Centrality and Power in Social Networks. A Game Theoretic Approach.

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Abstract

In this paper, a social network is modelized as a communication graph, which shows the possible direct communications between individuals. To reflect the interests that motivate the interactions, a cooperative game in characteristic function form is considered. From the graph and the game, the graph-restricted game is obtained. Shapley value in a game is considered as actor's power. The difference between actor's power in the new game and his/her power in the original one is proposed as a centrality measure. Conditions are given to reach desirable properties for this measure. Finally, a decomposition of this measure is proposed.

Key words: Social networks, game theory, centrality, Shapley value.
Classification code: C71.

1 Introduction

In a sociological context, it is usually assumed that a *social network* is given by a graph (N, Γ), where $N = \{1, 2, \ldots, n\}$ is a finite set of individuals (*nodes*) and Γ is a collection of (unordered) pairs $\{i, j\}$ of elements of N (*edges*), which shows the possible communications; i.e., individuals i and j can communicate directly if and only if $\{i, j\} \in \Gamma$. If i cannot communicate directly with j, it may still be possible for them to communicate indirectly if there is some k (an *intermediary*) with whom both can communicate, or more generally, a sequence of intermediaries.

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Centrality is a sociological notion which is not, however, clearly defined; it is frequently defined only in an indirect manner. For example, we are told, an individual, i, has centrality in a graph if :

(i) i can communicate directly with many other nodes, or

(ii) i is close to many other nodes, or

(iii) there are many pairs of nodes which need i (or, can use i) as intermediary in their communications.

Several approaches have been made in the past. A review of the recent literature about social networks (Hanneman, 1999 and Freeman, 1977, 2000) allows us to distinguish several different approaches to the centrality concept. The most relevant are:

(a) Degree centrality (Shaw, 1954, and Nieminen, 1974). This approach identifies centrality with the *degree* of a node, i.e., the number of edges incident on that node.

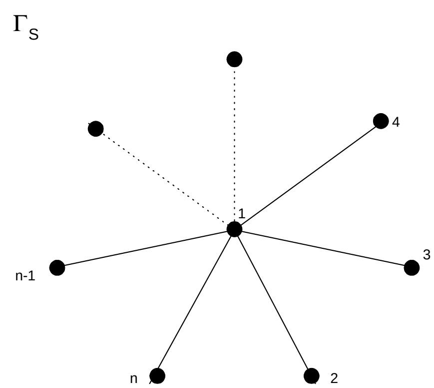


Figure 1: Star (N, Γ^S)

For instance, the hub of the star in the Figure 1 is less dependent on a specific node as long as it has a higher degree than the other nodes.

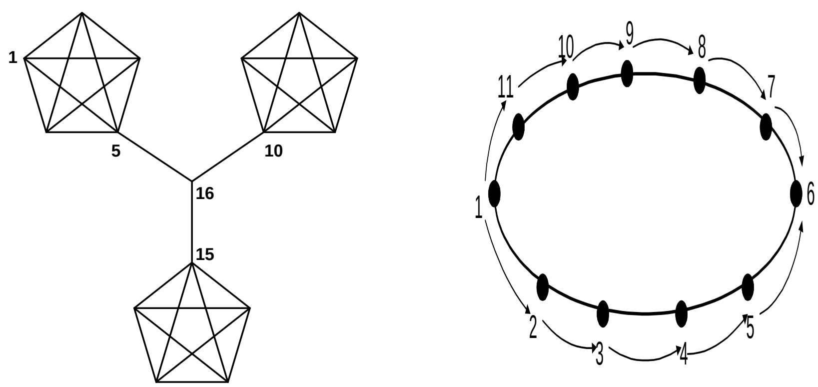
(b) Closeness centrality (Beauchamp, 1965 and Sabidussi, 1966). Another reason that makes the hub in the star more powerful is that it is *closer* to many more nodes than the rest of the nodes.

This approach considers the sum of the *geodesic* distances between a given node and the rest as a decentrality measure in the sense that, the lower is this sum, the greater is the centrality.

(c) Betweenness centrality (Bavelas, 1948 and Freeman, 1977). A third reason to consider the hub of a star in advantage is that it lies *between* all pair of nodes and no other node has this property.

In this approach all possible geodesic paths between pairs of points are considered. The centrality measure of a given node is then obtained by counting the number of such paths on which it lies.

Some critical analysis will show certain shortcomings for each of these approaches.



Figures 2(a) and 2(b)

Consider first the idea that centrality should be equivalent to the degree of a node. In Figure 2(a), there are three 5-person cliques ($1-5, 6-10, 11-15$), each with a herald ($5, 10$, and 15). Communications within cliques are perfect. For messages between two cliques, however, the herald must send the message to a switchboard (node 16) which relays it to another herald.

As may be seen, the switchboard has degree 3, each of the heralds has degree 5, and the remaining 12 players have degree 4 each. Yet it would be hard to suggest that node 16 is less central than, say, node 1. On the other hand, the sum of the geodesic distances between a given node and all the other nodes is less for node 16 (27) than for herald nodes $5, 10, 15$ (29) and even for each of the remaining nodes as for example node 1 (44). This suggests a different order for centralities. Node 1 is never in a position to relay a message, whereas the switchboard is always necessary for communications between cliques. We conclude that, at least for this graph, degree is not quite what we would want as a measure of centrality.

Consider now the idea that centrality should only take geodesic paths between two given nodes into account. In Figure 2(b), there are two possible paths between nodes 1 and 6. One path ($1-2-3-4-5-6$) has length 5, the other $(1-11-10-9-8-7-6)$ has length 6. Yet it would seem foolish to discard the possibility of using a longer path, simply because a (slightly) shorter one exists. However, we would tend to discard the long (10-edge) path from node 1 to 2, in favor of the one-edge direct path.

All sociologists would agree that *power* is a fundamental property of social structure but, again, there is much less agreement about what power is. For many years, social networks analysts have relied heavily on one fundamental concept to account for variation in actor's power: network centrality. Despite the once wide acceptance of the link between centrality and power, the extent to which both concepts are related is now an issue of intense controversy (e.g., Mizruchi and Potts, 1998, or Hanneman, 1999)

Our treatment below will give rise to a new measure of centrality based on power's variation due to the restrictions in the communications.

Moreover we analyze the extent to which our centrality measure satisfies the following *desiderata*:

(a) Any measure of centrality should be symmetric; i.e. if π is a permutation of N which preserves Γ, then a node i should have the same centrality as node $\pi(i)$.

(b) The centrality of a node in a disconnected graph should coincide with the centrality of that node in the connected subgraph to which it belongs.

(c) Isolated nodes should have minimal centrality.

(d) If Γ is a chain, centrality should increase from the end node to the median node.

(e) Of all connected graphs with n nodes, the minimal centrality should be attained by the end nodes in a chain.

(f) Of all graphs with n nodes, the maximal centrality should be attained by the hub of a star.

(g) Removing an edge should decrease (or at least, not increase) the centrality of both nodes incident on that edge.

The remainder of the paper is organized as follows: Section 2 is devoted to a game-theoretic approach to the concept of centrality and to define new measures that satisfy, for a wide class of games, the previous desiderata as it is proved in Section 3. In Section 4 some particular cases are analyzed and the corresponding discussion will give rise to an interesting decomposition of the introduced measures. Section 5 is devoted to this decomposition. Finally, some comments are included in Section 6.

2 A Game-theoretic Approach

We shall say that a social network (N, Γ) is connected if it is possible to join any two nodes i and j of N by a sequence of edges from Γ. We shall say that a subset S of N is *connected in* (N, Γ) if (S, Γ_S) is connected, where Γ_S is the set of those pairs $\{h, k\} \in \Gamma$ where both h and k are elements of S.

Let (N, v) be a n-personal game in characteristic function form, where $N = \{1, 2, \ldots, n\}$ is the players set and v is a real valued function defined on 2^N, satisfying $v(\emptyset) = 0$. No particular relation is assumed between the game (N, v) and the graph (N, Γ), other than the players of the game being the nodes of the graph. When there is no ambiguity with respect to N we will refer to the graph (N, Γ) and the game (N, v) as Γ and v respectively.

We can think of v as representing the economic possibilities of the several coalitions (subsets of N), whereas Γ tells us whether they can eventually communicate (and thus take advantage of the possibilities). Following Myerson (1977), we define a new game w, the *graph-restricted game*, by

$$w(S) = \sum_{T_k \in C_\Gamma(S)} v(T_k), \tag{1}$$

where $C_\Gamma(S)$ is the set of components of S in Γ. Note that, if S is connected in Γ, then $w(S) = v(S)$. The game w represents the economic possibilities taking the available communications into account.

If we consider some "reasonable outcome" for these two games, the differences between the corresponding outcomes can be considered as a result of the different positions which the players have in graph Γ. Clearly, the result will depend on the particular "reasonable outcome" which we use.

From now on we will use the Shapley value, φ, as an index of players' power in a given game, though we could just as easily use the Banzhaf-Coleman index of power (as in Grofman and Owen, 1982), or possibly the nucleolus or some other one-point solution concept. Then we can think of the difference between $\varphi_i(w)$ (the Shapley value to player i in the projected game w) and $\varphi_i(v)$ as a measure of the centrality of player i in the graph Γ, i.e.:

$$\gamma_i(v, \Gamma) = \varphi_i(w) - \varphi_i(v). \tag{2}$$

It represents the increase (or decrease) in $i's$ power due to its position in the graph. Note, however, that this depends also on the game v.

From Myerson (1977) we raise the following propositions.

Proposition 2.1 *If (N, v) is a super-additive game and (N, Γ) is a social network, then desideratum (g) is satisfied by $\gamma_i(v, \Gamma)$.*

Proposition 2.2 *If (N, v) is a game and (N, Γ) is a social network then, removing an edge of Γ will change the centrality of both incident nodes on that edge by an equal amount.*

This property is a direct consequence of choosing Shapley value as power index. If it is considered not necessary that a centrality measure satisfies Proposition 2.2, any other power index should be chosen.

In this paper we will analyze the case in which game v deals with all players symmetrically, so that the centrality measure depends on the graph Γ rather than on the particular role played by i in game v. Let us suppose, then, that v is symmetric, i.e.

$$v(S) = f(s), \quad S \subset N, \tag{3}$$

where s is the cardinality of S, and the function f satisfies $f(0) = 0$.

Assuming v is symmetric in this way, then of course $\varphi_i(v) = v(N)/n$ for all players. In this case, it is possible avoid the last term in (2). Since we are interested in comparing centrality, rather than in obtaining some absolute measure of centrality, we propose

$$\kappa_i[v, \Gamma] = \varphi_i(w), \tag{4}$$

as a measure for the centrality of i in Γ.

In order to calculate the centrality measure proposed, it will be useful to analyze the mapping P_Γ of G_N (the vector space of all games with players set N) into itself, defined by $P_\Gamma(v) = w$, where w is given by equation (1). This is a linear mapping and it is not difficult to verify that it is a projection, in the sense that $P_\Gamma \circ P_\Gamma = P_\Gamma$.

To do this, let us express the space G_N in terms of the *unanimity basis*. This basis consists of the $2^n - 1$ unanimity games. For each (non-empty) $S \subset N$, the unanimity game u_S is defined by

$$u_S(T) = \begin{cases} 1, & \text{if } S \subset T, \\ 0, & \text{if } S \not\subset T. \end{cases} \tag{5}$$

and it is not difficult to prove that game v can be expressed as

$$v = \sum_{S \subset N} \Delta(S) u_S, \tag{6}$$

where $\Delta(S)$, the *Harsanyi dividend* (of S in v), is given by

$$\Delta(S) = \sum_{T \subset S} (-1)^{s-t} v(T), \tag{7}$$

(s and t being the cardinalities of S and T respectively). Since G_N has dimension $2^n - 1$, it is clear that the games u_S form a basis.

We note that, if S is connected in Γ, then $P_\Gamma(u_S) = u_S$. If, on the other hand, S is not connected in Γ, we find that $P_\Gamma(u_S) = w_{S,\Gamma}$, where

$$w_{S,\Gamma}(T) = \begin{cases} 1, & \text{if there is some connected } K \text{ such that } S \subset K \subset T, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Thus, this last game, $w_{S,\Gamma}$, can be considered as the *connect* S *in* Γ *game*. This game can be quite complicated (depending on Γ), though there exists a special case easy to describe. If Γ is a *tree* (a connected graph with no cycles) then there is only one smallest connected K which contains S. We call it $H(S)$: the *connected hull* of S. In this case, it satisfies

$$w_{S,\Gamma} = u_{H(S)}. \tag{9}$$

A general expression for $w_{S,\Gamma}$ is given in Lemma 2.1 which uses the next definition:

Definition 2.1 *Given the social network* (N, Γ) *and* $S \subset N$*, we will say that* $S' \subset N$ *is a minimal connection set of* S *in* Γ *if there is no* $S'' \subset N$ *(*$S'' \neq S'$*) connected in* Γ *such that* $S \subset S'' \subset S'$*.*

Let us observe that for a given $S \subset N$, it could exist several minimal connection sets of S, one or none. $\mathcal{M}_\Gamma(S)$ will denote the collection of these sets.

From (8) $\mathcal{M}_\Gamma(S) = \emptyset$, implies $w_{S,\Gamma} = \mathbf{0}$ (the null vector of G_N). Therefore in next lemma we will only deal with the case $\mathcal{M}_\Gamma(S) \neq \emptyset$.

Lemma 2.1 *Given the social network* (N, Γ) *and* $S \subset N$*, if* $\mathcal{M}_\Gamma(S)$ *is nonempty and* $\mathcal{M}_\Gamma(S) = \{S_i\}_{i=1}^r$*, then*

$$P_\Gamma(u_S) = \mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u_{S_i}) \tag{10}$$

where $\mathbf{1}$ *is the game defined by* $\mathbf{1}(S) = 1,$ *for all* $S \neq \emptyset$*, i.e., the unit element of the standard inner product in* G_N*.*

Proof:

Let $T \subset N$, then

$$(\mathbf{1} - u_{S_i})(T) = \begin{cases} 0, & \text{if } S_i \subset T, \\ 1, & \text{if } S_i \not\subset T, \end{cases} \tag{11}$$

and thus

$$\left(\mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u_{S_i}) \right)(T) = \begin{cases} 1, & \text{if there is } S_i \subset T, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Then, (12) coincides with (8). ■

Next proposition characterizes the image of the mapping P_Γ. This result appears in Owen (1986), but we include a different proof, constructive and useful to the calculation of the centrality.

Proposition 2.3 *The image of the mapping P_Γ is the subspace of G_N spanned by the games u_T where T is connected in Γ.*

Proof:

From (10), given $S \subset N$ if $\mathcal{M}_\Gamma(S) \neq \emptyset$ and $\mathcal{M}_\Gamma(S) = \{S_i\}_{i=1}^r$, we obtain:

$$P_\Gamma(u_S) = \sum_{i=1}^{r} u_{S_i} - \sum_{i<j}^{r} u_{S_i} \cdot u_{S_j} + \sum_{i<j<k}^{r} u_{S_i} \cdot u_{S_j} \cdot u_{S_k} + \ldots + (-1)^{r+1} u_{S_1} \cdot u_{S_2} \cdots u_{S_r}.$$

Note that, if $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$,

$$u_{S_{i_1}} \cdot u_{S_{i_2}} \cdots u_{S_{i_k}} = u_{\cup_{j=1}^{k} S_{i_j}},$$

and thus,

$$P_\Gamma(u_S) = \sum_{i=1}^{r} u_{S_i} - \sum_{i<j}^{r} u_{S_i \cup S_j} + \sum_{i<j<k}^{r} u_{S_i \cup S_j \cup S_k} + \cdots + (-1)^{r+1} u_{\cup_{i=1}^{r} S_i}. \quad (13)$$

Since $S_{i_1}, \ldots, S_{i_k}$ are connected and not disjoint sets for all $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$, $\cup_{j=1}^{k} S_{i_j}$ is connected. Therefore, $P_\Gamma(u_S)$ is a linear combination of unanimity games u_T where T is connected. ■

Using (4) and (6) and the Shapley value linearity, we obtain

$$\kappa_i[v, \Gamma] = \sum_{S \subset N} \Delta(S) \varphi_i(w_{S,\Gamma}).$$

Therefore, we only need to know the Harsanyi dividends of v and the Shapley value for $w_{S,\Gamma}$, the projection of every unanimity game .

If $\mathcal{M}_\Gamma(S) = \{\hat{S}\}$, then $w_{S,\Gamma} = u_{\hat{S}}$ and the calculation of the value $\varphi(w_{S,\Gamma})$ is straightforward (essentially the dividend of S is evenly apportioned among all nodes of $\hat{S}$). In case of several options, calculations complexity increases. This, implicitly, means that if Γ is a tree (or a forest, defined as a union of disjoint trees) calculation of centrality will be easy. A method, based on generating functions, is described in detail in Owen (1986).

When Γ has cycles and there is a $S \subset N$ such that $\mathcal{M}_\Gamma(S) = \{S_l\}_{l=1}^r$, with $r > 1$, from Lemma 2.1, using (13) and linearity of Shapley value we obtain

$$\varphi_i(w_{S,\Gamma}) = \sum_{l=1}^{r} \varphi_i(u_{S_l}) - \sum_{l<j}^{r} \varphi_j(u_{S_l \cup S_j}) + \sum_{l<j<k}^{r} \varphi_i(u_{S_l \cup S_j \cup S_k}) -$$

$$- \cdots + (-1)^{r+1} \varphi_i(u_{\cup_{l=1}^r S_l}), \tag{14}$$

and thus,

$$\varphi_i(w_{S,\Gamma}) = \sum_{l=1}^{r} \frac{1}{s_l} \delta_{S_l}(i) - \sum_{l<j}^{r} \frac{1}{s_{l,j}} \delta_{S_{l,j}}(i) + \sum_{l<j<k}^{r} \frac{1}{s_{l,j,k}} \delta_{S_{l,j,k}}(i) -$$

$$- \cdots + (-1)^{r+1} \frac{1}{s_{1,\ldots,r}} \delta_{S_{1,\ldots,r}}(i), \tag{15}$$

where $s_{j_1,\ldots,j_k}$ is the cardinality of $\cup_{l=1}^k S_{j_l}$ and

$$\delta_{S_{j_1,\ldots,j_k}}(i) = \begin{cases} 1, & \text{if } i \in \cup_{l=1}^k S_{j_l}, \\ 0, & \text{otherwise .} \end{cases}$$

3 General results

In this section we will lay out general conditions for f under which $\kappa_i[v, \Gamma]$ satisfies the other *desiderata*.

To this end, we define two classes of characteristic functions. If v is symmetric,

(i) v is super-additive if $f(m+n) \geq f(m) + f(n)$, for all $m, n \in \mathbb{N}$

v is strictly super-additive if the above inequality holds strictly.

(ii) v is convex if f is convex in $\mathbb{N}$, i.e., $f(s+1) - f(s) \geq f(s) - f(s-1)$, for all $s \geq 1$.

v is strictly convex if the above inequality holds strictly.

From now on we will use

$$\mathcal{S}_N = \{v \in G_N : v \text{ symmetric }\}$$

$$\mathcal{A}_N = \{v \in G_N : v \text{ super-additive }\}$$

$$\mathcal{C}_N = \{v \in G_N : v \text{ convex }\}$$

Moreover, $\mathcal{G}_N$, will denote the set of all graphs with nodes set N.

Hereafter, if $v \in \mathcal{S}_N$, i.e.: $v(S) = f(s)$, we will use v or f equivalently.

The following Proposition 3.1, shows that, if v is symmetric, desideratum (a) is satisfied. Let us introduce first a definition.

Definition 3.1 *We will say that a permutation $\pi : N \to N$ preserves the graph (N, Γ) when $\{i, j\} \in \Gamma$ if and only if $\{\pi(i), \pi(j)\} \in \Gamma$.*

If we note $\pi\Gamma = \{\{\pi(i), \pi(j)\} : \{i, j\} \in \Gamma\}$, previous definition tells us that π preserves the graph (N, Γ) when $\pi\Gamma = \Gamma$.

Proposition 3.1 *Let $v \in \mathcal{S}_N$ and $\Gamma \in \mathcal{G}_N$. If π is a permutation on N that preserves Γ, $\kappa_i[v, \Gamma] = \kappa_{\pi(i)}[v, \Gamma]$, for all $i \in N$.*

Proof:

As $v \in \mathcal{S}_N$, $\pi v = v$.

It is straightforward to verify that:

$$T \in C_\Gamma(S) \text{ if an only if } \pi(T) \in C_{\pi\Gamma}(\pi(S)). \tag{16}$$

Let us show that $\pi w = w$, where $w = P_\Gamma(v)$. If $S \subset N$,

$$\pi w(S) = w(\pi(S)) = \sum_{H_k \in C_\Gamma(\pi(S))} v(H_k). \tag{17}$$

On the other hand, using (16)

$$w(S) = \sum_{T_k \in C_\Gamma(S)} v(T_k) = \sum_{\pi(T_k) \in C_\Gamma(\pi(S))} v(\pi(T_k)), \tag{18}$$

and then, $\pi w = w$.

Finally, by the symmetry of the Shapley value:

$$\varphi_i(w) = \varphi_{\pi(i)}(\pi w) = \varphi_{\pi(i)}(w),$$

and thus,

$$\kappa_i[v, \Gamma] = \kappa_{\pi(i)}[v, \Gamma].$$

■

Next proposition shows that the centrality of a node in a disconnected graph coincides with its centrality when it is considered as a node in the connected subgraph to which it belongs. So *desideratum* (b) is satisfied.

Proposition 3.2 *Let N_1 and N_2 be disjoint subsets of $\mathbb{N}$. Suppose $\Gamma^j \in \mathcal{G}_{N_j}$ is connected, $j = 1, 2$, and $\Gamma^1 \cup \Gamma^2 \in \mathcal{G}_{N_1 \cup N_2}$.*

If $v \in \mathcal{S}_{N_1 \cup N_2}$ and v_j is the restriction of v to N_j, $j = 1, 2$, then for $i \in N_j$,

$$\kappa_i[v, \Gamma^1 \cup \Gamma^2] = \kappa_i[v_j, \Gamma^j],\ j = 1, 2.$$

Proof:

Writing v in terms of the unanimity basis

$$v = \sum_{S \subset N_1 \cup N_2} \Delta(S) u_S.$$

Then, by the linearity of projection $P_{\Gamma^1 \cup \Gamma^2}$,

$$w = P_{\Gamma^1 \cup \Gamma^2}(v) = \sum_{S \subset N_1 \cup N_2} \Delta(S) w_{S, \Gamma^1 \cup \Gamma^2}$$

$$= \sum_{S\subset N_1} \Delta(S) w_{S,\Gamma^1\cup\Gamma^2} + \sum_{S\subset N_2} \Delta(S) w_{S,\Gamma^1\cup\Gamma^2} + \sum_{\substack{S\subset N \\ S\cap N_1\neq\emptyset \\ S\cap N_2\neq\emptyset}} \Delta(S) w_{S,\Gamma^1\cup\Gamma^2}.$$

As $N_1 \cap N_2 = \emptyset$, $\Gamma^1 \cup \Gamma^2$ is disconnected and then, by Lemma 2.1 and Proposition 2.1, u_S belongs to $\ker(P_{\Gamma^1\cup\Gamma^2})$ when $S \cap N_j \neq \emptyset$, for $j = 1, 2$.

Moreover, if $S \subset N_j$, $w_{S,\Gamma^1\cup\Gamma^2} = w_{S,\Gamma^j}$, $j = 1, 2$, because the elements of $\mathcal{M}_{\Gamma^1\cup\Gamma^2}(S)$ are contained in N_j. Then,

$$w = \sum_{S\subset N_1} \Delta(S) w_{S,\Gamma^1} + \sum_{S\subset N_2} \Delta(S) w_{S,\Gamma^2}.$$

Let us observe that if $i \in N_j$ and $S \subset N_k$, $k \neq j$, (15) shows that $\varphi_i(w_{S,\Gamma^k}) = 0$, because there are no sets in $\mathcal{M}_{\Gamma^k}(S)$ containing i. And then,

$$\kappa_i[v, \Gamma^1 \cup \Gamma^2] = \kappa_i[v_j, \Gamma^j].$$

■

Next proposition shows that, if v is symmetric and super-additive, isolated nodes have minimal centrality, so *desideratum* (c) is satisfied.

Proposition 3.3 *Let $v \in \mathcal{S}_N \cap \mathcal{A}_N$ and $\Gamma^0 \in \mathcal{G}_N$. If $i \in N$ is an isolated node in Γ^0, then, for all $\Gamma \in \mathcal{G}_N$, and for all $j \in N$*

$$\kappa_i[v, \Gamma^0] \leq \kappa_j[v, \Gamma]. \tag{19}$$

Proof:

By definition of P_Γ, the centrality of an isolated node of Γ is $f(1)$.

Let $j \in N$ with degree k in Γ. If j is not isolated then $1 \leq k \leq n-1$. Let us assume $\kappa_j[v, \Gamma] < f(1)$. The node j will become an isolated node by a stepwise elimination of the k edges incident on it. From Proposition 2.2, the sequence of centralities of node j is not increasing. Then, j would be an isolated node with centrality strictly less than $f(1)$. This contradiction proves the result. ■

Lemma 3.1, whose proof is straightforward, and Proposition 3.4 guarantee that, if v is symmetric and super-additive, of all graphs with n nodes the maximal centrality is attained by the hub of a star, so *desideratum* (f) is satisfied.

Lemma 3.1 *Let $v \in \mathcal{S}_N \cap \mathcal{A}_N$ and $\Gamma \in \mathcal{G}_N$, $w = P_\Gamma(v)$. If $i \in N$, then for all $S \subset N - \{i\}$*

(i) $w(S \cup \{i\}) \leq f(s+1)$,

(ii) $w(S) \geq sf(1)$,

(iii) $w(S \cup \{i\}) - w(S) \leq f(s+1) - sf(1)$.

Proposition 3.4 *Let $v \in \mathcal{S}_N \cap \mathcal{A}_N$. Let us suppose that $\Gamma^S \in \mathcal{G}_N$ is the star with n nodes where node 1 is the hub. Then, for all $\Gamma \in \mathcal{G}_N$ and for all $i \in N$*

$$\kappa_i[v, \Gamma] \leq \kappa_1[v, \Gamma^S]. \tag{20}$$

Proof:

Without lost of generality we can relabel the node i in Γ as node 1.

Using the usual Shapley value expression for $1 \in N$ we have

$$\kappa_1[v, \Gamma] = \varphi_1(w) = \sum_{S:1 \notin S} \frac{s!(n-1-s)!}{n!}(w(S \cup \{1\}) - w(S)),$$

and for the hub of the star

$$\kappa_1[v, \Gamma^S] = \sum_{S:1 \notin S} \frac{s!(n-1-s)!}{n!}(f(s+1) - sf(1)),$$

and, by Lemma 3.1

$$\kappa_1[v, \Gamma] \leq \kappa_1[v, \Gamma^S].$$

■

Proposition 3.5, which needs Lemma 3.2 and Lemma 3.3, proves that if v is symmetric and super-additive, of all connected graphs of n nodes, the minimal centrality is attained by the end nodes in a chain, so *desideratum* (e) is satisfied.

Lemma 3.2 *Let $\Gamma \in \mathcal{G}_N$, $S \subset N$ and Q a separated part of S in Γ, i.e., $Q \subset S$ and there are no edges joining a node in Q with a node in $S - Q$. If $w = P_\Gamma(v)$, then*

$$w(S) = w(Q) + w(S - Q). \tag{21}$$

Lemma 3.3 *Let $\Gamma \in \mathcal{G}_N$ and $S \subset N$. Let us suppose $i \in N - S$ and $w = P_\Gamma(v)$. Then we have*

$$w(S \cup \{i\}) - w(S) = v(Q) - w(Q - \{i\}) \tag{22}$$

where Q is the i-component of $S \cup \{i\}$, i.e., the component of $S \cup \{i\}$ in Γ which contains i.

Proof:

$Q - \{i\}$ may not be connected, but it is a separated part of S. Note that $S \cup \{i\} - Q$ and $S - (Q - \{i\})$ are both equal to $S - Q$.

As,

$$w(S \cup \{i\}) = w(Q) + w(S - Q),$$

and

$$w(S) = w(Q - \{i\}) + w(S - Q),$$

then,

$$w(S \cup \{i\}) - w(S) = w(Q) - w(Q - \{i\}).$$

As Q is connected, $w(Q) = v(Q)$. This proves the Lemma. ∎

Proposition 3.5 *Let $v \in \mathcal{S}_N \cap \mathcal{A}_N$. If $\Gamma^C \in \mathcal{G}_N$ is the chain with n nodes, where node 1 is an end node, then, for all connected $\Gamma \in \mathcal{G}_N$ and for all $i \in N$,*

$$\kappa_1[v, \Gamma^C] \leq \kappa_i[v, \Gamma]. \tag{23}$$

Proof:

Let us suppose that the nodes of Γ^C are labeled in the natural way. Then we relabel the nodes of Γ as follows. Let node i be relabeled as 1. Then, let 2 be any node which is adjacent to 1. Let 3 be any node which is adjacent to either 1 or 2 (or possibly both). Continuing in this manner let node k be any node which is adjacent to at least one of the nodes $1, \ldots, k-1$. Since Γ is connected, this process can be continued until all nodes have been numbered. We see then that every segment $\{1, \ldots, k\}$ is connected in Γ.

Let $w^C = P_{\Gamma^C}(v)$ and $w = P_\Gamma(v)$, then

$$\kappa_1[v, \Gamma^C] = \varphi_1(w^C) = \sum_{S: 1 \notin S} \frac{s!(n-1-s)!}{n!}(w^C(S \cup \{1\}) - w^C(S)), \tag{24}$$

whereas

$$\kappa_1[v, \Gamma] = \varphi_1(w) = \sum_{S: 1 \notin S} \frac{s!(n-1-s)!}{n!}(w(S \cup \{1\}) - w(S)). \tag{25}$$

We shall show that each summand in (25) is at least as large as the corresponding summand in (24).

In fact, for a given S, let K and Q be the 1-components of $S \cup \{1\}$ in Γ^C and in Γ, respectively. K must be of the form $\{1, \ldots, k\}$. By Lemma 3.3, we have

$$w^C(S \cup \{1\}) - w^C(S) = v(K) - w^C(K - \{1\}).$$

Since K has k elements, $v(K) = f(k)$. Now, as $K - \{1\}$ is connected in Γ^C, $w^C(K - \{1\}) = v(K - \{1\}) = f(k-1)$, and therefore

$$w^C(S \cup \{1\}) - w^C(S) = f(k) - f(k-1).$$

Next, by the relabeling method used for Γ, we see that K is connected in Γ. Thus we must have $K \subset Q$, which implies that q, the number of elements in Q, cannot be smaller than k. Then,

$$w(S \cup \{1\}) - w(S) = v(Q) - w(Q - \{1\}).$$

Since $v(Q) = f(q)$ and $w(Q - \{1\}) \leq v(Q - \{1\}) = f(q-1)$, we realize that

$$w(S \cup \{1\}) - w(S) \geq f(q) - f(q-1).$$

Since $q \geq k$, and v is convex, this leads to $f(q) - f(q-1) \geq f(k) - f(k-1)$. This proves that

$$w(S \cup \{1\}) - w(S) \geq w^C(S \cup \{1\}) - w^C(S).$$

Since this holds for every S, it proves the proposition. ∎

Finally, if Γ^C is a chain and v is symmetric and convex, centrality increases from the end node to the median node as it is shown by Lemma 3.4 and Proposition 3.6, so *desideratum* (d) is satisfied.

Lemma 3.4 *Let $n_1,\ n_2 \in \mathbb{N},\ n_1 < n_2$ and let $v \in \mathcal{S}_{N_2} \cap \mathcal{C}_{N_2}$. Let us suppose that Γ_i^C is a chain with n_i nodes ordered in the natural way. Then,*

$$\kappa_1[v, \Gamma_1^C] \leq \kappa_1[v, \Gamma_2^C]. \tag{26}$$

Proof:

We will show that the result is true for $n_2 = n_1 + 1$. We have

$$\kappa_1[v, \Gamma_2^C] - \kappa_1[v, \Gamma_1^C] = \sum_{S \subset N_2} \Delta(S)\varphi_1(w_S) - \sum_{S \subset N_1} \Delta(S)\varphi_1(w_S).$$

As 1 is an end node, $\varphi_1(w_S) = 0$ if $1 \notin S$, and then

$$\kappa_1[v, \Gamma_2^C] - \kappa_1[v, \Gamma_1^C] = \sum_{1 \in S \subset N_2} \Delta(S)\varphi_1(w_S) - \sum_{1 \in S \subset N_1} \Delta(S)\varphi_1(w_S) =$$

$$= \sum_{1, n_1+1 \in S \subset N_2} \Delta(S)\varphi_1(w_S).$$

There are $\begin{pmatrix} n_1 - 1 \\ s-2 \end{pmatrix}$ coalitions $S \subset N_2$ with cardinality s and such that $\{1, n_1+1\} \subset S$. For each one, $\varphi_1(w_S) = \frac{1}{n_1+1}$. Then, the value of the previous expresion is

$$\frac{1}{n_1+1} \sum_{s=2}^{n_1+1} \begin{pmatrix} n_1 - 1 \\ s-2 \end{pmatrix} \Delta(S).$$

From (4) and the symmetry of v

$$\sum_{s=2}^{n_1+1} \begin{pmatrix} n_1 - 1 \\ s-2 \end{pmatrix} \Delta(S) = \sum_{s=2}^{n_1+1} \begin{pmatrix} n_1 - 1 \\ s-2 \end{pmatrix} \sum_{t=1}^{s} \begin{pmatrix} s \\ t \end{pmatrix} (-1)^{s-t} f(t),$$

and after some algebraic manipulations we obtain

$$f(n_1+1) - 2f(n_1) + f(n_1 - 1),$$

which is nonnegative when v is convex. ∎

Proposition 3.6 *If $v \in \mathcal{S}_N \cap \mathcal{C}_N$ and Γ^C is a chain with n nodes numbered in the natural way, then for $1 \leq i \leq n/2$*

$$\kappa_i[v, \Gamma^C] \leq \kappa_{i+1}[v, \Gamma^C]. \tag{27}$$

Proof:

For each i, $1 \leq i \leq n/2$, removing the edge $\{i, i+1\}$, the nodes i, $i+1$ will become end nodes of chains that have i and $n-i$ nodes respectively ($n - i \geq i$). Let Γ_i^C and Γ_{n-i}^C be these chains. By Proposition 3.2 and Lemma 3.4

$$\kappa_i[v, \Gamma_i^C] \leq \kappa_{i+1}[v, \Gamma_{n-i}^C],$$

and from Proposition 2.2

$$\kappa_i[v, \Gamma^C] \leq \kappa_{i+1}[v, \Gamma^C].$$

■

Let us note that if in each of the previous results, the super-additiveness (convexity) condition is replaced by strict super-additiveness (strict convexity), then all inequalities become strict.

The above results show that if we choose a symmetric and convex game v, the measure of centrality induced by v in a social network satisfies the expected properties of what we have called centrality.

4 Some specific game functions

Granted that v is to be symmetric, defined by a function f which is to be super-additive and even convex, what function f should be chosen? Several come to mind:

- *The messages game* is defined by $v_1(S) = f_1(s) = s^2 - s$. We note that the Harsanyi dividends are given by

$$\Delta_1(S) = \begin{cases} 2, & \text{if } s = 2, \\ 0, & \text{if } s \neq 2. \end{cases}$$

  In other words, for this function, just the two-player coalitions render profit. In essence, this corresponds to the idea of sending messages: the coalition $\{i, j\}$ (where $i \neq j$) gives rise to two possible messages as each of i and j can send a message to the other. Then, the modified game w will tell us how much each player contributes to the delivery of messages.

- *The overhead game* is defined by $v_2(S) = f_2(s) = -1$ for all $s \neq 0$. The Harsanyi dividends are given by

$$\Delta_2(S) = (-1)^s, \ s \geq 1.$$

The projected game shows us the expenses (or incomes) that correspond to each player in a business.

- *The conferences game* is defined by $v_3(S) = f_3(s) = 2^s - s - 1$. In this case, the Harsanyi dividends are given by

$$\Delta_3(S) = \begin{cases} 0, & s = 0, 1, \\ 1, & s \geq 2. \end{cases}$$

In this game each coalition receives a unit for each possible meeting or conference among two or more of its members .

To illustrate the ideas we tabled above, let us calculate the centrality of different nodes in some particular connected graphs. In certain examples we will consider a general function v, whereas in others we will study only the three particular games we have enumerated.

a) We first analyze the case of a star with n nodes (Γ^S, Figure 1). If 1 is the hub of the star, then

$$\kappa_1[v, \Gamma^S] = \Delta(1) + \frac{1}{n} \sum_{k=2}^{n} \left(\begin{array}{c} n+1 \\ k+1 \end{array} \right) \Delta(k), \tag{28}$$

and for the other nodes $i \neq 1$,

$$\kappa_i[v, \Gamma^S] = \Delta(1) + \sum_{k=2}^{n} \frac{1}{k+1} \left[\left(\begin{array}{c} n-1 \\ k-1 \end{array} \right) + \frac{1}{k} \left(\begin{array}{c} n-2 \\ k-2 \end{array} \right) \right] \Delta(k). \tag{29}$$

If we particularize (28) and (29) to games $v_i, \ i = 1, 2, 3$, we obtain

$\kappa_1[v_1, \Gamma^S] = \frac{n^2-1}{3}$; $\qquad \kappa_i[v_1, \Gamma^S] = \frac{2n-1}{3}$, for $i \neq 1$,

$\kappa_1[v_2, \Gamma^S] = \frac{n-3}{2}$; $\qquad \kappa_i[v_2, \Gamma^S] = -\frac{1}{2}$, for $i \neq 1$;

$\kappa_1[v_3, \Gamma^S] = \frac{1}{n}\left[2^{n+1} - \frac{n^2+3n+4}{2}\right]$,

$\kappa_i[v_3, \Gamma^S] = \frac{2}{n(n-1)}\left[(n-2)2^{n-1} + 1\right] - \frac{1}{2}$, for $i \neq 1$.

For v_1, if we standardize the centrality measure dividing by $f_1(n) = n^2 - n$, (i.e., total centrality of the n nodes) we can study the asymptotic behaviour of the centrality for the hub and for the other nodes. If we note

$$\kappa_i^*[v_1, \Gamma^S] = \frac{\kappa_i[v_1, \Gamma^S]}{n^2 - n},$$

then we have,

$$\lim_{n\to\infty} \kappa_1^*[v_1, \Gamma^S] = \tfrac{1}{3},$$

$$\lim_{n\to\infty} \kappa_i^*[v_1, \Gamma^S] = 0,\ i \neq 1,$$

as in Grofman and Owen (1982).

b) Let us consider now a chain with 6 nodes, Γ^C (Figure 3).

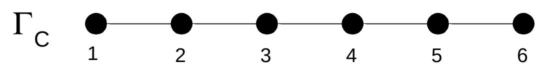


Figure 3: Chain (Γ^C)

We obtain for a general game v

$$\kappa_1[v,\Gamma^C] = \kappa_6[v,\Gamma^C] = \Delta(1)+\frac{29}{20}\Delta(2)+\frac{21}{10}\Delta(3)+\frac{37}{20}\Delta(4)+\frac{13}{15}\Delta(5)+\frac{1}{6}\Delta(6).$$

$$\kappa_2[v,\Gamma^C] = \kappa_5[v,\Gamma^C] = \Delta(1)+\frac{82}{30}\Delta(2)+\frac{53}{15}\Delta(3)+\frac{27}{20}\Delta(4)+\frac{16}{15}\Delta(5)+\frac{1}{6}\Delta(6).$$

$$\kappa_3[v,\Gamma^C] = \kappa_4[v,\Gamma^C] = \Delta(1)+\frac{199}{60}\Delta(2)+\frac{131}{30}\Delta(3)+\frac{59}{20}\Delta(4)+\frac{16}{15}\Delta(5)+\frac{1}{6}\Delta(6).$$

In the special games $v_i,\ i = 1, 2, 3$ we obtain

$$\kappa_1[v_1,\Gamma^C] = 2.9, \quad \kappa_2[v_1,\Gamma^C] = 5.47, \quad \kappa_3[v_1,\Gamma^C] = 6.63.$$

$$\kappa_1[v_2,\Gamma^C] = -0.5, \quad \kappa_2[v_2,\Gamma^C] = 0, \quad \kappa_3[v_2,\Gamma^C] = 0.$$

$$\kappa_1[v_3,\Gamma^C] = 6.43, \quad \kappa_2[v_3,\Gamma^C] = 10.2, \quad \kappa_3[v_3,\Gamma^C] = 11.87.$$

c) To illustrate the calculation of centrality on graphs with cycles, we consider the *kite* (Γ^K, Figure 4) and the messages game.

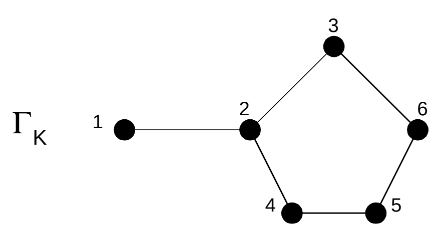


Figure 4: Kite (Γ^K)

Given that only $\Delta_1(2) = 2$ is different from zero, it is sufficient to consider coalitions of cardinality two. Table 1 below shows how the Harsanyi dividends are allocated among the six players.

Coalition S	Elements of $\mathcal{M}_{\Gamma^K}(S)$	$\Delta(S)\varphi_i(w_{S,\Gamma^K}), i = 1, \ldots, 6$					
$\{1,2\}$	$\{1,2\}$	1	1	0	0	0	0
$\{1,3\}$	$\{1,2,3\}$	0.666	0.666	0.666	0	0	0
$\{1,4\}$	$\{1,2,4\}$	0.666	0.666	0	0.666	0	0
$\{1,5\}$	$\{1,2,4,5\}$, $\{1,2,3,5,6\}$	0.566	0.566	0.066	0.166	0.566	0.066
$\{1,6\}$	$\{1,2,3,6\}$, $\{1,2,4,5,6\}$	0.566	0.566	0.166	0.066	0.066	0.566
$\{2,3\}$	$\{2,3\}$	0	1	1	0	0	0
$\{2,4\}$	$\{2,4\}$	0	1	0	1	0	0
$\{2,5\}$	$\{2,4,5\}$, $\{2,3,5,6\}$	0	0.766	0.1	0.266	0.766	0.1
$\{2,6\}$	$\{2,3,6\}$, $\{2,4,5,6\}$	0	0.766	0.266	0.1	0.1	0.766
$\{3,4\}$	$\{2,3,4\}$, $\{3,4,5,6\}$	0	0.266	0.766	0.766	0.1	0.1
$\{3,5\}$	$\{3,5,6\}$, $\{2,3,4,5\}$	0	0.1	0.766	0.1	0.766	0.266
$\{3,6\}$	$\{3,6\}$	0	0	1	0	0	1
$\{4,5\}$	$\{4,5\}$	0	0	0	1	1	0
$\{4,6\}$	$\{4,5,6\}$, $\{2,3,4,6\}$	0	0.1	0.1	0.766	0.266	0.766
$\{5,6\}$	$\{5,6\}$	0	0	0	0	1	1

Table 1

As mentioned above, when the cardinality of $\mathcal{M}_{\Gamma^K}(S)$ is greater than 1, the associated calculus become complex. To illustrate this situation let us consider the $S = \{1,6\}$ case in detail. For this coalition, $S_1 = \{1,2,3,6\}$ and $S_2 = \{1,2,4,5,6\}$ are the two minimal connected sets that contain S. Then, using (14)

$$\varphi_i(w_{S,\Gamma^K}) = \varphi_i(u_{S_1}) + \varphi_i(u_{S_2}) - \varphi_i(u_{S_1 \cup S_2}).$$

Taking (15) into account

$$\begin{aligned}
\varphi_1(w_{S,\Gamma^K}) &= 0.5 + 0.4 - 0.333 = 0.566, \\
\varphi_2(w_{S,\Gamma^K}) &= 0.5 + 0.4 - 0.333 = 0.566, \\
\varphi_3(w_{S,\Gamma^K}) &= 0.5 + 0 - 0.333 = 0.166, \\
\varphi_4(w_{S,\Gamma^K}) &= 0 + 0.4 - 0.333 = 0.066, \\
\varphi_5(w_{S,\Gamma^K}) &= 0 + 0.4 - 0.333 = 0.066, \\
\varphi_6(w_{S,\Gamma^K}) &= 0.5 + 0.4 - 0.333 = 0.566.
\end{aligned}$$

It may be seen that when there is no alternative, i.e., if $\mathcal{M}_{\Gamma^K}(S)$ has cardinality 1, an intermediary (e.g., 2 in path $1-2-3$) shares evenly with the sender and receiver of the message. When there are alternatives (e.g., 3 or 4-5 in the path from 2 to 6), these 'intermediate' players will receive much less due to the different paths available (competition).

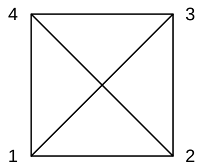
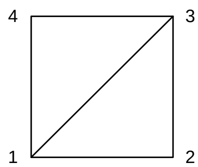
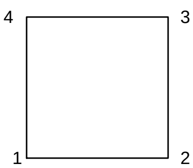
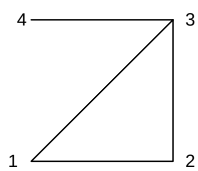
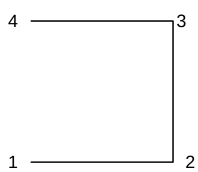
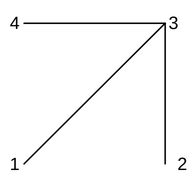
GRAPHS	NODES	v_1	v_2	v_3
Γ^1 	1	3	-1/4	11/4
	2	3	-1/4	11/4
	3	3	-1/4	11/4
	4	3	-1/4	11/4
Γ^2 	1	19/6	-1/6	17/6
	2	17/6	-1/3	8/3
	3	19/6	-1/6	17/6
	4	17/6	-1/3	8/3
Γ^3 	1	3	-1/4	11/4
	2	3	-1/4	11/4
	3	3	-1/4	11/4
	4	3	-1/4	11/4
Γ^4 	1	8/3	-1/3	5/2
	2	8/3	-1/3	5/2
	3	13/3	1/6	11/3
	4	7/3	-1/2	7/3
Γ^5 	1	13/6	-1/2	13/6
	2	23/6	0	20/6
	3	23/6	0	20/6
	4	13/6	-1/2	13/6
Γ^6 	1	7/3	-1/2	7/3
	2	7/3	-1/2	7/3
	3	5	1/2	4
	4	7/3	-1/2	7/3

Table 2

5 A decomposition of the measure of centrality

In Table 2 of previous section, we showed that each node in the network Γ^1 has the same centrality than the equivalent node in the network Γ^3 for each v_i, $i = 1, 2, 3$. This might be seen as a *not reasonable* result because the degree of connection in Γ^1 is higher than in Γ^3. Then it is interesting to try to obtain more information by decomposing the actual measure into two pieces.

Let us consider the messages game. As we mentioned in the introduction, centrality in this game can be considered as having two components: the ability to send and receive messages, and the ability to relay (or interrupt) messages between other individuals. So, from (30) and Table 1, we can see that, out of the 7.462 units of node 2, 4.532 come from the pairs $\{1,2\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$ and $\{2,6\}$ (messages individual 2 sends or receives) whereas 2.93 units come from the pairs $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{1,6\}$, $\{3,4\}$, $\{3,5\}$, $\{4,6\}$ (where 2 serves as intermediary). Doing the same with all players, we realize that the centrality vector $\kappa[v_1, \Gamma]$ can be expressed as the sum of two vectors:

$$\kappa^C[v_1, \Gamma] = (3.464, 4.532, 4.198, 4.198, 4.098, 4.098)$$

and

$$\kappa^B[v_1, \Gamma] = (0, 2.92, 0.698, 0.698, 0.532, 0.532)$$

where $\kappa^C[v_1, \Gamma]$ corresponds to sending or receiving messages (source/sink), while $\kappa^B[v_1, \Gamma]$ corresponds to relaying messages (intermediary).

More generally, for each node i in a network (N, Γ) and for $v \in \mathcal{S}_N$, we define the *communication centrality* of node i, $\kappa_i^C[v, \Gamma]$, as the portion of total centrality of node i corresponding to a payoff received as a member of different coalitions S, and the *betweenness centrality* of node i, $\kappa_i^B[v, \Gamma]$, as the payoff for i from coalitions in which i is not a member but it may be needed for the coalition to be connected.

Then, we propose the following decomposition

$$\kappa[v, \Gamma] = \kappa^C[v, \Gamma] + \kappa^B[v, \Gamma]. \tag{31}$$

If $\delta_S \in \mathbb{R}^n$ is the characteristic vector of $S \subset N$ and for $x, y \in \mathbb{R}^n$, the coordinates of $x \circ y \in \mathbb{R}^n$ are $x_i \cdot y_i$, $i = 1, \ldots, n$, the terms of the expression (31) above are given by

$$\kappa^C[v, \Gamma] = \sum_{S \subset N} \Delta(S)\ \varphi\left(\mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u_{S_i})\right) \circ \delta_S$$

and

$$\kappa^B[v, \Gamma] = \sum_{S \subset N} \Delta(S)\ \varphi\left(\mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u_{S_i})\right) \circ \delta_{N-S}.$$

Turning then to Table 2, for v_1 and Γ^1, the induced centrality in each node can be decomposed in

$$\kappa_i^C[v_1,\Gamma^1] = 3, \;\; \kappa_i^B[v_1,\Gamma^1] = 0, \; 1 \le i \le 4,$$

whereas in the network (v_1,Γ^3) we have

$$\kappa_i^C[v_1,\Gamma^3] = \frac{17}{6}, \;\; \kappa_i^B[v_1,\Gamma^3] = \frac{1}{6}, \; 1 \le i \le 4.$$

The resulting decomposition for the centrality induced by $v_i, \; i = 1,2,3$ in every connected graph with 4 nodes which is shown in Table 2 is

Graphs	Nodes	v_1		v_2		v_3	
		$\kappa_i^C[v_1,\Gamma]$	$\kappa_i^B[v_1,\Gamma]$	$\kappa_i^C[v_2,\Gamma]$	$\kappa_i^B[v_2,\Gamma]$	$\kappa_i^C[v_3,\Gamma]$	$\kappa_i^B[v_3,\Gamma]$
Γ^1	$1,2,3,4$	3	0	$-1/4$	0	$11/4$	0
Γ^2	$1,3$	3	$1/6$	$-1/4$	$1/12$	$11/4$	$1/12$
	$2,4$	$17/6$	0	$-1/3$	0	$8/3$	0
Γ^3	$1,2,3,4$	$17/6$	$1/6$	$-1/3$	$1/12$	$8/3$	$1/12$
Γ^4	$1,2$	$8/3$	0	$-1/3$	0	$5/2$	0
	3	3	$4/3$	$-1/4$	$5/12$	$11/4$	$11/12$
	4	$7/3$	0	$-1/2$	0	$7/3$	0
Γ^5	$1,4$	$13/6$	0	$-1/2$	0	$13/6$	0
	$2,3$	$8/3$	$7/6$	$-1/3$	$1/3$	$5/2$	$5/6$
Γ^6	$1,2,4$	$7/3$	0	$-1/2$	0	$7/3$	0
	3	3	2	$-1/4$	$3/4$	4	$5/4$

Table 3

Let us summarize the results corresponding to the introduced decomposition for the centrality induced by v_1 in the star with n nodes and in the graph of the cliques.

For the star with n nodes Γ^S (Figure 1), we obtain

$$\kappa_1^C[v_1,\Gamma^S] = n-1, \quad \kappa_1^B[v_1,\Gamma^S] = \tfrac{(n-1)(n-2)}{3},$$

$$\kappa_i^C[v_1,\Gamma^S] = \tfrac{2n-1}{3}, \quad \kappa_i^B[v_1,\Gamma^S] = 0, \; i \neq 1.$$

In graph Γ of Figure 2(a), node 16 (the switchboard) has centrality 44.2, while nodes $5,10$ and 15 (the heralds) have centrality 29.8 each. The remaining nodes have centrality 8.87 each.

With this decomposition, we obtain

$$\kappa_{16}^C[v_1,\Gamma] = 11, \qquad \kappa_{16}^B[v_1,\Gamma] = 33.2,$$

$$\kappa_5^C[v_1,\Gamma] = 10.33, \quad \kappa_5^B[v_1,\Gamma] = 19.47,$$

$$\kappa_1^C[v_1,\Gamma] = 8.87, \qquad \kappa_1^B[v_1,\Gamma] = 0.$$

As it was expected, given the definition of $\kappa_i^B[v,\Gamma]$, for every v and every Γ, all nodes of degree 1 are not able to intermediate and then, their betweenness centrality is zero.

6 Final Comments

As was announced in the introduction, we have obtained different centrality measures depending on the assumed interactions interests. Some comments seem to be outstanding:

(a) For a particular type of connected graphs, the trees (and even the forests), some of the centrality measures we have introduced are very related with the standard measures that are used in the study of social networks.

- The centrality induced by the game

  $$v_2(S) = f_2(s) = -1, \; s \geq 1$$

  on a tree Γ is

  $$\kappa_i[v_2,\Gamma] = \frac{1}{2}\theta_i(\Gamma) - 1, \tag{32}$$

  where $\theta_i(\Gamma)$ is the degree of node i in the graph Γ.

- The two components of the centrality measure induced by v_1 in a tree Γ are

  $$\kappa_i^C[v_1,\Gamma] = \sum_{j\neq i} \frac{2}{d(i,j)+1}, \tag{33}$$

  where $d(i,j)$ is the distance between nodes i,j measured as the number of edges in the (unique) geodesic path that join the two nodes i and j in the graph Γ, and

  $$\kappa_i^B[v_1,\Gamma] = \sum_{\substack{j<k \\ j,k\neq i}}^{n} \delta_{jk}(i)\frac{2}{d(j,k)+1}, \tag{34}$$

  where $\delta_{jk}(i) = \begin{cases} 1, & \text{if } i \text{ is in the geodesic path that join } j \text{ and } k, \\ 0, & \text{otherwise ,} \end{cases}$

being κ_i^C a closeness measure of centrality and κ_i^B a betweenness or intermediation measure.

When a graph has cycles, (33) and (34) (or their obvious extensions for this case) are not valid. In the example below (Figure 5) nodes 2 and 5 receive 7/15 as a payoff as intermediate in communication between 1 and 6, whereas nodes 3 and 4 have 1/15 each. From the extension of (34), 2 and 5 would receive 2/5 each, whereas 3 and 4 would receive 1/5 each. When there are alternatives, Shapley value is less generous than (33) and (34) are.

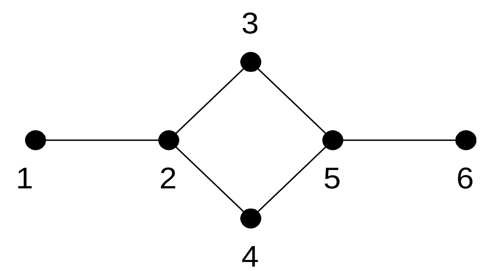


Figure 5

(b) The centrality measures induced by different games, though they have some similar patterns (e.g. the maximal centrality is attained by the hub of a star), will not order in the same way, in general, the centralities of the different nodes of a graph. For the *comet* Γ^{CT} (Figure 6) we obtain

$$\kappa_2[v_2, \Gamma^{CT}] = \frac{1}{2}\theta_2(\Gamma^{CT}) - 1 = -\frac{1}{2},$$

$$\kappa_{29}[v_2, \Gamma^{CT}] = \frac{1}{2}\theta_{29}(\Gamma^{CT}) - 1 = 0,$$

whereas

$$\kappa_2[v_1, \Gamma^{CT}] = 15.69,$$

$$\kappa_{29}[v_1, \Gamma^{CT}] = 14.02.$$

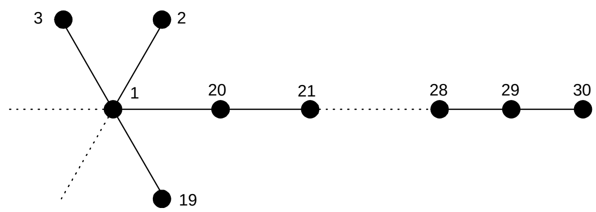


Figure 6: Comet (Γ^{CT})

Moreover this example shows that a node of degree 1 (node 2) is not necessarily the node with least centrality in a graph as perhaps it should be inferred from previous examples.

(c) In previous examples the two components of the centrality tend to go together in the sense that they order the nodes in a graph in the same way. This is not true in general. In the comet we have

$$\kappa_2^C[v_1, \Gamma^{CT}] = \kappa_2[v_1, \Gamma^{CT}] = 15.69, \qquad \kappa_2^B[v_1, \Gamma^{CT}] = 0,$$

whereas,

$$\kappa_{29}^C[v_1, \Gamma^{CT}] = 8.04, \qquad \kappa_{29}^B[v_1, \Gamma^{CT}] = 5.98.$$

(d) It is interesting to point out that the problem of computation associated with the defined centrality is NP-hard. For every $S \subset N$ we must find all elements of $\mathcal{M}_\Gamma(S)$.

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