Non-Gaussianity in the CMB

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Plan of the talk

What is (primordial) non-Gaussianity and why it is important?

Primordial non-Gaussianity from Inflation and alternative scenarios

What the CMB is telling us about NG?

CMB anisotropies at second-order

Conclusions
A basic formula

A phenomenological way of parametrizing the possible presence of non-Gaussianity is the formula

\[ \Phi = \Phi_L + f_{NL} \left( \Phi_L^2 - \langle \Phi_L^2 \rangle \right) + \ldots \]

where \( \Phi \) is the large-scale (Bardeen) gravitational potential, \( \Phi_L \) its linear Gaussian part and \( f_{NL} \) is the so called non-linearity parameter.

This Non-Gaussian feature is then transferred to the large-scale CMB anisotropies through the Sachs-Wolfe effect

\[ \frac{\Delta T}{T} = \frac{1}{3} \Phi \]
Gravitational potential perturbation in the perturbed Friedmann-Robertson-Walker metric

\[ ds^2 = a^2(\tau)[-(1 + 2\phi) d\tau^2 - \omega_i d\tau dx^i + ((1 - 2\psi)\delta_{ij} + h_{ij})dx^i dx^j] \]
What if the perturbations are (not) Gaussian?

If the fluctuations $\delta(x)$ are Gaussian distributed then their statistical properties are completely characterized by the two-point correlation function, $\langle \delta(x_1)\delta(x_2) \rangle$ or its Fourier transform, the power-spectrum.

Thus a non-vanishing three point function, or its Fourier transform, the bispectrum

$$\langle \delta(x_1)\delta(x_2)\delta(x_3) \rangle \neq 0$$

is an indicator of non-Gaussianity

e.g. : $\Phi(k)$-bispectrum for $f_{\text{NL}}=$cost.

$$\langle \Phi(k_1)\Phi(k_2)\Phi(k_3) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \times [2f_{\text{NL}}P_\Phi(k_1)P_\Phi(k_2) + \text{cycl}]$$

where $\langle \Phi_L(k)\Phi_L(k') \rangle = (2\pi)^3 P_\Phi(k_1) \delta^{(3)}(k + k')$ is the linear power spectrum.
Non-Gaussian maps

Planck resolution 5'
$l_{\text{max}} = 3000$

Liguori, Matarrese, Moscardini, 2003
Why non-Gaussianity is important?
• In general different models for the generation of the primordial perturbations predict:

- some departure from scale invariance
- some amount of tensor perturbations
- some amount of non-Gaussianity

\[ P_\zeta = A \, k^{n-1} \]
\[ P_T \propto H_*^2 \]
\[ f_{\text{NL}} \]

• In fact most models predict negligible amount of tensor perturbations and negligible departure from scale-invariance

Even a precise measurement of the spectral index might not allow to discriminate among competing scenarios; if no G.W. is detected (e.g. via CMB polarization) no efficient discrimination

However these models predict different NG
Going beyond linear order

- Collection of independent harmonic oscillators (no mode-mode coupling)

- NG requires more than linear theory
  (Various techniques available: second-order perturbation theory, fully non-linear approach a la Salopek/Bond 1991 (or the so called $\delta N$ formalism), quantum field theory)
Primordial non-Gaussianity from Inflation and alternative scenarios
Scenarios to generate the primordial density perturbations

♦ **Standard single field models of inflation:** perturbations due to the fluctuations of the inflaton field itself leading to primordial adiabatic perturbations

♦ **Curvaton scenario:** initial isocurvature perturbations associated to quantum fluctuations of a light scalar field different from the inflaton -- the *curvaton* $\sigma$ -- with negligible energy density during inflation. The curvaton isocurvature perturbations are transformed into adiabatic when the curvaton decays into radiation after the end of inflation

  K. Enqvist & M.S. Sloth, 2001
  T. Moroi & T. Takahashi, 2001
  D. Lyth & D. Wands, 2001

♦ **Modulated (Inhomogeneous) reheating:** fluctuations in the decay rate of the inflaton field $\delta \Gamma_\phi$ due to some other light field. Adiabatic perturbations in the final reheating temperature in different regions of the universe.

  G. Dvali, A. Gruzinov, M. Zaldarriaga, 2003
  Kofman, 2003
  (see also T. Hamazaki and H. Kodama, 1996)
…. and some alternative models of inflation

♦ Single field models of inflation with non-canonical kinetic term
  $L = P(\phi, X), \ X = (\partial \phi)^2$, like
  **DBI Inflation:** based on the Dirac-Born-Infield action
  Silverstein and Tong (2004)

♦ **Ghost Inflation:** ghost scalar field, derivatively coupled,
  with $\phi = \text{const.}$
  Arkani-Hamed et al. (2004)

♦ **NG from preheating** (Enqvist et al. (2004+), Barnaby and Cline 2006),
  **multiple models of inflation** (N.B. et al. 2002; Van-Tent et al. 2004; Vernizz Wands)
  **single field inflation with a feature in the inflaton potential**
  (Wang/Kamionkowski 2000)
  and many more (see tables)
Primordial non-Gaussianity

It is useful to characterize it in terms of the non-linearities present in the curvature perturbation $\zeta$ (in the uniform energy density slice)

$$g_{ij} = a^2(t) e^{-2\zeta} \gamma_{ij} = a^2(t, x) \gamma_{ij}$$

Think of $\zeta \approx \frac{\delta \rho}{\rho}$

Note that

a) this is a **fully non-linear definition** for the curvature perturbation (Salopek/Bond ’91; Kolb et al. ‘05; Lyth et al. ‘05) and at linear order it reduces to the usual definition

$$\zeta^{(1)} = -\Psi - H \frac{\delta \rho}{\dot{\rho}}, \quad \text{where} \quad \zeta = \zeta^{(1)} + \frac{1}{2} \zeta^{(2)} + \ldots$$
b) At second-order the gauge-invariant curvature perturbation is (N.B., Matarrese, Riotto 2002; Malik & Wands 2003)

\[
\zeta = \zeta^{(1)} + \frac{1}{2} \zeta^{(2)}
\]

\[
\zeta^{(2)} = -\psi^{(2)} - \mathcal{H} \frac{\delta^{(2)} \rho}{\rho'} + 2\mathcal{H} \frac{\delta^{(1)} \rho'}{\rho'} \frac{\delta^{(1)} \rho}{\rho'} + 2 \frac{\delta^{(1)} \rho}{\rho'} \left( \psi^{(1)} + 2\mathcal{H} \psi^{(1)} \right) - \left( \frac{\delta^{(1)} \rho}{\rho'} \right)^2 \left( \frac{\mathcal{H} \rho''}{\rho'} - \mathcal{H}' - 2\mathcal{H}^2 \right)
\]

See also Lyth & Wands, 2003; Rigopoulos & Shellard 2003

From the energy-continuity equation at second-order on superhorizon scales

\[
\zeta^{(2)'} = -\frac{\mathcal{H}}{\rho + P} \delta^{(2)} P_{\text{nad}}
\]

\[= -\frac{2}{\rho + P} \left[ \delta^{(1)} P_{\text{nad}} - 2(\rho + P)\zeta^{(1)} \right] \zeta^{(1)'}
\]

and

\[
\zeta^{(1)'} = -\frac{\mathcal{H}}{\rho + P} \delta^{(1)} P_{\text{nad}}
\]

where

\[
\delta^{(1)} P_{\text{nad}} = \delta^{(1)} P - c_s^2 \delta^{(1)} \rho
\]

is the non-adiabatic pressure perturbation

Malik & Wands 2003; see also Lyth & Wands, 2003; N.B., Matarrese, Riotto 2002; Rigopoulos & Shellard 2003
The key point is that $\zeta$ remains \textit{constant} on superhorizon scales after it has been generated and possible isocurvature (entropy) perturbations are no longer present.

$\zeta^{(2)}$ \textit{provides all the information about the primordial level of the NG generated during inflation, as in the standard scenario, or after inflation, as in the curvaton scenario.}

The value of $\zeta^{(2)}$ is different for different scenarios

$$\zeta^{(2)} = 2a_{NL} \left(\zeta^{(1)}\right)^2$$

It is not directly connected to the measurable quantity, the CMB anisotropy.
Case 1:

Standard single-field models of inflation

Acquaviva, N.B, Matarrese and Riotto (2002); Maldacena (2002); Lidsey and Seery (2004)
Primordial Non-Gaussianity in standard single field models of Inflation

- Non-Gaussianity generated during inflation:
  
  Accounting for the inflaton self-interactions and metric fluctuations at second-order in the perturbations brings

  \[ a_{NL} = -\frac{1}{4} (n - 1) + n_T f(k_1, k_2) \]

  Acquaviva, N.B., Matarrese, Riotto (2002); Maldacena (2002)

Primordial non-Gaussianity for single-field models of slow-roll inflation is tiny \( \sim O(\varepsilon, \eta) \)

\[ |n - 1| = |2\eta - 6\varepsilon| \ll 1; \quad n_T = 2\varepsilon \quad \varepsilon \equiv \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2, \quad \eta \equiv \frac{1}{8\pi G} \frac{V''}{V} \]

- Notice that in the squeezed limit \( k_1 \ll k_2 \approx k_3 \) \( f(k_1, k_2) \) goes to zero (Maldacena 2001)
Case 2: Local models

$f_{NL} = \text{constant}$

NG local in real space: non-linearities develop outside the horizon
Radiation density can be perturbed by other scalar fields during or after inflation

- Curvaton decay after inflation
  Mollerach (1990)
  Linde & Mukhanov (1997)
  Enqvist & Sloth; Lyth and Wands, Moroi and Takahashi (2001)

- Inhomogeneous/modulated reheating or preheating
  Dvali, Gruzinov & Zaldarriaga; Kofman (2003); Enqvist et al. (2003); Kolb, Riotto and Vallinotto (2004)
NG from the curvaton decay

\[ \rho_\sigma = \frac{m^2}{2} \sigma^2 = \frac{m^2}{2} \left( \sigma_0^2 + 2\sigma_0 \delta \sigma + \delta \sigma^2 \right) \]

\[ \zeta_{\text{late times}} = \sum_i \frac{\dot{Y}_i}{Y} \zeta_i \approx (1 - r) \zeta_y + r \zeta_\sigma \]

\[ \zeta_\sigma \approx \left( \frac{\delta \sigma}{\sigma} \right) + \left( \frac{\delta \sigma}{\sigma} \right)^2 \]

\[ a_{NL} = \frac{3}{4r} - \frac{r}{2} + 1 \quad \left( = \frac{3}{4r} \quad \text{if} \quad r \ll 1 \right) \]

\[ r = \left( \frac{\rho_\sigma}{\rho} \right)_{\text{decay}} \]

\[ \zeta_1 \approx r \left( \frac{\delta \sigma}{\sigma} \right) \]

Lyth, Ungarelli and Wands (2002); N.B, Matarrese and Riotto (2004); Lyth, Rodríguez, Malik (2005+)
Non-Gaussianity shape

The bispectrum peaks for **squeezed configuration**: \(k_1 \ll k_2 \sim k_3\)

\[
\langle \Phi(k_1) \Phi(k_2) \Phi(k_3) \rangle = (2\pi)^3 \delta^3(k_1 + k_2 + k_3) F(k_1, k_2, k_3)
\]

\[
F^{\text{loc}}(k_1, k_2, k_3) = f_{NL}^{\text{loc}} \left( \frac{\Delta^2_{\phi}}{k_1^{4-n_s} k_2^{4-n_s}} \right) + \text{(symm.)}
\]

\[
\zeta^{(1)} = -\frac{5}{3} \Phi^{(1)}
\]
Most signal expected in the squeezed configuration

$x_3 \equiv k_3/k_1$ and $x_2 \equiv k_2/k_1$

Babich et al. (2004)
Case 3: equilateral models

NG non-local in real space: non-linearities generated by operators with gradients

Such as Ghost inflation, N. Arkani-Hamed et al., 2003
DBI inflation, E. Silverstein and D. Tong, 2003
‘Equilateral’ models:

- Models where NG comes from higher derivative interactions such as

$$\frac{1}{8\Lambda^4} (\nabla \phi)^2 (\nabla \phi)^2$$

- The bispectrum peaks for *equilateral configuration: $k_1 = k_2 = k_3$* since in this case the correlation is among modes that exit the horizon at the same time

$$F_{eq}(k_1, k_2, k_3) = f_{NL}^{eq} \left[ -3 \frac{\Delta^2 \Phi}{k_1^{4-n_s} k_2^{4-n_s}} 
-2 \frac{\Delta^2 \Phi}{k_1^{2(4-n_s)/3} k_2^{2(4-n_s)/3} k_3^{2(4-n_s)/3}} 
+6 \frac{\Delta^2 \Phi}{k_1^{4-n_s} k_2^{4-n_s} k_3^{4-n_s}} \right] + \text{(symm.)}$$
In this case an equilateral non-liearity parameter $f_{NL}^{\text{equil}}$ is used to characterize the amplitude of the bispectra. It is defined so that

$$F(k,k,k) = 6 f_{NL}^{\text{equil}} P_\Phi(k)$$

- **Ghost inflation**, N. Arkani-Hamed et al., 2003

$$S = \int d^4 x \left( \frac{1}{2} \dot{\pi}^2 - \frac{\alpha^2}{2M^2} (\nabla^2 \pi)^2 - \frac{\beta}{2M^2} \dot{\pi} (\nabla \pi)^2 + \cdots \right)$$

$$f_{NL}^{\text{equil}} = -0.85 \alpha \beta^{-8/5}$$

- **DBI inflation**, E. Silverstein and D. Tong, 2003

$$L_{\text{eff}} = -\frac{1}{g_s} \left( f(\varphi)^{-1} \sqrt{1 + f(\varphi) g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi)} \right)$$

$$f_{NL}^{\text{equil}} = -0.2 \gamma^2$$

with $\gamma > 5$
Higher Deriv.

$x_3 \equiv k_3/k_1$ and $x_2 \equiv k_2/k_1$

Babich et al. (2004)
“...the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible ...”
(Sachs & Wolfe 1967)
From primordial NG to CMB anisotropies

- Evaluate the non-Gaussianity generated during inflation (or immediately after as in the curvaton scenario): primordial “input”

\[ \zeta^{(2)} = 2a_{NL} (\zeta^{(1)})^2 \]

- After inflation, follow the evolution of the non-linearities on large scales by matching the conserved quantity \( \zeta^{(2)} \) to the initial input, plus Einstein equations at second-order: non-linearities in the gravitational potentials

- The non-linearities in the gravitational potentials are then transferred to the observable \( \Delta T/T \) fluctuations: additional non-linearities are acquired

Post-inflationary nonlinear gravitational dynamics is common to all scenarios
An example: dark matter on large scales

(0-0)-Einstein equations

\[ \phi^{(2)} = -\frac{1}{2} \delta^{(2)} \frac{\rho}{\rho} + 4 \left( \psi^{(1)} \right)^2 \]

From the trace of (i-j)-Einstein equations

\[ \psi^{(2)} - \phi^{(2)} = -4 \left( \psi^{(1)} \right)^2 - \nabla^2 \left( 2 \partial^i \psi^{(1)} \partial_i \psi^{(1)} + 3(1 + w)H^2 v_{(1)i} v_{(1)i} \right) \]

\[ + 3 \nabla^4 \partial_i \partial_j \left( 2 \partial^i \psi^{(1)} \partial^j \psi^{(1)} + 3(1 + w)H^2 v_{(1)i} v_{(1)j} \right) \]
Large-angular scales

3 main effects:  a) Sachs-Wolfe effect  
b) Early and late Integrated Sachs-Wolfe  
c) Tensor contributions

In this limit we do not really need to solve the Boltzmann equations. The relevant effects comes from gravity (at the last scattering surface and from the last scattering surface to the observer).

For the Early, Late Integrated Sachs-Wolfe effects, and for the tensor contributions see expressions available in  
B. N., Matarrese S., Riotto A., 2005, JCAP 0605
Fully non-Linear Sachs Wolfe effect

Non-linear perturbations

\[ ds^2 = -e^{2\Phi} \, dt^2 + a^2(t) e^{-2\Psi} \delta_{ij} \, dx^i \, dx^j \]

\[ T_{\Omega} = \frac{\Omega_{\Omega}}{\Omega_{\epsilon}} \]  

\[ \delta_{np} T \left( \begin{array}{c} \frac{T}{T} \end{array} \right) = e^{\Phi/3} - 1 \]

Non-perturbative extension of the Sachs-Wolfe effect

\[ \frac{\delta^{(1)} T}{T} = \frac{1}{3} \Phi^{(1)} \]

\[ \frac{\delta^{(2)} T}{T} = \frac{1}{3} \Phi^{(2)} + \frac{1}{9} (\Phi^{(1)})^2 \]

see expression in N.B., Komatsu, Matarrese, Riotto (2004)
Second-order Sachs-Wolfe effect

\[ \frac{\Delta T}{T} = \frac{1}{3} \phi^{(1)}_{\varepsilon} + \frac{1}{18} \left( \phi^{(1)}_{\varepsilon} \right)^2 - \frac{K}{10} - \frac{5}{9} a_{NL} \left( \phi^{(1)}_{\varepsilon} \right)^2 \]


Post-inflation non-linear evolution of gravity: order unity NG

\[ K = 10^{-4} \partial_i \partial_j \left( \partial^i \phi^{(1)}_{\varepsilon} \partial^j \phi^{(1)}_{\varepsilon} \right) - \frac{10}{3} \nabla^2 \left( \partial^i \phi^{(1)}_{\varepsilon} \partial^i \phi^{(1)}_{\varepsilon} \right) \]

Initial conditions set during or after inflation

\[ \zeta^{(2)} = 2a_{NL} \left( \zeta^{(1)} \right)^2 \]

\[ a_{NL} = \begin{cases} O(\varepsilon, \eta) & \text{standard scenario} \\ \frac{3}{4r} - \frac{r}{2} - 1 & \text{curvaton scenario} \end{cases} \]

Such an expression allows to single out the non-linearities of large-scale CMB anisotropies at last scattering. Evolving them inside the horizon requires the full radiation transfer function at second-order in the perturbations.
Extracting the non-linearity parameter $f_{\text{NL}}$

$$\frac{\Delta T}{T} = \frac{1}{3} \left( \phi_{(1)} + f_{\text{NL}} \phi_{(1)}^2 \right)$$

$$f_{\text{NL}}(k_1, k_2) = \frac{5}{3} a_{\text{NL}} - \frac{1}{6} + K(k_1, k_2)$$

Connection between theory and observations

This is the proper quantity measurable by CMB experiments, via the phenomenological analysis by Komatsu and Spergel (2001)

$$K(k_1, k_2) = 3 \frac{(k_1 \cdot k)(k_2 \cdot k)}{k^4} - \frac{(k_1 \cdot k_2)}{k^2}$$

$k = |k_1 + k_2|$
Inflation models and $f_{NL}$


<table>
<thead>
<tr>
<th><strong>model</strong></th>
<th>$f_{NL}(k_1,k_2)$</th>
<th><strong>comments</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard inflation</td>
<td>$O(\varepsilon,\eta) - 1/6 + K(k_1,k_2)$</td>
<td>K universal, goes to zero in squeezed limit</td>
</tr>
<tr>
<td>curvaton</td>
<td>$5/4 \Gamma -11/6 - 5\Gamma/6 + K(k_1,k_2)$</td>
<td>$r \sim (\rho_\sigma/\rho)_{\text{decay}}$</td>
</tr>
<tr>
<td>Modulated reheating</td>
<td>$-17/12 - I + K(k_1,k_2)$</td>
<td>I = $-5/2 + 5\Gamma / (12 \alpha \Gamma_i)$ I = 0 (minimal case)</td>
</tr>
<tr>
<td>multi-field inflation</td>
<td>may be large ?</td>
<td>order of magnitude estimate of the absolute value</td>
</tr>
</tbody>
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“unconventional” Inflation set-ups

<table>
<thead>
<tr>
<th><strong>model</strong></th>
<th><strong>comments</strong></th>
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<tbody>
<tr>
<td>Warm inflation</td>
<td>typically $10^{-1}$; second-order corrections not included;</td>
</tr>
<tr>
<td>ghost inflation</td>
<td>$-85 \beta \alpha^{-3/5}$; post-inflation corrections not included; <em>equilateral configuration</em></td>
</tr>
<tr>
<td>DBI</td>
<td>$-0.2 \gamma^2$; post-inflation corrections not included; <em>equilateral configuration</em></td>
</tr>
<tr>
<td>model</td>
<td>$f_{NL}(k_1,k_2)$</td>
</tr>
<tr>
<td>--------------------------------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>Preheating scenarios</td>
<td>e.g. $(M_{Pl}/\varphi_0) e^{Nq/2} \sim 50$</td>
</tr>
<tr>
<td>Inhomogeneous preheating And inhom. hybrid inflation</td>
<td>e.g. $(5/6) \lambda_\varphi (M_{Pl}/m_\chi)$</td>
</tr>
<tr>
<td>Generalized single field inflation (including k-and brane inflation)</td>
<td>$-(35/108) (1/c_s^2-1) + (5/81) (1/c_s^2-1-2\lambda/\Sigma)$</td>
</tr>
<tr>
<td>Warm Inflation (II)</td>
<td>$-15L(r) &lt; f_{NL} &lt; L(r)$</td>
</tr>
<tr>
<td>Generalized slow-roll inflation (higher-order kinetic terms)</td>
<td>$f_{NL} &gt;&gt; +1$</td>
</tr>
<tr>
<td>Excited initial states+derivative interactions</td>
<td>$\sim (6.3 \times 10^{-4} M_{Pl}/M) \sim (1-100)$</td>
</tr>
<tr>
<td>Ekpyrotric models</td>
<td>$-50 &lt; f_{NL} &lt; 200$</td>
</tr>
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</table>
“As yet, all the tests carried out are consistent with initial perturbations, which are Gaussian, though the strength of this statement is unclear given the lack of so far well-motivated and calculable non-Gaussian models”

A. Liddle & D. Lyth in “Cosmological Inflation and Large-Scale Structure”
On large scales:

NG = NG from gravity (universal) + NG primordial

Gravity itself is non-linear
Non-linear (second-order) GR perturbations in the standard cosmological model introduce some order unity NG:

\[ \sqrt{\text{we would be in trouble if NG turned out to be very close to zero}} \]

\[ \sqrt{\text{such non-linearities have a non-trivial form: their computation } \equiv \text{ core of the (large-scale) radiation transfer function at second-order}} \]
What the **CMB** is telling about **NG**?
What is it needed?

- We need to know the **predicted form of statistical tools as a function of model parameters** to fit the data.
- There are three statistical tools for which the analytical predictions are known:

  - The angular bispectrum
    Komatsu & Spergel (2001); Babich & Zaldarriaga (2004); Creminelli et al. (2006)

  - Minkowski functionals
    Hikage, Komatsu & Matsubara (2006)

  - The angular trispectrum (harmonic transform of the 4-point correlation function)
    Okamoto & Hu (2002); Kogo & Komatsu (2006)
Angular Bispectrum, $B_{lmn}$

$$a_{lm} = \int d^2 n \frac{\Delta T(n)}{T(n)} Y^*_l m(n)$$

$$a_{lm} = 4\pi (-i)^l \int \frac{d^3 k}{(2\pi)^3} \Phi(k) g_{Tl}(k) Y^*_l m(\hat{k}),$$

$$\Phi(x) = \Phi_L(x) + f_{NL}(\Phi^2_L(x) - \langle \Phi^2_L(x) \rangle)$$

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} \equiv \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle,$$

$$B_{l_1 l_2 l_3} = \sum_{all \, m} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{l_1 l_2 l_3}^{m_1 m_2 m_3},$$

$$B_{l_1 l_2 l_3} = \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} b_{l_1 l_2 l_3}$$
Most signal coming from the decoupling epoch

\[ b_{lll}^{\text{prim}} = 2 \int_0^\infty r^2 dr \left[ b_{ll1}^L(r) b_{l2}^L(r) b_{l3}^L(r) + b_{l1}^L(r) b_{l2}^L(r) b_{l3}^L(r) \right. \\
\left. + b_{l1}^{NL}(r) b_{l2}^L(r) b_{l3}^L(r) \right]. \]

\[ b_i^L(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk P_\Phi(k) g_{T1}(k) j_i(kr), \]

\[ b_i^{NL}(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 dk f_{NL} g_{T1}(k) j_i(kr). \]

\[ b_{lll}^{\text{prim}} \sim l^{-4} \left[ 2r_*^2 \Delta r_* \left( l^2 b_i^L \right)^2 b_i^{NL} \times 3 \right] \sim l^{-4} \times 2 \times 10^{-17} f_{NL}. \]

Komatsu & Spergel (2001)
- Primordial
  - Inflation
  - Second-order PT
- Secondaries (dominate at l>2000)
  - Gravitational lensing
  - Sunyaev-Zel’dovich effect
- Nuisance
  - Radio point sources
  - Our Galaxy

\[
\left( \frac{S}{N} \right)_{\text{prim}} \sim l \times 10^{-4} f_{NL}.
\]

Babich & Zaldarriaga (04)

<table>
<thead>
<tr>
<th>Experiment</th>
<th>TTT</th>
<th>EEE</th>
<th>TTT,TTE</th>
<th>TTT,TTE,TEE</th>
<th>All</th>
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<tr>
<td>WMAP</td>
<td>13.3</td>
<td>314</td>
<td>11.2</td>
<td>10.9</td>
<td>10.9</td>
</tr>
<tr>
<td>Planck</td>
<td>4.7</td>
<td>8.9</td>
<td>3.4</td>
<td>3.0</td>
<td>2.9</td>
</tr>
<tr>
<td>Ideal</td>
<td>3.5</td>
<td>2.6</td>
<td>2.2</td>
<td>1.8</td>
<td>1.6</td>
</tr>
</tbody>
</table>

Minimum values $f_{NL}$ detectable with signal to noise ratio of one

Komatsu & Spergel (2001)
WMAP limits on $f_{NL}$ (Local models)

From an analysis of the bispectrum of CMB temperature anisotropies

$-58 < f_{NL} < 134$ (95%) (1yr)

$-54 < f_{NL} < 114$ (95%) (3yr)

Komatsu et al. (2003; 2007)

See also Creminelli et al. (2007)

$-36 < f_{NL} < 100$ (95%) (3yr)

Fig. 1.— The non-linear coupling parameter $f_{NL}$ as a function of the maximum multipole $l_{\text{max}}$, measured from the $Q$+$V$+$W$ co-added map using the cross (bispectrum) estimator (Eq. (8)). The best constraint is obtained from $l_{\text{max}} = 365$. The distribution is cumulative, so that the error bars at each $l_{\text{max}}$ are not independent.
NG DETECTED VIA THE BISPECTRUM?

Yadav & Wandelt 2008

FIG. 1: We show the measured value of the non-linear coupling parameter $f_{NL}$ using WMAP 3-year maps, and the corresponding 95% error bars derived from the Gaussian simulations. For this analysis the WMAP Kp0 mask was used. The analysis is done for 4 combinations of the frequency channels: coadded Q+V+W, coadded V+W, V, and W.

$27 < f_{NL} < 137$ (95%) (3yr); using V+W channel, Kp0 mask $l_{\text{max}} = 750$

N.B.: They use the KSW estimator with the linear term, introduced in Creminelli et al. (‘07), correcting for a factor of 2 normalization missing in Creminelli et al., and leading to exploit higher multipoles. The linear term helps in reducing the excess variance due to inhomogeneous noise and thus allow to include more multipoles.

TABLE I: Non-linear coupling parameter $f_{NL}$ using the V+W, Q, and Q+V+W WMAP 3-year raw maps, as a function of maximum multipole used in the analysis $l_{\text{max}}$ and mask Kp12, Kp2, Kp0, and Kp0+ (corresponding $f_{sky}$ is stated in the text and the masks are shown in Fig 2). The last row (750*) shows the mean $f_{NL}$ estimated from Gaussian simulations including the WMAP foreground model. Foreground contamination biases $f_{NL}$ negatively by similar amounts in both the data and the model.
WMAP 5yr CONSTRAINTS

\[-9 < f_{NL} < 111 \ (95\%)\]

- from the template-cleaned V+W channel;
- using a larger mask than Yadav&Wandelt;
- accounting for a bias from unresolved point sources
- for \(l_{\text{max}} = 500\)

<table>
<thead>
<tr>
<th>Band</th>
<th>Mask</th>
<th>(l_{\text{max}})</th>
<th>(f_{NL}^{\text{local}})</th>
<th>(\Delta f_{NL}^{\text{local}})</th>
<th>(b_{\text{src}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>V+W</td>
<td>KQ85</td>
<td>400</td>
<td>50 ± 29</td>
<td>1 ± 2</td>
<td>0.26 ± 1.5</td>
</tr>
<tr>
<td>V+W</td>
<td>KQ85</td>
<td>500</td>
<td>61 ± 26</td>
<td>2.5 ± 1.5</td>
<td>0.05 ± 0.50</td>
</tr>
<tr>
<td>V+W</td>
<td>KQ85</td>
<td>600</td>
<td>68 ± 31</td>
<td>3 ± 2</td>
<td>0.53 ± 0.28</td>
</tr>
<tr>
<td>V+W</td>
<td>KQ85</td>
<td>700</td>
<td>67 ± 31</td>
<td>3.5 ± 2</td>
<td>0.34 ± 0.20</td>
</tr>
<tr>
<td>V+W</td>
<td>Kp0</td>
<td>500</td>
<td>61 ± 26</td>
<td>2.5 ± 1.5</td>
<td>0.34 ± 0.20</td>
</tr>
<tr>
<td>V+W</td>
<td>KQ75p1</td>
<td>500</td>
<td>53 ± 28</td>
<td>4 ± 2</td>
<td>0.34 ± 0.20</td>
</tr>
<tr>
<td>V+W</td>
<td>KQ75</td>
<td>400</td>
<td>47 ± 32</td>
<td>3 ± 2</td>
<td>0.34 ± 0.20</td>
</tr>
<tr>
<td>V+W</td>
<td>KQ75</td>
<td>500</td>
<td>55 ± 30</td>
<td>4 ± 2</td>
<td>0.15 ± 0.51</td>
</tr>
<tr>
<td>V+W</td>
<td>KQ75</td>
<td>600</td>
<td>61 ± 36</td>
<td>4 ± 2</td>
<td>0.53 ± 0.30</td>
</tr>
<tr>
<td>V+W</td>
<td>KQ75</td>
<td>700</td>
<td>58 ± 36</td>
<td>5 ± 2</td>
<td>0.38 ± 0.21</td>
</tr>
</tbody>
</table>

\(\text{\footnotesize{\textsuperscript{\textit{a}}}}\text{This mask replaces the point-source mask in KQ75 with the one that does not mask the sources identified in the WMAP K-band data.}}\)

Komatsu et al. 2008
WMAP limits on Equilateral models

\(-151 < f_{\text{NL}}^{\text{equil}} < 253\) (95%) for \(l_{\text{max}} = 700\)

See also Creminelli et al. (2007)

\(-256 < f_{\text{NL}} < 332\) (95%) (3yr) for \(l_{\text{max}} = 475\)

### TABLE 7

**Clean-map estimates and the corresponding 68% intervals of the equilateral form of primordial non-Gaussianity, \(f_{\text{NL}}^{\text{equil}}\), and Monte-Carlo estimates of bias due to point sources, \(\Delta f_{\text{NL}}^{\text{equil}}\)**

<table>
<thead>
<tr>
<th>Band</th>
<th>Mask</th>
<th>(l_{\text{max}})</th>
<th>(f_{\text{NL}}^{\text{equil}})</th>
<th>(\Delta f_{\text{NL}}^{\text{equil}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>V+W</td>
<td>(KQ75)</td>
<td>400</td>
<td>77 ± 146</td>
<td>9 ± 7</td>
</tr>
<tr>
<td>V+W</td>
<td>(KQ75)</td>
<td>500</td>
<td>78 ± 125</td>
<td>14 ± 6</td>
</tr>
<tr>
<td>V+W</td>
<td>(KQ75)</td>
<td>600</td>
<td>71 ± 108</td>
<td>27 ± 5</td>
</tr>
<tr>
<td>V+W</td>
<td>(KQ75)</td>
<td>700</td>
<td>73 ± 101</td>
<td>22 ± 4</td>
</tr>
</tbody>
</table>

Komatsu et al. 2008
SOME WARNINGS

The main concern for detecting primordial NG is the contamination from foregrounds emission.

To be sure of a detection of Primordial NG wait for a confirmation from other statistical estimators.

Check for any kind of systematics: different statistical tools are sensitive to different systematics.
CMB anisotropies at second-order
Motivations

Planck and future experiments may be sensitive to non-Gaussianity (NG) at the level of second- or higher order perturbation theory.

Crucial to provide accurate theoretical predictions for the statistics of the CMB anisotropies.

Fundamental: a) what is the relation between primordial NG and non-linearities in the CMB anisotropies on different angular scales?  
  b) what is the non-linear dynamics of the baryon-photon fluid and CDM+ gravity near recombination?
First step: calculation of the full 2-nd order radiation transfer function on large scales (low-l), which includes:

- **NG initial conditions**
- non-linear evolution of gravitational potentials on large scales
- second-order SW effect (and second-order temperature fluctuations on last-scattering surface)
- second-order ISW effect, both early and late
- ISW from second-order tensor modes (unavoidably arising from non-linear evolution of scalar modes), also accounting for second-order tensor modes produced during inflation

Second step: solve Boltzmann equation at 2-nd for the photon, baryon and CDM fluids which allows to follow CMB anisotropies at 2-nd order at all scales; this includes both scattering and gravitational secondaries, like:

- Thermal and Kinetic Sunyaev-Zel'dovich effect
- Ostriker-Vishniac effect
- Inhomogeneous reionization
- Further gravitational terms, including gravitational lensing (both by scalar and tensor modes), Rees-Sciama effect, Shapiro time-delay, effects from second-order vector (i.e. rotational) modes, etc. ...

(N.B, Matarrese & Riotto 2005+)
WHAT ABOUT SMALLER SCALES?

Aim: - have a full radiation transfer function at second-order for all scales

- in particular:
  compute the CMB anisotropies generated by the non-linear dynamics of the photon-baryon fluid for subhorizon modes at recombination (acoustic oscillations at second-order)
  see N.B., Matarrese, Riotto JCAP 2006, 2007

Remember: crucial to extract information from the bispectrum are the scales of acoustic peaks according to the phenomenological analysis of Komatsu and Spergel (2001)
2nd-order CMB Anisotropies on all scales

Apart from gravity account also for:

a) Compton scattering of photons off electrons
b) baryon velocity terms $\mathbf{v}$

Boltzmann equation for photons

$$\frac{df}{d\eta} = \alpha C[f]$$

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i} \frac{dx_i}{d\eta} + \frac{\partial f}{\partial p} \frac{dp}{d\eta} + \frac{\partial f}{\partial n^i} \frac{dn_i}{d\eta}$$

Gravity effects

+ Boltzmann equations for baryons and CDM+ Einstein equations
Metric perturbations

Poisson gauge

\[
d s^2 = a^2(\eta) \left[ -e^{2\Phi} d\eta^2 + 2\omega_i dx^i d\eta + (e^{-2\Psi} \delta_{ij} + \chi_{ij}) dx^i dx^j \right]
\]

\[\Phi = \Phi^{(1)} + \Phi^{(2)}, \quad \psi = \psi^{(1)} + \frac{1}{2} \psi^{(2)}\]

Examples: using the geodesic equation for the photons

\[\frac{1}{p} \frac{dp}{d\eta} = -\mathcal{H} + \Psi' - \Phi_{,i} n^i e^{\Phi+\Psi} - \omega_i' n^i - \frac{1}{2} \chi_{ij} n^i n^j\]

Redshift of the photon  
**Sachs-Wolfe and ISW effects**

\[\frac{dn^i}{d\eta} = (\Phi_{,k} + \Psi_{,k}) n^k n^i + \Phi_{,i} - \Psi_{,i}\]

Direction of the photons changes due to gravitational potentials  
**lensing effect** (it arises at second-order)

PS: Here the photon momentum is  \( p = p n^i \) with  \( p^2 = g_{ij} P^i P^j \) and  
\( P^\mu = dx^\mu(\lambda) / d\lambda \) quadri-momentum vector
Photon Boltzmann equation (I)

Expand the distribution function in a linear and second-order parts around the zero-order Bose-Einstein value

\[ f(x^i, p, n^i, \eta) = f^{(0)}(p, \eta) + f^{(1)}(x^i, p, n^i, \eta) + \frac{1}{2}f^{(2)}(x^i, p, n^i, \eta) \]

\[ f^{(0)}(p, \eta) = 2 \frac{1}{\exp \left\{ \frac{p}{T(\eta)} \right\} - 1} \]

Left-hand side \( df / d\eta \)

\[
\frac{df}{d\eta} = \frac{df^{(1)}}{d\eta} + \frac{1}{2} \frac{df^{(2)}}{d\eta} - p \frac{\partial f^{(0)}}{\partial p} \frac{d}{d\eta} \left( \Phi^{(1)} + \frac{1}{2} \Phi^{(2)} \right) + p \frac{\partial f^{(0)}}{\partial p} \frac{\partial}{\partial \eta} \left( \Phi^{(1)} + \Psi^{(1)} + \frac{1}{2} \Phi^{(2)} + \frac{1}{2} \Psi^{(2)} \right) \\
- p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \omega_i}{\partial \eta} n^i - \frac{1}{2} p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \chi_{ij}}{\partial \eta} n^i n^j ,
\]
The collision term $C[f]$

- Up to *recombination* photons are tightly coupled to electrons via Compton scatterings $e(q) \gamma(p) \leftrightarrow e(q') \gamma(p')$. The collision term governs *small scale anisotropies and spectral distortions*.

- Important also for *secondary scatterings*: reionization, kinetic and thermal Sunyaev-Zeldovich and Ostriker-Vishniac effects.

- **Crucial points to compute the second-order collision term:**
  1) Little energy $\delta\varepsilon/T$ is transferred $\rightarrow$ expand in the perturbations *and* in $\delta\varepsilon/T \approx q/m_e$. At linear order the Boltzmann equations depend only on $n_i$, *at second-order there is energy exchange* ($p$ dependency) *and thus spectral distortions*.
  
  2) Take into account *second-order baryon velocity* $v^{(2)}$.
  
  3) Take into account *vortical components* of (first-order $\times$ first-order).
The 2nd-order brightness equation

Integrating over $p$

$$\Delta^{(2)'} + n^i \frac{\partial \Delta^{(2)}}{\partial x^i} - \tau' \Delta^{(2)} = S$$

with $\tau' = -\bar{n}_e \sigma_T a$ optical depth

Source term $S = S^{(2)} + S^{(I \times I)}$

Sachs-Wolfe effect

$$S^{(2)} = -4\Psi^{(2)'} + 4n^i \Phi_i^{(2)} n^i + 8 \omega' + 4 \chi^{n} n^i n^j - \tau' \left[ \Delta^{(2)}_{00} - \Delta^{(2)} - \frac{1}{2} \sum_{m=-2}^{2} \frac{\sqrt{4\pi}}{5^{3/2}} \Delta^{(2)}_{2m} Y_{2m}(\mathbf{n}) + 4v^{(2)} \cdot \mathbf{n} \right]$$

Gravitational lensing

$$S^{(I \times I)} = -8 \Delta^{(1)} \left( \Psi^{(1)'} - \Phi^{(1),i} n^i \right) + 2n^i (\Phi^{(1)} + \Psi^{(1)}) \partial_i (\Delta^{(1)} + 4\Phi^{(1)})$$

$$+ \left[ (\Phi^{(1)} + \Psi^{(1)}) n^i n^j + (\Phi^{,i} - \Psi^{,i}) \right] \frac{\partial \Delta^{(1)}}{\partial n^i}$$

$$- \tau' \left[ 2\delta_c^{(1)} \left( 4\mathbf{v} \cdot \mathbf{n} + \Delta^{(1)} - \Delta^{(1)} + \frac{1}{2} \Delta^{(1)}_2 P_2(\hat{\mathbf{v}} \cdot \mathbf{n}) \right) \right.$$

$$\left. + 2(\mathbf{v} \cdot \mathbf{n}) \left[ \Delta^{(1)}_0 + 3\Delta^{(1)}_0 - \Delta^{(1)}_2 \left( 1 - \frac{5}{2} P_2(\hat{\mathbf{v}} \cdot \mathbf{n}) \right) \right] - v \Delta^{(1)}_1 (5 + 4P_2(\hat{\mathbf{v}} \cdot \mathbf{n})) + 14(\mathbf{v} \cdot \mathbf{n})^2 - 2v^2 \right]$$

Coupling velocity and linear photon anisotropies
Boltzmann equations for massive particles

The Source term requires to know the evolution of baryons and CDM

Left-hand side

\[ \frac{dg}{d\eta} = \frac{\partial g}{\partial \eta} + \frac{\partial g}{\partial x^i} \frac{dx^i}{d\eta} + \frac{\partial g}{\partial q^i} \frac{dq^i}{d\eta} \]

just extend to a massive particle with mass \( m \) and energy \( E = (m^2 + q^2)^{1/2} \)

Example:

\[
\frac{dq^i}{d\eta} = -(\mathcal{H} - \Psi') q^i + \Psi_k q^i q^k E \Phi^k \Psi - \Phi^i E e^k \Phi^k E^i + \Psi_i q^2 E \Phi^k \Psi - E (\omega^{ij} + \mathcal{H} \omega^i) - (\chi^i_{jk} + \omega^i_{jk} - \omega^i_k) E \\
+ \left[ \mathcal{H} \omega^i \delta_{jk} - (\chi^i_{jk} + \chi^i_{kj} + \chi^i_{jk}) \right] \frac{q^j q^k}{E}.
\]

Collision terms: electrons are coupled to protons via Coulomb scatt.

driving \( \delta_e = \delta_p = \delta_b \) and \( v_e = v_p \equiv v \ ("baryons") \);

\[
\frac{dg_e}{d\eta}(x, q, \eta) = \langle c_{ep} \rangle QQ' q' + \langle c_{e\gamma} \rangle pp' q' \\
\frac{dg_p}{d\eta}(x, Q, \eta) = \langle c_{ep} \rangle qq' Q'
\]
Momentum continuity equation (I)

\[
\frac{\partial (\rho_b v^i)}{\partial \eta} + 4(\mathcal{H} - \Psi')\rho_b v^i + \Phi^i e^{\Phi+\Psi} \rho_b + e^{\Phi+\Psi} \left( \frac{T_b}{m_p} \right)^i + e^{\Phi+\Psi} \frac{\partial}{\partial x^j} (\rho_b v^j v^i) + \frac{\partial \omega^i}{\partial \eta} \rho_b + \mathcal{H} \omega^i \rho_b
\]

\[
= -n_e \sigma_T a \tilde{\rho}_\gamma \left[ \frac{4}{3} (v^{(1)i} - v^{(1)\gamma i}) + \frac{4}{3} \left( \frac{v^{(2)i} - v^{(2)\gamma i}}{2} \right) + \frac{4}{3} \delta^{(1)} (v^{(1)i} - v^{(1)\gamma i}) + v^{(1)} \Pi^{ij} \right]
\]

Photon velocity

\[
(\rho_\gamma + p_\gamma) v^i_\gamma = \int \frac{d^3 p}{(2\pi)^3} f p^i
\]

2nd-order velocity

\[
\frac{4}{3} \frac{v^{(2)\gamma i}}{2} = \frac{1}{2} \int \frac{d\Omega}{4\pi} \Delta^{(2)} n^i - \frac{4}{3} \delta^{(1)} (v^{(1)i})
\]

Quadrupole moments of photon distribution

\[
\Pi^{ij}_\gamma = \int \frac{d\Omega}{4\pi} \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) \left( \Delta^{(1)} + \frac{\Delta^{(2)}}{2} \right)
\]
Acoustic oscillations at second-order

In the tight coupling limit the energy and momentum continuity equations for photons and baryons at second-order describes the **non-linear dynamics of the photon-baryon fluid at recombination**

\[
\begin{align*}
\left(\Delta_{00}^{(2)} - 4 \Psi^{(2)}\right)'' + H \frac{R}{1+R} \left(\Delta_{00}^{(2)}' - 4 \Psi^{(2)}'\right) - c_s^2 \nabla^2 \left(\Delta_{00}^{(2)} - 4 \Psi^{(2)}\right) = \\
\frac{4}{3} \nabla^2 \left(\Phi^{(2)} + \frac{\Psi^{(2)}}{1+R}\right) + S'_{\Delta} + H \frac{R}{1+R} S'_{\Delta} - \frac{4}{3} \partial_i S^i_v
\end{align*}
\]

\[
\Delta_{00}^{(2)} = \delta_{\gamma}^{(2)}
\]

\[
c_s = \frac{1}{\sqrt{3(1+R)}}, \quad R = \frac{3 \rho_b}{4 \rho_{\gamma}}
\]
Non-linear dynamics at recombination

Modes entering the horizon during radiation epoch \((k > \eta_{\text{EQ}}^{-1})\)

\[
\Delta^{(2)}_{00}(k, \eta) = \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta^{(3)}(k_1 + k_2 - k) f(k_1, k_2, k, \eta) \Psi^{(1)}(0, k_1) \Psi^{(1)}(0, k_2)
\]

Acoustic oscillations of primordial non-Gaussianity

\[
f(k_1, k_2, k, \eta) \approx -18(a_{\text{NL}} - 1) \cos(k c_s \eta)
\]

\[
+ f_1(k_1, k_2, k) \left( \cos(k_1 c_s \eta) \cos(k_2 c_s \eta) - \cos(k_3 c_s \eta) \right)
\]

\[
+ f_2(k_1, k_2, k) \sin(k_1 c_s \eta) \sin(k_2 c_s \eta)
\]

Non-linear evolution at recombination

In this case the driving force is the quadrupole \(\Pi \propto v^2\gamma\)

\[
\Delta^{(1)}_{00} = 6 \Psi^{(1)}(0, k) \cos(k c_s \eta) \quad v^{(1)i}_{\gamma} = -i \frac{k^i}{k} \frac{9}{2} \Psi^{(1)}(0, k) \sin(k c_s \eta) c_s
\]
Linear vs. full radiation transfer function

\[ f(k_1, k_2, k, \eta) \approx -18(a_{NL} - 1) \cos(k_c \eta) + f_1(\cos(k_1 c_s \eta)) \cos(k_2 c_s \eta) - \cos(k_3 c_s \eta)) \]

\[ + f_2(k_1, k_2, k) \sin(k_1 c_s \eta) \sin(k_2 c_s \eta) \]

primordial non-Gaussianity is transferred linearly: Radiation Transfer function at first order

Non-linear evolution of gravity and non-linear dynamics at recombination \( \equiv \) the core of the 2nd order transfer function

\( \text{(how these contributions mask the primordial signal? how do they fit into the analysis of the bispectrum?)} \)

(from Komatsu & Spergel 2001)
Linear vs. **full** radiation transfer function

- A full numerical evaluation of the second-order Boltzmann equations is needed.

- Preliminary results: from non-linear dynamics at recombination and non-linear evolution of gravity, we get $f_{NL} \approx 5$

  (in progress N.B., Komatsu, Matarrese, Nitta, Riotto)
POSSIBLE IMPLICATIONS
If large NG is detected, are single field models of inflation ruled out?

It depends what “large” means

- If $|f_{NL}| >> 1$, yes
- If $|f_{NL}| << 1$, we are in trouble (remember that order unity NG is expected from gravity)
- Have to compute exactly this order unity number (is it 0.2? Is it 5?).
Is the sign of $f_{NL}$ a discriminator?

Yes: it is important to rule out some scenarios

- e.g., Typically $f_{NL}^{\text{equil}} \ll -1$ for models with derivative interactions, like DBI, k-inflation, etc.

- Typically $f_{NL}^{\text{local}} > 0$ for the curvaton models (if NG is large)

- There exist models which can accommodate a primordial NG which can be positive or negative depending on the parameters of the models (but still you can cut out some regions in the parameter space): e.g., multiple field inflation, or Ekpyrotic models
Mimicking a local $f_{NL}$?

Yes: it is possible, and such effects can be important in a intermediate region $1 \leq |f_{NL}| \leq 10$

- e.g. ISW-lensing bispectrum can contribute a bias $\Delta f_{NL}^{loc} = 9$
  (Smith/Zaldarriaga 2006)

- Effects from non-linear dynamics from recombination can give an effective $f_{NL} \approx 5$
  (Bartolo, Komatsu, Matarrese, Nitta, Riotto, in progress)

- Point sources can contribute a bias $\Delta f_{NL}^{loc} = 1.3$
  (Babich, Pierpaoli, 2008)
If the primordial $f_{NL}$ is compatible with zero, are primordial perturbations Gaussian distributed?

NO!!!

- One can have models with zero bispectrum and non-vanishing trispectrum 

$$\langle \Phi(k_1)\Phi(k_2)\Phi(k_3)\Phi(k_4) \rangle$$

- e.g., this is what happens in some configurations of the curvaton models
\[ \Phi = \Phi_L + f_{NL} \ast \left( \Phi_L^2 - \langle \Phi_L^2 \rangle \right) + g_{NL} \ast \Phi_L^3 \]

(e.g., Okamoto & Hu, 2002)

Also in this case primordial cubic non-linearities + non-linearities coming from the post-inflationary evolution
Theoretical Predictions for the trispectrum

<table>
<thead>
<tr>
<th>Model</th>
<th>$g_{NL}(k_1, k_2)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slow-roll inflation (including multiple fields)</td>
<td>$O(\varepsilon, \eta)$</td>
<td>$\varepsilon, \eta$ : slow-roll parameters</td>
</tr>
<tr>
<td>Curvaton scenario</td>
<td>$(9/4 \ r^2) \ (g^2 \ g''/g'^3 + 3 g \ g''/g'^2) +$ $- (2/r) \ (1+ 3 g \ g''/g'^2)$</td>
<td>$g''$ : deviation from a quadratic potential</td>
</tr>
<tr>
<td>Inhomogeneous reheating</td>
<td>$(5/3) \ f_{NL}^2 + (25/24) \ (Q''(x)/ Q^3(x))$</td>
<td>$X=\Gamma/H$ at the end of inflation</td>
</tr>
<tr>
<td>Ekpyrotic models</td>
<td>$</td>
<td>g_{NL}</td>
</tr>
</tbody>
</table>
- See expressions in D’Amico, N.B., Matarrese, Riotto (JCAP 08) for cubic non-linearities in the CMB anisotropies on all scales from gravitational effects.

- See N.B, Matarrese and Riotto (JCAP 05) for the expressions of the measurable $g_{NL}$ entering in the CMB anisotropies, allowing for generic NG initial conditions.
Results: trispectrum

For alternative scenarios and generic momenta configuration we can determine the non-linearity parameter $g_{NL}$ which enters into the trispectrum of the CMB anisotropies

\[
\frac{\Delta T}{T} = \frac{1}{3} \Phi
\]

\[
\Phi = \Phi_L + f_{NL} \ast (\Phi_L)^2 + g_{NL} \ast (\Phi_L)^3
\]

\[
g_{NL} = \frac{25}{9} b_{NL} + \frac{5}{9} a_{NL} \left[ 5 A(k_1,k_2,k_3) - 1 \right] + \frac{1}{54} + \frac{25}{9} C(k_1,k_2,k_3)
\]

\[
-\frac{1}{3} \left( \frac{\mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2)(\mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{\left| \mathbf{k}_1 + \mathbf{k}_2 \right|^2} - \frac{1}{3} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{\left| \mathbf{k}_1 + \mathbf{k}_2 \right|^2} + \text{cycl.} \right)
\]

N.B., Matarrese & Riotto 2005
D’Amico, N.B., Matarrese & Riotto 2008

\[
\zeta = \zeta_L + a_{NL} \zeta_L^2 + b_{NL} \zeta_L^3 + ....
\]
Conclusions

√ Both a positive measurement of the non-Gaussianity or an upper limit on its amplitude will represent a crucial observational discriminant between competing models for the primordial perturbation generation.

√ A detection of $f_{\text{NL}} \sim 10$ would rule out all standard single-field models of inflation.

√ To assess a detection of primordial NG: take care of foregrounds and any secondary signal that can mimic a primordial NG signal.

√ Up to now a lot of attention focused on the bispectrum of the curvature perturbation $\zeta$. However the physical quantity which is observed is the CMB anisotropy: one would like to have the *full second-order radiation transfer function to provide an accurate theoretical prediction of the CMB NG in terms of the primordial NG seeds.*

(SW, Integrated SW, tensor modes, Boltzmann equations at second-order)