ \mathbb{Z}^2 \rightarrow \{ \text{contact elements of } \mathbb{R}^3 \} \]

such that any two neighboring elements share a sphere. These spheres (associated with the edges of \( \mathbb{Z}^2 \)) are curvature spheres.

**Questions:**

- Relation of this Definition to circular and conical nets?
- Integrability?
Spheres are represented as points of $\mathbb{P}(\mathbb{R}^{4,2})$. Basis of $\mathbb{R}^{4,2}$:

\[ e_1, \ldots, e_6, \quad \langle e_i, e_j \rangle = \begin{cases} 
1, & i = j \in \{1, 2, 3, 4\} \\
-1, & i = j \in \{5, 6\} \\
0, & i \neq j
\end{cases}, \]

\[ e_0 = \frac{1}{2}(e_5 - e_4), \quad e_\infty = \frac{1}{2}(e_5 + e_4), \]

so that

\[ \langle e_0, e_0 \rangle = \langle e_\infty, e_\infty \rangle = 0, \quad \langle e_0, e_\infty \rangle = -\frac{1}{2}. \]
Lie model of Lie geometry

- Oriented sphere with center $c \in \mathbb{R}^3$ and signed radius $r \in \mathbb{R}$:
  $$\hat{s} = c + e_0 + (|c|^2 - r^2)e_\infty + re_6.$$ 

- Oriented plane $\langle v, x \rangle = d$ with $v \in S^2$ and $d \in \mathbb{R}$:
  $$\hat{p} = v + 0 \cdot e_0 + 2de_\infty + e_6.$$ 

- Point $x \in \mathbb{R}^3$:
  $$\hat{x} = x + e_0 + |x|^2e_\infty + 0 \cdot e_6.$$ 

- Infinity $\infty$:
  $$\hat{\infty} = e_\infty.$$ 

All these points lie in the Lie quadric $\mathbb{P}(\mathbb{L})$, where

$$\mathbb{L} = \{ \xi \in \mathbb{R}^{4,2} : \langle \xi, \xi \rangle = 0 \}.$$ 

Lie sphere transformations are projective transformations of $\mathbb{P}(\mathbb{R}^{4,2})$ which preserve the Lie quadric $\mathbb{P}(\mathbb{L})$. 

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Further,

- Contact element \((x, P)\): \(\text{span}(\hat{x}, \hat{p}) = \ell \subset \mathbb{P}(\mathbb{L})\), an *isotropic* line.

Therefore, our definition turns into:

**Definition.** Discrete curvature line parametrized surface is a discrete congruence of isotropic lines

\[ \ell: \mathbb{Z}^2 \to \{\text{isotropic lines in } \mathbb{P}(\mathbb{L})\} \]

such that any two neighboring lines intersect.
Discrete line congruences

Thus, we are in the framework of *discrete line congruences* [Doliwa, Santini, Mañas ’00], i.e., maps

\[ \ell : \mathbb{Z}^2 \rightarrow \{ \text{lines in } \mathbb{RP}^N \} \]

such that any two neighboring lines intersect.

Features:

- Multidimensionally consistent (integrable).
- Line congruences can be restricted to (Lie) quadric.
- Focal surfaces of discrete line congruences (consisting of intersection points of neighboring lines) are Q-nets.
Curvature line net. Projective model

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MÖBIUS

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Curvature line net. Euclidean model
Circular and conical nets. Relation

[Bobenko, Suris ’06], [Pottmann ’06]

► Given a conical net $P$ there exists a two-parameter family of circular nets $x$ such that $(x, P)$ is curvature line parametrized.

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